PRINCIPLES OF STELLAR STRUCTURE

VOLUME I: PHYSICAL PRINCIPLES

BY JOHN P. COX

in collaboration with R. Thomas Giuli

Joint Institute for Laboratory Astrophysics and
Department of Physics and Astrophysics
University of Colorado

GORDON AND BREACH

Principles of Stellar Structure

Volume 1

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Physical Principles, Volume 1 of the two-volume series **Principles of Stellar Structure** treats the basic physics that underlies present theories of stellar interiors. The book will be useful as a reference work for research workers studying stellar structure, as well as for first-year graduate students and advanced undergraduates with strong backgrounds in the physical sciences. This volume contains eighteen chapters, of which Chapter 0 is an "Introduction and Survey of Observations," Chapter 1 discusses "Physical Conditions in Stellar Interiors," and Chapters 2 through 8 discuss the transfer of radiation in stellar interiors and closely related phenomena. The effects of dispersion on these phenomena are consistently included. Particularly noteworthy in this section is Chapter 7 which discusses local thermodynamic equilibrium, based on the simultaneous solution of the equations of radiative transfer and the statistically steady state for the atomic energy level population. Also of special note in Volume 1 are the discussions of the mixing-length theory of convection in Chapter 14 and of departures from the perfect gas equation of state resulting from electrostatic inter-

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Physical Principles

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To

MARSHAL H. WRUBEL

who first inspired my serious interest
in the subject of stellar interiors

Preface

Several books on stellar interiors have appeared in recent years, among which are *Nuclear Transformations, Stellar Interiors, and Nebulae*, by L. H. Aller (1954); *Structure and Evolution of the Stars*, by M. Schwarzschild (1958); *Physical Processes in Stellar Interiors*, by D. Frank-Kamenetskii (1962); *Stellar Interiors*, by D. H. Menzel, P. L. Bhatnagar, and H. K. Sen (1963); Vol. *51* of the *Handbuch der Physik*, ed. by S. Flügge (1958); *Stellar Structure*, ed. by L. H. Aller and D. H. McLaughlin (1965); and *Stellar Physics*, by H.-Y. Chiu (1967); in addition to a number of symposium proceedings on stellar structure and evolution. The present book is intended not to compete with or to supersede these books but, rather, to complement them. The subject matter dealt with in stellar interiors covers a very broad area indeed, and is almost yearly expanding even further into fields that were previously not considered a part of the proper subject matter of stellar interiors – fields such as stellar atmospheres, the interstellar medium, aerodynamics, high-energy particle physics, relativistic astrophysics, cosmology, even solid state physics. Moreover, what with the very rapid growth of knowledge in the subject in recent years, much of this recent material is scattered throughout the rather extensive astrophysical literature. In view of these considerations, it was felt that there was a need to bring together into one place much of this material, and to have a fairly extensive and somewhat uniform presentation of the principal present-day theoretical and factual knowledge relevant to the study of stellar structure. It was primarily in an attempt to fill this need that the present book was written.

The present book is based largely on lecture notes developed for stellar interiors courses given by the writer over somewhat more than the past decade, both at Cornell University and at the University of Colorado. The transformation of the notes into a book was begun in the fall of 1963 and required approximately three years. Much of the planning for this transformation was done jointly with the writer's collaborator, R. Thomas Giuli, without whose dedication, encouragement, and untiring efforts the transformation might not have been effected.

The material has undergone many revisions and updatings during this period. The most recent revision and updating of the entire book were

carried out during the academic year 1965–66, and the last two chapters (Chaps. 26 and 27 on, respectively, stellar evolution and pulsating stars) were completed in their present forms in mid-1966, which defines the cut-off date for the references for the main text. Although no claim for completeness of the references is made, it is believed that most of the chapters are reasonably up-to-date (at least as far as published literature is concerned) through the end of 1965; and some are reasonably up-to-date through mid-1966.

Unfortunately, because of the delays associated with the publication proceedings, it was impossible to include in the main text the more recent results from mid-1966 through early 1968. However, as a partial remedy for this defect a Supplement has been prepared, written in May–June, 1968, which summarizes some of this work. Since results obtained over the past two years or so have clarified a number of points which were still somewhat unsettled at the time the main text was completed (and corrected a few errors), the reader is advised to consult the relevant parts of the Supplement before leaving the topic of present interest in the main text.

The rather large size of the book is in part a result of our deliberate desire to maintain a fairly leisurely style. This desire is consistent with some of the other main goals we have striven for. These were clarity of presentation, ready comprehensibility, adequate explanation and discussion of results, presentation of alternative viewpoints in some cases, correctness, careful attention to details, and some degree of thoroughness and generality. Many derivations can be simply "read," alternative derivations are sometimes given, many equations are written down several times for the convenience of the reader, and an effort has been made to keep the number of statements starting out with "It can be shown..." to a minimum. (However, the reader will find that statements which start out with "It can *easily* be shown..." are generally correct.) It remains to be seen whether these goals have been realized, and, if so, whether their realization will have justified the size of the book. In a few places, on the other hand (especially in parts of Chaps. 20, 24, and 27), considerable work on the part of the reader may be required to follow some of the derivations in detail.

It is hoped that the book will be useful in at least two contexts. On the one hand, the attention to detail and the leisurely style may be helpful to the beginning reader approaching the subject of stellar interiors for the first time. On the other hand, it is hoped that the book contains sufficient information to be useful as a reference work for the more advanced reader. Some new (unpublished) research results are included, especially in Chaps. 2 through 8 and in Chaps. 25 and 27. The writer takes full responsibility if future research should show that these results are incorrect.

The text consists of 28 chapters, which are grouped into two large divisions, or parts. Part I, comprising Volume 1, consists of Chaps. 0 through 17 and contains most of the basic physics which underlies present theories of stellar interiors. Part II, comprising Volume 2, consists of Chaps. 18 through 27 and is devoted primarily to applications of the basic physics to problems of stars and stellar interiors. The outstanding exception to this organization is the discussion in Chap. 24 of semi-degenerate equations of state. This discussion should logically be in Part I, but was placed in its present location largely because of the immediate and extensive applications in the discussion in Chap. 25 of the theory of white dwarf stars. Additional useful information is contained in five appendices, and the Supplement follows the appendices.

This "basic physics" consists mostly of "classical" physics, with frequent applications of quantum mechanics and special relativity in appropriate places. General relativity as such is considered outside the scope of this book. However, this subject is important in certain very late stages of stellar evolution, and some applications of general relativity are made in this context (see Chap. 26).

By "stars" we mean, loosely, self-gravitating, self-luminous (usually) objects having masses between, roughly, 10^{-2} and 10^{+2} solar masses. Reasons for this choice of mass range are given in Chap. 26, on stellar evolution. In addition, as is pointed out in the introduction to this chapter, we restrict consideration in this book (except in Sect. S. 26.6 of the Supplement) only to single, non-rotating, non-magnetic, spherically symmetric stars, although some general effects of rotation and magnetic fields are considered semi-quantitatively in parts of Chap. 17, on stellar energy sources, and in parts of Chaps. 25 and 26. Stellar pulsation, generally regarded as a "perturbation" in the same sense as are stellar rotation and stellar magnetic fields, is dealt with at some length in Chap. 27, in part because of the considerable progress which has been made in recent years in our understanding of the causes and nature of this phenomenon.

Some comments are perhaps in order regarding the contents of certain chapters and the treatment of the appropriate subjects.

Chapters 2 through 8 are all closely interrelated, and form, essentially, a "unit" which is concerned, mostly, with the general problems of the transfer of radiation in stellar interiors, the thermodynamic state of the stellar interior, and certain approximations valid under these conditions. The treatment in these chapters differs in two respects from previous published treatments of which the writer is aware. First, the effects of dispersion (*i.e.*, of a non-unity and frequency dependent refractive index, see Sect. 2.10) are carried through in a consistent manner, within the limitations of the basic assumptions and of present-day knowledge about this subject. These effects are not important

in most applications of interest, and may easily be ignored by the reader who is not interested in these effects. Secondly, our arguments for the validity of "local thermodynamic equilibrium" (LTE) in stellar interiors (see Chap. 7) are based on the ideas, developed mostly by R. N. Thomas and collaborators, involving the simultaneous solution of the transfer equation and the statistically steady state equations for the population of atomic energy levels. It is only on the basis of these ideas, we feel, that a rigorous approach to this problem may be found. While our actual arguments are not rigorous and are somewhat schematic, nevertheless we feel that these arguments are more convincing than the usual ones based on a more intuitive approach.

In Chap. 12 the concept of polytropic changes is generalized over that found in usual treatments to apply to *any* material equation of state.

Chapter 13, on the stability of the stellar material against convection, is based on the conventional Schwarzschild criterion; however, the treatment includes not only the usual case of uniform chemical composition, but also the case of a spatially variable composition.

Chapter 14, on the mixing length theory of convection, is in part along the lines developed mostly by Mrs. Böhm-Vitense. However, some discussion regarding solutions of the equations is given, and some interesting order-of-magnitude numerical relations pertaining to convection in stars are derived.

In Chap. 15, on the excitation-ionization equilibrium of matter in stellar interiors, some discussion is given of effects of electrostatic interactions, both with respect to the "lowering of the ionization potential" and to their effects on the gas pressure.

Included in Chap. 17, on stellar energy sources, are, among other things, the derivation of a very general form of the virial theorem (which is applied here and in several other places throughout the book), and the calculation of the reaction rate for the triple alpha reaction both from a direct "rate" approach and also from use of the Saha equation (after the manner originally followed by E. E. Salpeter).

Chapters 18 and 19, dealing, respectively, with the Vogt-Russell "Theorem" and some special cases in stellar structure, serve the purpose of an introduction to Part II. In Chap. 18 it is emphasized that the Vogt-Russell "theorem," as usually stated, is not a theorem at all. Some of the conditions under which this "theorem" may be valid are discussed.

Chapter 20 is concerned primarily with a fairly detailed discussion of the surface boundary conditions, how they lead to deep convective envelopes in the cooler stars, and with the structure of convective and radiative envelopes. Most of the ideas discussed here have been developed and their profound consequences worked out only since about 1960.

Chapters 21, 22, and 23 are all rather closely related, as they deal with various aspects of stellar models and their computation. In Chap. 21, on computation of stellar models, a section is included which outlines briefly some of the more modern techniques of stellar model calculation, such as the "Henyey" techniques and variations thereof. In Chap. 22 the concept of "permanent homology" is introduced, and conditions on the constitutive equations for permanent homology are derived. In Chap. 23, in addition to a discussion of the more usual types of idealized stellar models, the luminosity of completely convective stars, conditions required for a star to be completely convective, and effects of inhomogeneities in chemical composition on stellar properties are discussed.

Chapter 24, on semi-degenerate equations of state, is actually concerned with the whole temperature-density plane, and a discussion of effects of electron-positron pairs comprises the last section.

In Chap. 25, on the theory of white dwarf stars, the usual "classical" theory of zero-temperature stars occupies only the first section. Most of the chapter is devoted to consideration of the thermal properties of white dwarfs, approach to the white dwarf state, energy sources of white dwarf stars, secular stability, effect of "perturbations" on properties of white dwarfs, and the role of these stars in stellar evolution.

Chapter 26, on stellar evolution, is the one which will become obsolete first. This chapter is to a certain extent self-contained. Although numerous references to other parts of the book are given, the chapter can be read and most of it understood (at least qualitatively) without reference to other parts of the book. It is hoped that this chapter provides a fair overall picture of the status of stellar evolution theory as of mid-1966.

Chapter 27, on pulsating stars, contains a fairly thorough discussion of stellar pulsational stability, a summary of recent ideas and calculations related to some of the causes of pulsational instability in stars, and a summary of recent (as of mid-1966) non-linear calculations of stellar pulsation. This chapter is also almost self-contained.

The level at which the book has been aimed is about that of the first year graduate student in physical sciences, although most of the book can probably be read with understanding by an advanced undergraduate. A knowledge of calculus, differential equations, and vector analysis is assumed on the part of the reader.

Although the book has not necessarily been intended exclusively as a text (for example, no problems are included), it may nevertheless be useful in that capacity. The writer has found that there is more than sufficient material contained in the book for a two-semester course in stellar structure. (Part I might be covered in the first semester, Part II in the second.)

No bibliographies have been compiled, but it is believed that sufficient references to the literature have been given in the text and in the Supplement to enable the reader to seek further information on almost any given topic. The list of references for the main text also serves as an author index.

Cross-references to other parts of the book are generally to chapter, section, figure, table, or equation number. The numbering of sections, figures, tables, and equations is always by chapter, and the appropriate number is always preceded by the chapter number followed by a period; *e.g.*, 17.3 means (Section, Figure, Table, Equation) number 3 in Chap. 17. Subsections are numbered by affixing additional symbols after the section number. Thus, *e.g.*, Sect. 24.9b1 means subsubsection 1 in subsection b in Section 9 of Chap. 24. References to sections, figures, and tables are always indicated explicitly; *e.g.*, Sect. 2.10a, Fig. 14.1, Table 26.5. References to equations are generally indicated simply by enclosing the appropriate equation number in parentheses; thus (14.6a) means equation 6a in Chap. 14.

Finally, the writer takes great pleasure in acknowledging the many persons who have contributed, directly or indirectly, to this book. First and foremost is the writer's collaborator, R. Thomas Giuli, mentioned earlier in this preface, for whose contributions the writer's gratitude knows no bounds. Most of the equations have been checked by him, as well as most of the text for clarity and correctness. Many thanks go to Richard N. Thomas for his encouragement, enthusiasm, and interest throughout the entire undertaking. Yueh-Ying Cheng, who uncomplainingly calculated most of the numbers appearing in the book, some of the equations in Sect. 9.18, and many of the expansions in Sect. 24.7, is warmly thanked, as are also Margaret Herz, who prepared the index and assisted in numerous other ways, and J. Craig Wheeler, who assisted in the preparation of the Reference List and Author Index and in many other ways. The writer is also deeply grateful to Edward Langer for reading practically the entire manuscript and pointing out a number of errors, typographical and otherwise, and to Kenneth Ziebarth, John C. Stewart, and Edward U. Condon for reading and commenting on parts of the manuscript.

Various persons, in addition to those mentioned above, have read and commented on various sections and chapters of the book, and the writer is deeply grateful to them for their efforts. Among these are the following: Joachim Oxenius, who read an earlier version of Chap. 2; Mahendra S. Vardya, who read earlier versions of most of Chaps. 1 through 14 and who offered a number of useful general suggestions; T. Neil Divine, who read earlier versions of Chaps. 0 through 9 and Chaps. 14 and 15 and who offered numerous valuable suggestions regarding the book; Richard N. Thomas, who read an earlier version of Chap. 7; Arthur N. Cox, who read an earlier

version of Sect. 21.7; Douglas Gough, who read an earlier version of Chap. 27; and Richard L. Sears, who read and commented on Chap. 26.

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Sincere thanks are also extended to the numerous persons who have sent the writer preprints of their work over the past few years. Some of this work could not have been incorporated into the book, had it been necessary for the writer to wait for publication of the papers.

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John P. Cox, June 1968

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Principles of Stellar Structure

VOLUME 1. PHYSICAL PRINCIPLES

Introduction to Part I

Volume 1 contains a discussion of the basic physics related to the conditions existing in stellar interiors. The student who encounters this text in a course in stellar interiors undoubtedly will have previously been exposed to much of the material contained in this part. However, some of the concepts dealt with here may be unfamiliar to him (*e.g.*, general specific heats, stellar opacity, radiative transfer). It is therefore strongly recommended that the student who contemplates doing further work in the field of stellar structure become thoroughly familiar with the basic material in Volume 1 before proceeding to the application of this material in Volume 2.

CHAPTER 0

Introduction and Survey of Observations

The basic observational material of relevance for stellar interior studies has been concisely summarized by Schwarzschild [Sc58b, Chap. 1], by Frank-Kamenetskii [Fr62, Chap. 1], and by Menzel, Bhatnagar, and Sen [Me63, Chap. 1]. A more recent summary is given by Eggen [Eg65]. More detailed discussions may be found in Vol. *51* of the *Handbuch der Physik* (ed. S. Flügge, 1958), and in Strand [St63a]. Here we shall review briefly only certain aspects of the data which we feel are especially relevant.

We assume that the reader is familiar with the basic terminology of observational astrophysics (see, for example, Aller [Al63b, especially Chap. 1]). Also, we do not discuss the methods by which the relevant empirical data are obtained (see, for example, Baker [Ba64c]).

In Sect. 0.1 we give values of luminosity, mass, and radius for some representative stars, and we discuss the ranges in values of these parameters for observed stars. In Sect. 0.2 we discuss three important characteristic stellar time scales (derived in later chapters), which are referred to frequently throughout much of this book. Finally, in Sects. 0.3 and 0.4 we discuss two important correlations among the observable parameters of stars: the (L,T_e) correlation (the "Hertzsprung-Russell diagram") (Sect. 0.3) and the (L,M) correlation (the "empirical mass-luminosity relation") (Sect. 0.4).

0.1 Luminosities, Masses, and Radii of Stars

The parameters conventionally adopted for the specification of the physical properties of a star are the luminosity L (normally taken to be the total rate of loss of energy by radiation from the stellar surface), the mass M, and the radius R. Values of these parameters can often be determined empirically or semi-empirically. In the case of variable stars L and R are normally taken to be average values over times long compared to the

characteristic time of the variations in luminosity and radius. In such stars other parameters, such as the period of the variation, could of course also be given. From L and R follows the *effective temperature* T_e, which is *defined* by the relation

$$L = 4\pi R^2 \sigma T_e^4 , \tag{0.1}$$

where σ is the Stefan-Boltzmann constant (see Sect. 3.7). For the sun we have (Allen [A163a])

$$\begin{aligned} L_\odot &= 3.90\times 10^{33} \text{ erg/sec} \\ M_\odot &= 1.989\times 10^{33} \text{ gm} \\ R_\odot &= 6.960\times 10^{10} \text{ cm} \\ T_e &= 5800^\circ\text{K} . \end{aligned} \tag{0.2}$$

Solar units are frequently used for L, M, and R.

These three parameters show a tremendous range of values for the various stars. Taking extreme limits, we have

$$\begin{aligned} &10^{-6}L_\odot \lesssim L \lesssim 10^6 L_\odot\text{: factor of } \sim 10^{12} \text{ in } L; \\ &(1/800)R_\odot \lesssim R \lesssim 1500 R_\odot\text{: factor of } \sim 10^6 \text{ in } R; \\ &(1/20)M_\odot \lesssim M \lesssim 50 M_\odot\text{: factor of } \sim 10^3 \text{ in } M; \\ &2000^\circ\text{K} \lesssim T_e \lesssim 100{,}000^\circ\text{K: factor of } \sim 50\ (\approx 10^{1.7}) \text{ in } T_e .* \end{aligned} \tag{0.3}$$

Are the values of L, M, and R completely independent of one another? In view of the large ranges in values of these basic parameters, any correlations among them would be of great interest. A three-dimensional plot of L, M, R shows that, indeed, the points, rather than being uniformly distributed, tend to form into definite sheets, or volumes, with some regions completely empty. Field stars in the general vicinity of the sun would be mostly concentrated in a conspicuous narrow band running diagonally from one corner of the cube (small L, M, R) to the other (large L, M, and R). This band would be found to contain the vast majority of all such stars (well over 90 per cent). This band is called the *main sequence*. The density toward the lower end is several powers of ten larger than that at the upper end, reflecting the fact that there are many more small, faint, small-mass stars per unit volume of space than large, bright, large-mass stars. The main sequence band should have only a finite length, however, corresponding to $M \approx 0.08 M_\odot$ at the lower end (*cf.* Chaps. 25 and 26) and to $M \approx 60 M_\odot$ at the upper end

* In this book we do not consider the recently discovered "quasars" (see, *e.g.*, Robinson, Schild, and Schucking [Ro65]) as stars; their nature is as yet unknown. We also do not regard the postulated "supermassive" objects ($M \ll 100\ M_\odot$, say), which have received a great deal of theoretical attention recently (see, *e.g.*, Vols. *140*ff of the *Astrophysical Journal*), as possible models for stars (see next paragraph and Sect. 26.3).

(*cf.* Chaps. 26 and 27). Evolutionary processes have evidently favored only certain combinations of L, M, R, and no others. It is one task of the theory of stellar structure to explain this remarkable grouping of values of L, M, and R.

Table 0.1

SOME PROPERTIES OF REPRESENTATIVE STARS

Star	MKK *Sp.Type*	T_e(°K)	$L/L_\odot$	$R/R_\odot$	$M/M_\odot$	$\bar{\rho}$(gm/cm^3)	*Remarks*
α Sco A (Antares)	M0 Ib	3300	34,000	530	19:	1.7×10^{-7}	Supergiant
α Boo (Arcturus)	K2 III	3970	130	26	4.2:	3.2×10^{-4}	Giant
η Ori	B1 V	23,000	13,000	7.2	13.7	0.052	
α Ca Ma A (Sirius)	A1 V	9700	61	2.4	3.3	0.81	Main Sequence
Sun	G2 V	5740*	1	1	1	1.410	
Barnard's Star	M5 V	3000:	0.015	0.50	0.38	4.3	
α C Ma B (Sirius B)	A5 V	8200	0.0026	0.026	0.96	7.7×10^{4}	White Dwarf

* This value of T_e, as well as the others in this column, are temperatures of the MKK spectral types (*cf.* Table 0.2) and not necessarily of the actual stars listed.

Table 0.1 gives values of L, M, R, T_e, and mean density $\bar{\rho} = M/[(4/3)\pi R^3]$ for some representative stars. The radii given here were computed from L and T_e (except in the case of the sun), and the masses were computed from an assumed mass-luminosity relation (except in the cases of the sun and Sirius A and B). Such a relation may not be applicable to giants and supergiants (*cf.* Sects. 0.4 and 26.1b). The entries in this table accordingly may not be very accurate and are to be taken only as rough indications of actual values. The monotonic decreases in the values of L, M, and R, and the monotonic *increase* in the value of $\bar{\rho}$, in descending the main sequence should be noted. It may also be noted, parenthetically, that the radius given for Antares exceeds the radius of Mars' orbit, and the radius given for Sir. B is only about 2.5 earth radii.

Were we to examine the distribution of points on our three-dimensional plot of various selected *groups* of stars, we would in general find that this

distribution is different for different groups. For example, the distribution for stars in *globular clusters* (representatives of Baade's [Ba44] Population II) would be found to be fundamentally different from that for stars in *galactic clusters* (representatives of Population I); even different galactic clusters would be found to have different distributions (see Figs. 26.1, 26.2, and 26.3 below). These differences can be understood in terms of aging, or evolution, and of differences in "initial" chemical composition (*cf.* Chap. 26). The study of these differences between stars in different types of clusters furnishes a very powerful tool in the observational approach to stellar evolution (*cf.* Sect. 26.1b).

The study of stellar evolution introduces the *time* variable into our considerations, even with regard to stars which are in hydrostatic equilibrium. The question of stellar evolution concerns how the points in our three-dimensional plot of L, M, and R move about over long periods of time, where the meaning of a "long" period of time will be discussed in the next section.

0.2 Stellar Time Scales

The most important cause of stellar evolution in single stars is assumed to be the gradual change in the chemical composition of the stellar interior, brought about by the nuclear transmutations associated with the nuclear reactions occurring within the star. In some stars dynamical effects, such as rotation, mass loss, or tidal interaction with a companion, can significantly affect the evolution. Such effects will not be considered in detail in this book. During times when the star is consuming nuclear fuels, it is normally very nearly in *thermal equilibrium* (*cf.* Chap. 5), with the energy lost from the stellar surface very nearly compensated for by the release of thermonuclear energy in the interior.* The relevant time for questions of stellar evolution is then in most cases the "nuclear" time t_{nuc}, which may be defined as the time required for the properties of a star to change significantly as a result of nuclear transmutations occurring in the stellar interior. For example, for a star deriving its energy from hydrogen "burning" (conversion of H^1 into He^4) we have, to order of magnitude,

$$t_{\text{nuc}} \sim \left(\frac{1}{10}\right)\frac{0.007\,Mc^2}{L} \sim 10^{10}\left(\frac{M}{M_\odot}\right)\left(\frac{L_\odot}{L}\right)\text{years}, \qquad (0.5)$$

where c is the velocity of light in vacuo and where we have assumed that approximately 10 per cent of the mass of the star is available for hydrogen

* For exceptions to this statement, see Chap. 26, especially Sect. 26.4.

"burning." The factor 0.007 is the fraction of the mass that is converted to energy when a given mass of H^1 is transformed completely into He^4 (*cf.* Chap. 17). Points in our three-dimensional L, M, R diagram would be expected to move significantly in times of the order of or larger than t_{nuc}.

The representative point for a star which has *no* effective nuclear fuel (for example, a star which has exhausted a particular nuclear fuel or which has not yet become hot enough to "ignite" a nuclear fuel) will move significantly in the three-dimensional L, M, R diagram in a time of the order of t_{K}, the "Kelvin" time. The Kelvin time is, essentially, the "relaxation time" for departures of a star from energy balance (*i.e.*, thermal equilibrium); it is also approximately the "e-folding" time for radius changes in a gravitationally contracting star (*cf.* Sect. 17.4). In this case the energy lost through the stellar surface is supplied by the store of thermal energy in the star, which in turn is related (for a star in hydrostatic equilibrium) to the gravitational energy of the star via the virial theorem (*cf.* Sect. 17.2). To order of magnitude, we may write

$$t_{\text{K}} \sim E_{\text{th}}/L, \tag{0.6}$$

where E_{th} is the total internal thermal energy of the star. If the star is, on the average at least, in hydrostatic equilibrium, possesses no magnetic fields, and if its internal energy is made up mostly of non-relativistic, monatomic particles, then the virial theorem (*cf.* Sect. 17.2) can be written as $E_{\text{th}} = -(1/2)\Omega = (1/2)qGM^2/R$, where G is the constant of gravitation, Ω is the gravitational potential energy of the star (*cf.* Sect. 17.1), and q is a numerical factor of order unity ($q \approx 3/2$ for main sequence stars). Taking $q = 3/2$, we have

$$t_{\text{K}} \sim \frac{3}{4}\frac{GM^2}{LR} \approx 2 \times 10^7 \left(\frac{M}{M_\odot}\right)^2 \left(\frac{L_\odot}{L}\right)\left(\frac{R_\odot}{R}\right) \text{years}. \tag{0.7}$$

More accurate expressions for t_{K} will be derived in Sects. 17.4, 25.5, and 27.5b.

For stars which are not in *hydrostatic equilibrium* (*i.e.*, pressure balance, *cf.* Chap. 1), the relevant time scale is t_{ff}, the "free-fall" time scale. Pulsating stars (see Chap. 27) are a good example of this kind of star. The free-fall time is of the order of magnitude of the time required for the dynamical collapse of a star; as will be shown in Chap. 27, t_{ff} is also of the order of the pulsation period of a pulsating star. It will be shown in Chap. 1 (*cf.* also Sect. 26.2b and Chap. 27) that

$$t_{\text{ff}} \sim 2\sqrt{\frac{R^3}{GM}} = \frac{2}{(4\pi G/3)^{1/2}}\left(\frac{1}{\bar{\rho}}\right)^{1/2} \approx 0^{\text{d}}.04\,(\bar{\rho}_\odot/\bar{\rho})^{1/2}, \tag{0.8}$$

so that $t_{\text{ff}} \sim 1$ hour for the sun.

In the case of exploding stars, such as novae or supernovae, the appropriate time scale is not necessarily t_{ff}, although it may be in many cases. The time t_{ff} is also not necessarily the appropriate time scale for magnetic variables and possibly for other types of variable stars.

In connection with questions of stellar evolution, then, a "long" time would normally be one comparable with t_{nuc} or, in some cases, t_K; but at any rate large compared with t_{ff}. In most cases we find that $t_{ff} \ll t_K \ll t_{nuc}$.

It is useful in a number of contexts, in fact, to note the orders of magnitude of some ratios of these three time scales. We have, first,

$$\frac{t_{nuc}}{t_K} \sim 10^{-3} \frac{Mc^2}{GM^2/R} \sim 10^3 \left(\frac{R}{R_\odot}\right)\left(\frac{M}{M_\odot}\right)^{-1}, \tag{0.8'}$$

so that this ratio is of the order of the ratio of the total available nuclear energy of a star to its gravitational potential energy (*cf.* Sect. 17.1). Next, we have

$$\frac{t_K}{t_{ff}} \sim \frac{GM^2/LR}{(R^3/GM)^{1/2}} = \frac{G^{3/2}M^{5/2}}{LR^{5/2}} \sim 10^{12} \left(\frac{M}{M_\odot}\right)^{2.5} \left(\frac{L}{L_\odot}\right)^{-1} \left(\frac{R}{R_\odot}\right)^{-2.5}. \tag{0.8''}$$

These equations confirm the validity of the inequality in the preceding paragraph for "normal" stars.

0.3 Empirical (L,T_e) Correlation: Hertzsprung-Russell Diagram

If the *absolute visual magnitudes*, M_V, are plotted vs. *spectral types* for several thousands of stars in the solar neighborhood (representatives of field Population I stars), a distribution of points is obtained similar to that shown in Figs. 0.1 (from Keenan [Ke63, p.106]) and 0.2 (from Keenan [Ke63, pp. 91–92]), where the lines only indicate *average* loci for the various MKK (Morgan, Keenan, and Kellman [Mo43]) luminosity classes. The conspicuous narrow band running diagonally across the diagram (solid line in Fig. 0.2) is the *main sequence* (luminosity class V), and it contains most of the points. The location of the *subgiants* (class IV), *giants* (class III), *bright giants* (class II), and *less bright supergiants* (class Ib), is also indicated schematically in Fig. 0.2. The location of the white dwarfs is also indicated in Figs. 0.1 and 0.2. (*Cf.* Fig. 26.8, which shows the theoretically computed "zero-age" main sequence (defined in Sect. 26.3)). Also, theory and observation are compared in the review article by Eggen [Eg65].

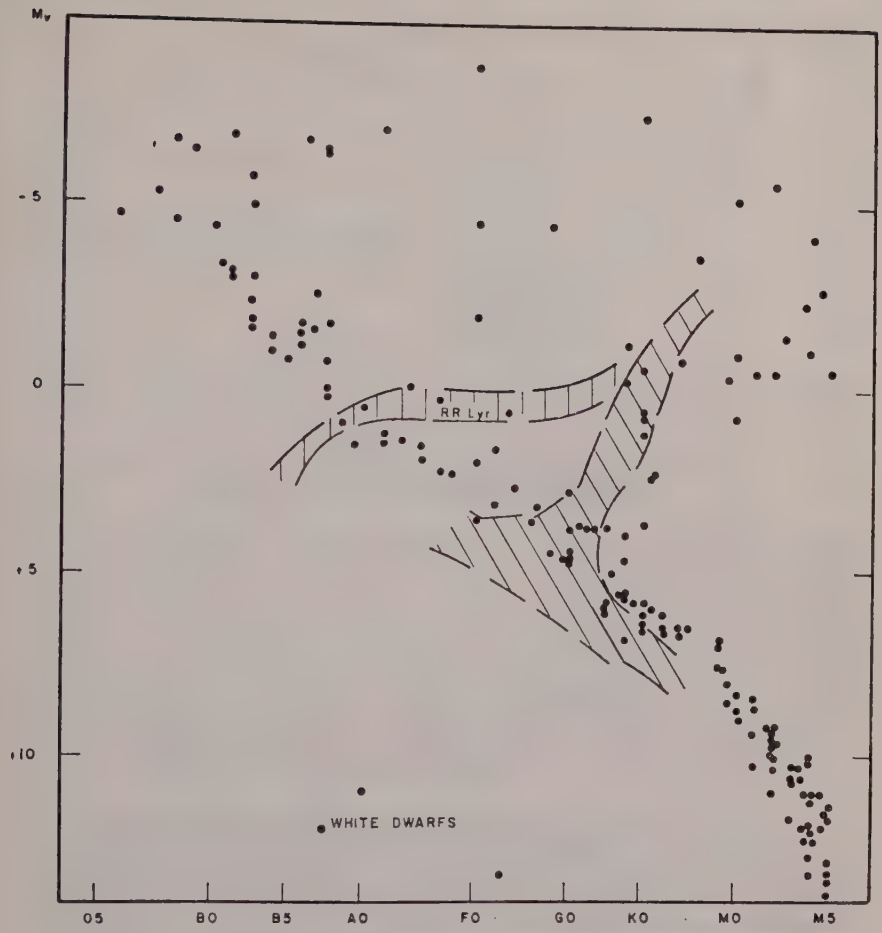


Fig. 0.1 The Hertzsprung-Russell diagram for stars of Population I (*filled circles*) and Population II (*cross-hatched areas*). (From Keenan [Ke63], reproduced by permission of the University of Chicago Press, © copyright 1963 by the University of Chicago Press. All rights reserved).

The modern counterpart of the diagram shown in Figs. 0.1 and 0.2 is the "color-magnitude" diagram, in which M_V is plotted against (intrinsic) photoelectric color index $B-V$ (see Fig. 0.3). This kind of diagram is convenient for plotting observational (especially photoelectric) results.

The "theoretical" counterpart of the above two diagrams is the M_{bol}-log T_e diagram (see Fig. 0.3), as this is the one on which the results of theoretical calculations can be directly plotted (see Fig. 26.8 below).

All three of the above types of diagrams are loosely referred to as the "H-R diagram."

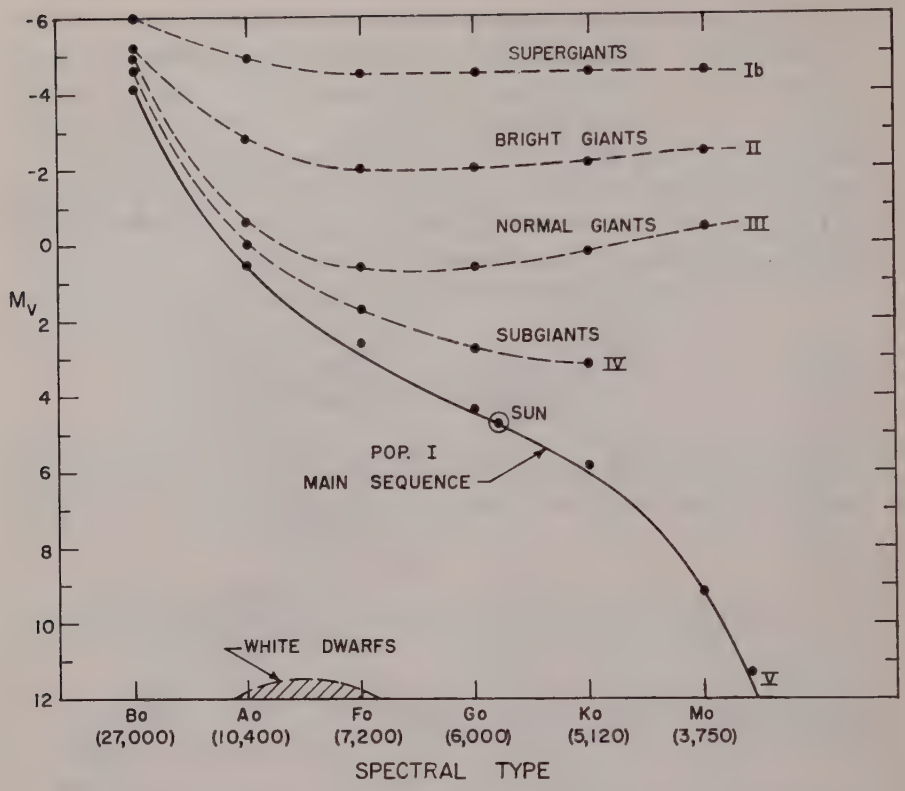


Fig. 0.2 The Hertzsprung-Russell diagram (schematic) for Population I stars, showing the MKK spectral types and luminosity classes. The lines indicate *average* loci for the various luminosity classes (*cf.* Fig. 0.1). Numbers in parentheses below the spectral types on the abscissa are effective temperatures for luminosity class V (from Table 0.2).

For convenience, we present here the relation between M_{bol} and L (see any text on introductory astronomy):

$$M_{bol} = -2.5 \log (L/L_\odot) + M_{bol}(\odot) , \qquad (0.9a)$$

where we adopt the value

$$M_{bol}(\odot) = +4.77 . \qquad (0.9b)$$

The *bolometric correction* (*B.C.*), which depends strongly on spectral type (*i.e.*, on T_e or $B-V$) and less strongly on luminosity class (*i.e.*, on M_V or M_{bol}), supplies the relation between M_V and M_{bol}:

$$M_{bol} = M_V + B.C. \qquad (0.10)$$

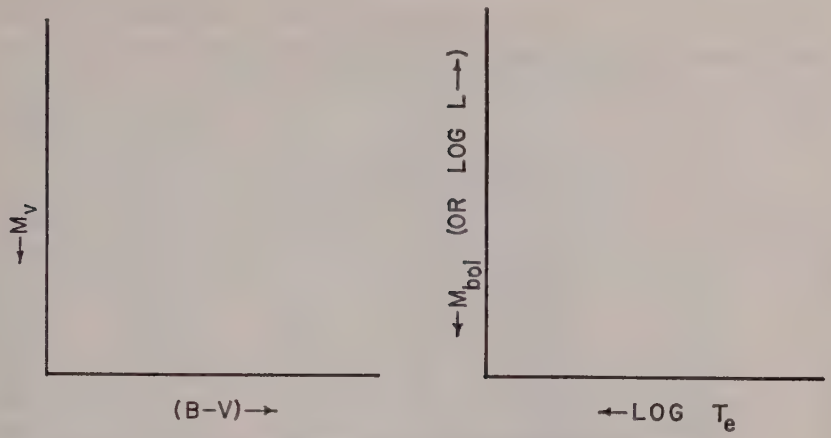


Fig. 0.3 *Left:* Coordinates of the color-magnitude diagram.
Right: Coordinates of the M_{bol}-logT_e diagram.

Unfortunately, the transformation from one set of coordinates to another (particularly from the "theoretical" H-R diagram to either one of the others) involves many uncertainties, and such transformations cannot at present be effected very accurately, particularly for very hot and very cool stars, for which much of the radiation is either in the ultraviolet or in the infra-red. It is clear that T_e, spectral type, and $B-V$ are all closely related to one another, as they all depend primarily on the *surface temperature* (or the *color*) of the star. However, even for given T_e, the values of $B-V$ and spectral type will depend on the details of the spectrum of the star, and these in turn depend on the *surface gravity* (which is the physical parameter underlying luminosity class) and on the *chemical composition* of the outer stellar layers (which may not be exactly the same for all stars under consideration). It is also clear that M_V and M_{bol} are closely related to each other, as both are in some sense a measure of a star's intrinsic brightness. But, because M_V is a measure of a star's intrinsic brightness in a certain spectral region, whereas M_{bol} corresponds to the intrinsic brightness in *all* wavelengths, the relation between M_V and M_{bol} depends fairly strongly on the effective temperature (or color) of the star. Rather than pursue this discussion further, we refer the reader to published summaries of basic methods of obtaining the needed relationships (see, *e.g.*, Keenan and Morgan [Ke51], Arp [Ar58], and Keenan [Ke63]). We also present some tables which assist in effecting some of the transformations among the various forms of the H-R diagram.

Tables 0.2 and 0.3 (from Keenan [Ke63]) give values of M_V and T_e for the various MKK spectral types. Table 0.4 gives values of the photoelectric color index $B-V$ and bolometric correction ($\equiv M_{bol}-M_V$) for the MKK spectral types. This table is a composite of the tables given by Arp [Ar58,

p. 83] Johnson [Jo63, p. 214], and Harris, Strand, and Worley [Ha63a, p. 269]. Finally, Table 0.5 gives M_V *vs.* $B-V$ for the "standard zero-age main sequence" (from Johnson [Jo63, p. 216]).

Table 0.2

VISUAL ABSOLUTE MAGNITUDES*

Sp. Type	V	IV	III	II	Ib	Ia
O9	−4.8	−5.4	−6.0	–	–	–
B0	−4.1	−4.6	−5.0	−5,6:	−6.2	−7.0:
B1	−3.5	−3.9	−4.4	−5.1:	−6.0	−7.0:
B2	−2.5	−3.0	−3.6	−4.4	−5.9	−7.0:
B3	−1.7	−2.3	−2.9	−3.9	−5.8	−7.0:
B5	−1.1	−1.6	−2.2	−3.7	−5.7	−7.0:
B7	−0.6	−1.0	−1.6	−3.6	−5.6	−7.0:
B8	−0.2	−0.6	−1.2	−3.4	−5.5	−7.0:
B9	+0.2	−0.3	−0.8	−3.1	−5.4	−7.0:
A0	+0.6	0.0	−0.6	−2.8	−4.9	−7.0:
A1	+1.2	+0.3	−0.4	−2.6	−4.8	−7.0:
A3	+1.7	+0.9	0.0	−2.3	−4.6	−7.0:
A5	+2.1	+1.2	+0.3	−2.1	−4.5	−7.0:
A7	+2.4	+1.5	+0.5	−2.0	−4.5	−7.0:
F0	+2.6	+1.7	+0.6	−2.0	−4.5	−7.0:
F2	+3.0	+1.9	+0.6	−2.0	−4.5	−7.0:
F5	+3.4	+2.1	+0.7	−2.0	−4.5	−7.0:
F6	+3.7	+2.2	+0.7	−2.0	−4.5	−7.0:
F8	+4.0	+2.4	+0.6	−2.0	−4.5	−7.0:
G0	+4.4	+2.8	+0.6	−2.0	−4.5	−7.0:
G2	+4.7	+3.0	+0.4	−2.1	−4.5	−7.0:
G5	+5.2	+3.2	+0.3	−2.1	−4.5	−7.0:
G8	+5.6	+3.2	+0.3	−2.1	−4.5	−7.0:
K0	+5.9	+3.2	+0.2	−2.1	−4.5	−7.0:
K2	+6.3	–	−0.1	−2.2	−4.5	−7.0:
K3	+6.9	–	−0.2	−2.3	−4.5	−7.0:
K5	+8.0	–	−0.3	−2.3	−4.5	−7.0:
M0	+9.2	–	−0.4	−2.4	−4.5	−7.0:
M1	+9.7	–	−0.5	−2.4	−4.5	−7.0:
M2	+10.1	–	−0.5	−2.4:	−4.5	−7.0:
M3	+10.6	–	−0.5	−2.4:	−4.5	–
M4	+11.3	–	−0.5:	−2.4:	−4.5	–
M5	+12.3	–	–	–	–	–
M6	+13.4	–	–	–	–	–

* From Keenan [Ke63, p. 92]. (Reproduced by permission of the University of Chicago Press.)

Table 0.3

EFFECTIVE TEMPERATURES FOR MK TYPES*

Sp. Type	T_e(°K)
B0	27,000
B1	23,000
B2	20,000
B3	18,000
B5	16,000
B6.5	14,000
B8	12,500
B9	11,200
A0	10,400
A1	9,700
A2	9,100
A3	8,500
A5	8,200
A7	7,600
F0	7,200

	V	IV	III	II	Ib	Ia
F2	6900	6830	6800	6700	6600	
F5	6700	6600	6500	6350	6200	
F6	6500	6370	6250	6020	5800	
F8	6200	6050	5900	5720	5450	
G0	6000	5720	5500	5350	5050	
G2	5740	5420	5100	4950	4750	
G5	5520	5150	4800	4650	4500	
G8	5320	4950	4600	4450	4300	
K0	5120	4750	4400	4350	4100	
K1	4920	4550	4150	4000	3850	
K2	4760	4400	3970	3860	3750	
K3	4600	—	3820	3720	3600	
K5	4350	—	3700	3600	3500	
K6	4000	—	—	—	—	
M0	3750	—	3500	3400	3300	
M1	3600	—	3300	3150:	3050	
M2	3350:	—	3100	2050:	—	
M3	3100:	—	2900:	—	—	
M4	—	—	2700:	—	—	

* Adapted from Keenan [Ke63, p. 91].

Table 0.4

INTRINSIC COLORS AND BOLOMETRIC CORRECTIONS*

Spectral Type	*B–V*			*B.C.*	
	Luminosity Class			*Luminosity Class*	
	V	III	I	V	III
05	(−0.32)	(−0.32)	(−0.32)	[−4.31]	
09	(−0.31)	(−0.31)	(−0.28)	−3.34	
09.5	(−0.30)	(−0.30)	(−0.27)	[−3.68]	
B0	(−0.30)	(−0.30)	(−0.24)	−3.17	
B1	(−0.26)	(−0.26)	(−0.19)	−2.50	
B2	(−0.24)	(−0.24)	(−0.17)	−2.23	
B3	(−0.20)	(−0.20)	(−0.13)	−1.77	
B5	(−0.16)	(−0.16)	(−0.09)	−1.39	
B6	(−0.14)	(−0.14)	(−0.07)	−1.21	
B7	(−0.12)	(−0.12)	(−0.05)	−1.04	
B8	(−0.09)	(−0.09)	(−0.02)	−0.85	
B9	(−0.06)	(−0.06)	(0.00)	−0.66	
A0	(0.00)	(0.00)	(+0.01)	−0.40	
A1	(+0.03)	(+0.03)	(+0.01)	−0.32	
A2	(+0.06)	(+0.06)	(0.00)	−0.25	
A3	(+0.09)	–	(0.00)	−0.20	
A5	(+0.15)	(+0.15)	(+0.07)	−0.15	
A7	(+0.20)	–	(+0.13)	−0.12	
F0	(+0.30)	–	(+0.24)	−0.08	
F2	(+0.38)	–	(+0.34)	−0.06	
F5	(+0.45)	–	(+0.45)	−0.04	
F6	+0.47	[+0.48]	–	−0.04	
F7	+0.50	–	–	−0.04	
F8	+0.53	–	[+0.68]	−0.05	
G0	+0.60	–	[+0.83]	−0.06	
G2	+0.64	–	–	−0.07	
G4	–	–	[+0.97]	–	
G5	+0.68	+0.86	–	−0.10	
G8	+0.72	+0.93	–	−0.15	
K0	+0.81	+1.01	–	−0.19	
K2	+0.92	+1.16	[+1.37]	−0.25	

* Entries in square brackets are from Arp [Ar58]; in parentheses from Johnson [Jo63]; unmarked from Harris [Ha63].

Spectral Type	*B–V*			*B.C.*	
	Luminosity Class			*Luminosity Class*	
	V	III	I	V	III
K3	+0.98	+1.29	–	−0.35	
K4	–	+1.40	–	–	
K5	+1.18	+1.52	[+1.45]	−0.71	
K7	+1.38	–	–	−1.02	
M0	–	–	–	–	
M1	+1.48	–	–	[−1.70]	
M2	–	–	[+1.67]	[−2.03]	
M3	+1.49	–	–	[−2.35:]	
M4	–	–	–	[−2.7:]	
M5	+1.69	–	–	[−3.1:]	
M6	–	–	–	–	

Table 0.5

THE STANDARD ZERO-AGE MAIN SEQUENCE*

B–V	M_V	*U–B*	M_V
−0.25	−2.10	−0.90	−1.98
−0.20	−1.10	−0.80	−1.50
−0.15	−0.30	−0.70	−1.03
−0.10	+0.50	−0.60	−0.59
−0.05	+1.10	−0.50	−0.13
0.00	+1.50	−0.40	+0.27
+0.05	+1.74	−0.30	+0.66
+0.10	+2.00	−0.20	+1.02
+0.20	+2.45	−0.10	+1.30
+0.30	+2.95	0.00	+1.50
+0.40	+3.56		
+0.50	+4.23		
+0.60	+4.79		
+0.70	+5.38		
+0.80	+5.88		
+0.90	+6.32		
+1.00	+6.78		
+1.10	+7.20		
+1.20	+7.66		
+1.30	+8.11		

* From Johnson [Jo63]. (Reproduced by permission of the University of Chicago Press.)

0.4 Empirical (L,M) Correlation: Empirical Mass-Luminosity Relation

Just as a strong correlation between L and T_e (or, equivalently, L and R) exists for most observed stars, so a strong correlation between L and M is found to exist for stars whose masses can be reliably determined (this requirement limits one almost exclusively to visual binaries and to "double-

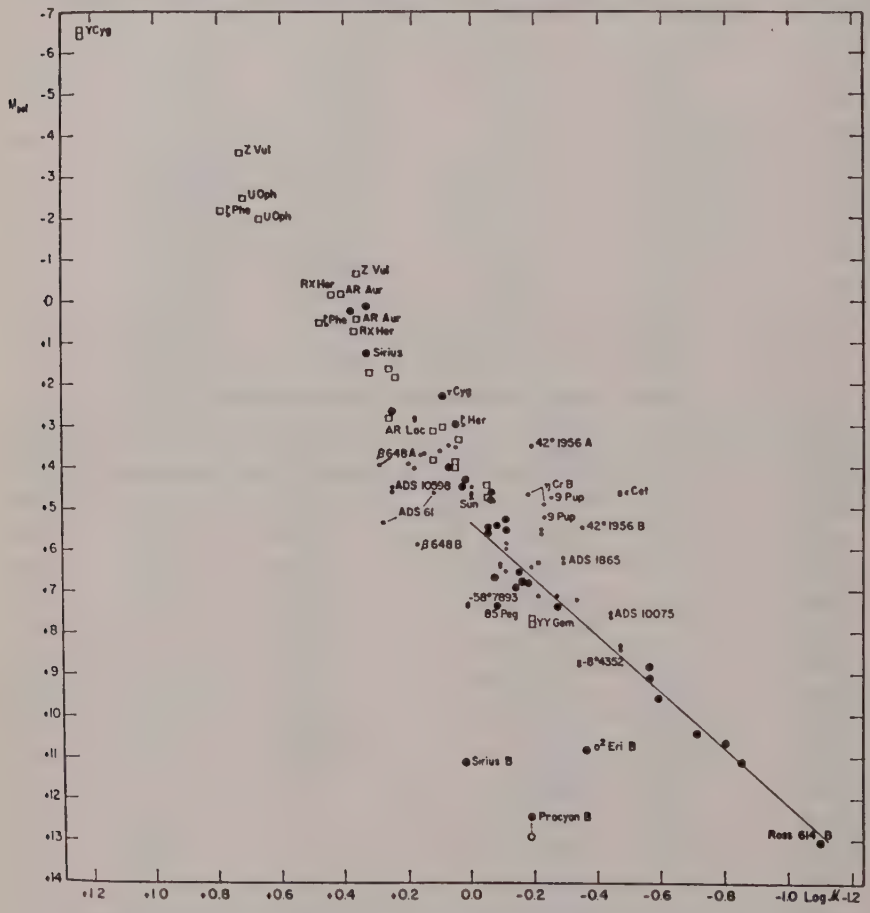


Fig. 0.4 Empirical mass-luminosity relation. Ordinate is absolute bolometric magnitude; abscissa is logarithm of stellar mass in solar units. Filled circles indicate visual binaries (large and small size denote, respectively, high and low weight), while unfilled squares indicate eclipsing binaries. The solid line is a plot of (0.11b) for the late type dwarfs. (From Harris, Strand, and Worley [Ha63a], reproduced by permission of the University of Chicago Press, © copyright 1963 by the University of Chicago Press. All rights reserved).

lined" eclipsing spectroscopic binaries). This correlation, called the empirical mass-luminosity (*M-L*) relation, is such that L and M tend to increase together, and is illustrated in Fig. 0.4, in which M_{bol} is plotted against $\log M$.

According to Harris, Strand, and Worley [Ha63a], the empirical mass-luminosity relation for main-sequence stars in the solar neighborhood (based on visual binaries) can be represented by the relations

$$\left.\begin{aligned} M_{bol} &= 4.6 - 10.0 \log (M/M_\odot) \\ \text{or} \\ L/L_\odot &= 1.2\,(M/M_\odot)^{4.0} \end{aligned}\right\} \quad (0 \leq M_{bol} \leq +7.5), \tag{0.11a}$$

$$\left.\begin{aligned} M_{bol} &= 5.2 - 6.9 \log (M/M_\odot) \\ \text{or} \\ L/L_\odot &= 0.67\,(M/M_\odot)^{2.76} \end{aligned}\right\} \quad (+7.5 \leq M_{bol} \leq +11), \tag{0.11b}$$

where we have taken $M_{bol}(\odot) = +4.77$. These authors state, on the basis of less reliable eclipsing double-lined spectroscopic binaries, that the relation (0.11a) can probably be extended without significant alteration to main sequence stars with M_{bol} up to (-6) or (-7). Thus, approximately, $L \propto M^{3 \text{ or } 4}$. This relation applies specifically to main sequence stars which are members of binaries, and it is generally assumed to apply also to single main sequence stars. There is considerable scatter of the points about the curve defined by the above relations, particularly for the more luminous stars (see Fig. 0.4). The white dwarf stars do not obey this relation, as their luminosities are far too small for their masses (see Chap. 25). Giants and supergiants (particularly of Population II) probably also do not obey the relation, as they are evidently too luminous for their masses (see Chap. 26). (*Cf.* Fig. 26.9, which is a theoretically computed *M-L* relation for zero-age main sequence stars.)

CHAPTER 1

Physical Conditions in Stellar Interiors

In this chapter we wish to obtain a rough idea as to the physical conditions of matter in stellar interiors, by making only a few simple and plausible assumptions. The most important of these assumptions is that the star is in *hydrostatic equilibrium*. The existence of hydrostatic equilibrium implies that none of the physical (macroscopic) variables is changing rapidly with time. More precisely, it implies that significant changes in any of the physical variables can occur only in times long compared with $t_{\rm ff}$, the "free-fall" time (see Chaps. 0, 27, and Sect. 1.3). We assume, also, complete spherical symmetry for the star (thus it is not rotating) and that the star possesses no magnetic fields.

In Sect. 1.1 we write down the mathematical expression of the condition of hydrostatic equilibrium. In Sect.1.2 we derive estimates, based on our assumptions and on observed properties of stars, of interior values of pressure and temperature. Finally, in Sect.1.3 we examine the consequences of a departure from hydrostatic equilibrium in a star.

1.1 Statement of Condition of Hydrostatic Equilibrium

Because of our assumption that the star is in hydrostatic equilibrium, the pressure at each point in the star must be just sufficient to balance the weight of the overlying layers. We can obtain the mathematical expression of this condition by considering an infinitesimal element of mass dm at the distance r from the center of the star (see Fig. 1.1). The condition of hydrostatic equilibrium requires that the sum of the outward force on the element due to the pressure gradient and the inward force on the element due to gravity be zero. If the element of mass dm has the cross sectional area dA perpendicular to r, then this condition is expressed by the relation

$$-(dP)(dA)-g\,dm = 0\,, \tag{1.1}$$

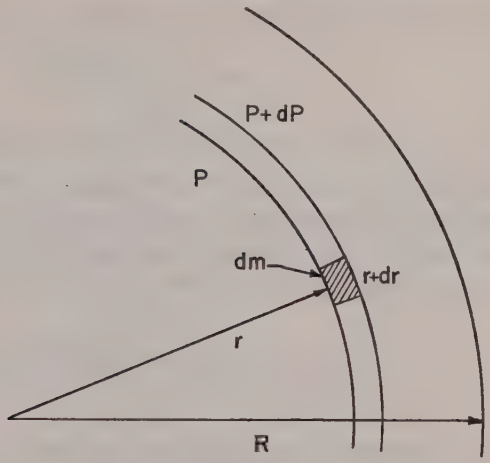


Fig. 1.1
Illustration of hydrostatic equilibrium.

where P denotes the total pressure arising from all sources and g is the magnitude of the gravitational acceleration at r:

$$g = \frac{GM(r)}{r^2} . \tag{1.2}$$

Here $M(r)$ is the mass enclosed within a sphere of radius r and G is the constant of gravitation. Since $dm = \rho(dr)(dA)$, where dr is the radial thickness of the element and ρ is its density, we may write the equation of *hydrostatic equilibrium* for spherical symmetry in the form

$$dP = -\frac{GM(r)}{r^2}\rho dr . \tag{1.3}$$

Following Zel'dovich and Novikov [Ze64], we may interpret the quantity on the right side of (1.3) as the element of "gravitational pressure", since this is the amount by which the pressure must increase inward in the distance dr in order that the element of mass of thickness dr shall be supported against gravity. Equation (1.3) then says that the element of "material" (gas plus radiation, say) pressure must be equal to the element of "gravitational" pressure for hydrostatic equilibrium to obtain.

1.2 Estimates of Interior Values of Pressure and Temperature

In order to obtain the desired estimates, we shall find it convenient to write (1.3) in an alternative form. It is clear that we can write for the mass of a spherical shell of matter of density ρ, radius r, and thickness dr

$$dM(r) = 4\pi r^2 \rho dr . \tag{1.4}$$

We now combine (1.3) and (1.4) to obtain

$$-\mathrm{d}P=\frac{G}{4\pi}\frac{M(r)}{r^4}\,\mathrm{d}M(r)\,, \tag{1.5}$$

which is the desired form of the equation of hydrostatic equilibrium.

An integral expression for the *central pressure* P_c can be obtained immediately by integrating (1.5) over the whole stellar mass. In so doing we assume (as we do throughout this chapter) that $P = 0$ at $r = R$, where R is the radius of the star. The desired expression is then

$$P_c=\frac{G}{4\pi}\int_0^M\frac{M(r)\mathrm{d}M(r)}{r^4}\,, \tag{1.6}$$

where M is the total mass of the star. This formula expresses the fact that for a star in hydrostatic equilibrium the central pressure must be great enough to support the weight (per unit area) of all the overlying layers. This weight (per unit area) of all the overlying layers, given by the right side of (1.6), may be interpreted as the central value of the "gravitational pressure" (*cf.* Zel'dovich and Novikov [Ze64] and Sect. 1.1).

We define the *mean pressure* $\bar{P}$ by the relation

$$\bar{P}=\frac{1}{M}\int_0^M P\mathrm{d}M(r)\,. \tag{1.7}$$

Integrating by parts, noting that the integrated term vanishes since $P(R) = 0$ and $M(0) = 0$, and using (1.5), we obtain

$$\bar{P}=\frac{G}{4\pi M}\int_0^M\frac{M^2(r)\mathrm{d}M(r)}{r^4}\,. \tag{1.8}$$

We define, finally, the *mean temperature* $\bar{T}$ by the relation

$$\bar{T}=\frac{1}{M}\int_0^M T\mathrm{d}M(r)\,. \tag{1.9}$$

Before we can proceed further in the evaluation of $\bar{T}$, we must make an assumption regarding the *equation of state* of the stellar material. For example, we may write with a high degree of generality

$$P = p_g+p_r\,, \tag{1.10}$$

where p_g and p_r denote gas and radiation pressure, respectively. It can be shown (*cf.* Sect. 15.5b) that the gas pressure for (non-degenerate) stellar

material is given under most conditions to a high order of accuracy by the perfect gas law. We write the perfect gas law in the form

$$p_g = \frac{\mathscr{R}}{\mu}\rho T, \tag{1.12}$$

where ρ and T are, respectively, density and absolute temperature; $\mathscr{R}$ is the universal gas constant per mole; and μ is the mean molecular weight (*i.e.*, mass per mole, *cf.* Chap. 15). By using the relation

$$\mathscr{R} = N_0 k\,,$$

where N_0 is Avogadro's number and k is Boltzmann's constant, we can write (1.12) also in the form

$$p_g = \frac{k}{\mu H}\rho T, \tag{1.13}$$

where we have put $H \equiv 1/N_0$ ($\approx$ mass of the hydrogen atom). As will be shown later (*cf.* Chap. 6), the radiation pressure in the stellar interior is given under most conditions to sufficient accuracy by $p_r = (1/3)aT^4$. Here, however, we shall neglect radiation pressure (the importance of radiation pressure for stellar interiors will be considered in Chap. 11).*

Thus we shall write for the total pressure

$$P = \frac{k}{\mu H}\rho T. \tag{1.14}$$

Expressing T in terms of ρ and P from (1.14), (1.9) becomes

$$\overline{T} = \frac{1}{M}\frac{\mu H}{k}\int_0^M \frac{P}{\rho}\,\mathrm{d}M(r)\,,$$

where we have assumed μ to be constant throughout the star.† Using the relation $\mathrm{d}M(r) = 4\pi r^2 \rho \mathrm{d}r$, we obtain

$$\overline{T} = \frac{1}{M}\frac{\mu H}{k}4\pi\int_0^R Pr^2\,\mathrm{d}r = \frac{4\pi}{M}\frac{\mu H}{k}\int_0^R P\mathrm{d}(r^3/3)\,.$$

* The effects of radiation pressure may be formally taken into account by replacing μ in the equations of this chapter by $\beta\mu$, where $\beta \equiv P_g/P \leq 1$ is the ratio of gas to total pressure.

† When μ stands for $\beta\mu$ (see preceding footnote), then it is of course $\beta\mu$ which is being assumed constant throughout the star. While β and μ are not in general strictly constant in stars, they are both generally slowly varying functions of position (cf. Chaps. 23 and 15). Whenever μ or $\beta\mu$ are taken outside of an integral sign they should really be regarded as mean values for the whole star.

Integrating by parts and noting that the integrated term vanishes if $P(R) = 0$, we have

$$\bar{T} = \frac{4\pi}{3M}\frac{\mu H}{k}\int_{P_c}^{0} r^3 \mathrm{d}P.$$

Using (1.5) gives

$$\bar{T} = \frac{G}{3M}\frac{\mu H}{k}\int_{0}^{M}\frac{M(r)\mathrm{d}M(r)}{r}. \tag{1.15}$$

These three expressions for P_c, $\bar{P}$, and $\bar{T}$ can all be written, following a variant of Chandrasekhar's [Ch51] treatment, in terms of the dimensionless integral

$$I_{\sigma,\nu} \equiv \int_{0}^{1}\left[\frac{M(r)}{M}\right]^{\sigma}\left[\frac{r}{R}\right]^{-\nu}\mathrm{d}\left[\frac{M(r)}{M}\right], \tag{1.16}$$

where, as we shall see, we must have

$$3(\sigma+1) > \nu. \tag{1.17}$$

Hence we have for P_c, $\bar{P}$, and $\bar{T}$

$$P_c = \frac{GM^2}{4\pi R^4} I_{1,4}, \tag{1.18a}$$

$$\bar{P} = \frac{GM^2}{4\pi R^4} I_{2,4}, \tag{1.18b}$$

$$\bar{T} = \frac{GM}{3R}\frac{\mu H}{k} I_{1,1}. \tag{1.18c}$$

The factors multiplying the $I_{\sigma,\nu}$ in (1.18) immediately give the general orders of magnitude of P_c, $\bar{P}$, and $\bar{T}$, since the $I_{\sigma,\nu}$, being dimensionless integrals, are normally of order of magnitude unity.

We now define the mean density interior to r:

$$\bar{\rho}(r) = \frac{M(r)}{(4\pi r^3/3)} = \left[\frac{M(r)}{M}\right]\left(\frac{r}{R}\right)^{-3}\bar{\rho}(R), \tag{1.19}$$

where $\bar{\rho}(R) \equiv M/(4\pi R^3/3)$ is the mean density of the whole star. Thus (1.16) becomes

$$I_{\sigma,\nu} = \int_{0}^{1}\left[\frac{\bar{\rho}(r)}{\bar{\rho}(R)}\right]^{\nu/3}\left[\frac{M(r)}{M}\right]^{\sigma-\nu/3}\mathrm{d}\left[\frac{M(r)}{M}\right]. \tag{1.20}$$

We now assume that $\bar{\rho}(r)$ does not increase outward; *i.e*, we assume that

$$\frac{\mathrm{d}\bar{\rho}(r)}{\mathrm{d}r} \leq 0. \tag{1.21}$$

A corollary of this assumption is that

$$\bar{\rho}(r) \geq \rho(r). \tag{1.22}$$

Equation (1.22) follows from (1.21) by using (1.19) and (1.4).

An *upper* limit to the value of $I_{\sigma,\nu}$ is obtained by setting

$$\frac{\bar{\rho}(r)}{\bar{\rho}(R)} = \frac{\bar{\rho}(0)}{\bar{\rho}(R)} = \frac{\rho_c}{\bar{\rho}(R)},$$

where ρ_c denotes the *central* density of the star. A *lower* limit to the value of $I_{\sigma,\nu}$ is obtained by setting

$$\frac{\bar{\rho}(r)}{\bar{\rho}(R)} = \frac{\bar{\rho}(R)}{\bar{\rho}(R)} = 1.$$

Physically, the above statements mean that P_c, for example, is greatest when the mass concentration is greatest, consistent with the foregoing assumption about $\bar{\rho}(r)$.

Thus we can now write

$$\left[\frac{\rho_c}{\bar{\rho}(R)}\right]^{\nu/3} \int_0^1 \left[\frac{M(r)}{M}\right]^{\sigma-\nu/3} \mathrm{d}\left[\frac{M(r)}{M}\right] \geq I_{\sigma,\nu} \geq \int_0^1 \left[\frac{M(r)}{M}\right]^{\sigma-\nu/3} \mathrm{d}\left[\frac{M(r)}{M}\right].$$

Integrating, we have

$$\int_0^1 \left[\frac{M(r)}{M}\right]^{\sigma-\nu/3} \mathrm{d}\left[\frac{M(r)}{M}\right] = \frac{3}{3(\sigma+1)-\nu},$$

which shows that σ and ν must satisfy the inequality (1.17) for convergence of the integral.

Finally, the above inequality involving $I_{\sigma,\nu}$ can be written in the form

$$\left[\frac{\rho_c}{\bar{\rho}(R)}\right]^{\nu/3} \frac{3}{3(\sigma+1)-\nu} \geq I_{\sigma,\nu} \geq \frac{3}{3(\sigma+1)-\nu} \quad [3(\sigma+1) > \nu]. \tag{1.23}$$

In particular we have, considering only the lower limit,

$$I_{1,4} \geq 3/2, \qquad I_{2,4} \geq 3/5, \qquad I_{1,1} \geq 3/5, \tag{1.24}$$

which give, respectively,

$$P_c \geq \frac{3}{8\pi} G \frac{M^2}{R^4}, \tag{1.26a}$$

$$\bar{P} \geq \frac{3}{20\pi} G \frac{M^2}{R^4}, \tag{1.26b}$$

$$\bar{T} \geq (1/5) \frac{\mu H}{k} G \frac{M}{R}. \tag{1.26c}$$

Inserting numerical values, we have

$$P_c \geq 1.326 \times 10^9 \left(\frac{M}{M_\odot}\right)^2 \left(\frac{R}{R_\odot}\right)^{-4} \text{atmospheres},$$

$$\geq 1.342 \times 10^{15} \left(\frac{M}{M_\odot}\right)^2 \left(\frac{R}{R_\odot}\right)^{-4} \text{dynes/cm}^2, \tag{1.27a}$$

$$\bar{P} \geq 5.30 \times 10^8 \left(\frac{M}{M_\odot}\right)^2 \left(\frac{R}{R_\odot}\right)^{-4} \text{atmospheres},$$

$$\geq 5.37 \times 10^{14} \left(\frac{M}{M_\odot}\right)^2 \left(\frac{R}{R_\odot}\right)^{-4} \text{dynes/cm}^2, \tag{1.27b}$$

$$\bar{T} \geq 4.582 \times 10^6 \mu \left(\frac{M}{M_\odot}\right) \left(\frac{R}{R_\odot}\right)^{-1} {}^\circ\text{K}, \tag{1.27c}$$

where the equality signs apply to the case of the *homogeneous* star, $\rho(r) = \bar{\rho}(R)$.

We note that the derivation of the expressions for the minimum values of P_c and $\bar{P}$ required no information concerning the equation of state or the composition of the stellar material, but only the mass distribution, through the condition of hydrostatic equilibrium. The temperature expression came about by relating the temperature to the density and pressure through the perfect gas equation of state. Great central pressure implies high central temperature, assuming the central density to have some reasonable (finite) value. Thus high central temperatures are required for stars obeying the perfect gas equation of state to maintain the high central pressures which in turn must be large enough to support the weight of the overlying layers. Note that the effect of radiation pressure, which varies as T^4, is to *decrease* the lower limit to $\bar{T}$ (see the first footnote in this section).

We note a simple, approximate picture, used by Struve [St59, p. 176], which makes clear intuitively the dependence of interior pressure on the quantity GM^2/R^4.

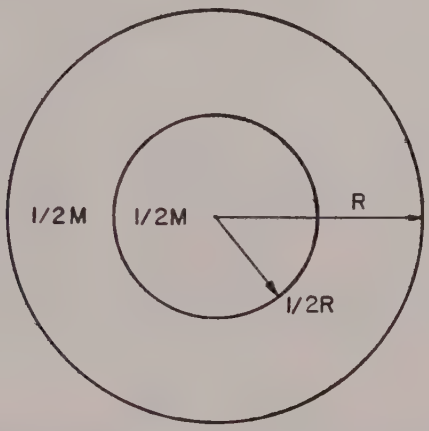


Fig. 1.2 Schematic picture of a star for estimating interior pressure.

We divide the star into two regions (see Fig. 1.2), each containing about half the total mass of the star. Treating these two regions as two masses separated by the distance $\sim R$, say, we have for the force of attraction between them

$$F \approx G \frac{(1/4)M^2}{R^2}.$$

This force will be distributed over the surface of the inner mass, say of radius $\sim R/2$. Thus the pressure on the surface of the inner mass is, to rough order of magnitude,

$$P \sim G \frac{M^2/4}{R^2 \cdot 4\pi (R/2)^2} = \frac{G}{4\pi} \frac{M^2}{R^4},$$

which is in order-of-magnitude agreement with (1.26 a,b).

We note, in addition, that the assumption of hydrostatic equilibrium was necessary in order to obtain the foregoing results. This assumption is amply justified in view of the observed apparent stability of most actual stars. Geologic evidence suggests that the sun has not altered its condition appreciably in at least a billion years. As a more dramatic argument justifying the assumption of hydrostatic equilibrium, we shall examine in the next section the consequences of a small departure from strict hydrostatic equilibrium.

1.3 Effect of Departures from Hydrostatic Equilibrium

Consider a point a distance r from the center of a star. If the material is not in hydrostatic equilibrium such that, say, $\left|\frac{1}{\rho}\frac{\partial P}{\partial r}\right| < G\frac{M(r)}{r^2}$, then the material will experience an inward acceleration given by

$$-\ddot{r} = \frac{GM(r)}{r^2} + \frac{1}{\rho}\frac{\partial P}{\partial r} = \frac{GM(r)}{r^2}\left[1 - \frac{\frac{1}{\rho}\left|\frac{\partial P}{\partial r}\right|}{\frac{GM(r)}{r^2}}\right]. \tag{1.28}$$

Suppose that the acceleration is constant during the time, t, during which the material moves a distance αR, where α is a number less than one. Then we have

$$\alpha R = \frac{1}{2}(-\ddot{r})t^2,$$

whence

$$t = \left(\frac{2\alpha R}{-\ddot{r}}\right)^{(1/2)} = \left(\frac{2\alpha r^2 R}{GM(r)}\frac{1}{[\ ]}\right)^{(1/2)},$$

where [ ] denotes the quantity in square brackets in (1.28).

To obtain an order-of-magnitude estimate of the value of t, we set $r = (1/2)R$, $M(r) = (1/2)M$, whence we have

$$t \approx \left(\frac{R^3}{GM}\right)^{(1/2)}\left(\frac{\alpha}{[\ ]}\right)^{(1/2)},$$

or

$$t \simeq 1.6 \times 10^3 \left(\frac{\alpha}{[\ ]}\right)^{(1/2)}\left(\frac{R}{R_\odot}\right)^{(3/2)}\left(\frac{M_\odot}{M}\right)^{(1/2)} \text{sec}.$$

Taking $(1/\rho)\,|\partial P/\partial r| = 0.99\ GM(r)/r^2$, or [ ] = 0.01, and $\alpha = 0.1$ gives for the sun

$$t \sim 5 \times 10^3 \text{ sec} \approx 1 \text{ hour},$$

which is approximately equal to the "free-fall" time t_{ff}.* Thus a departure

* One possible definition of t_{ff} is the following: It is the time required for a satellite circling a star just above the stellar surface (at radial distance R from the stellar center) to complete, say, one-quarter of a revolution; this gives

$$t_{\text{ff}} = \frac{\pi}{2}\sqrt{R^3/GM}. \tag{1.29}$$

See Chap. 27 and Sect. 26.2b for further discussion of t_{ff}.

from hydrostatic equilibrium by as little as 1 per cent would lead to a 10 per cent change in the solar radius in something like an hour's time. Since the sun's radius has not changed appreciably in several hundred millions of years, it follows that the condition of hydrostatic equilibrium must be satisfied to a high order of accuracy in the sun and, in fact, in all other stars which are at least in "quasi equilibrium."

Indeed, we can easily estimate the average departure of a star from strict hydrostatic equilibrium during, for example, a phase when the star is contracting gravitationally. As we stated in Chap. 0 (see also Sect. 17.4), the "Kelvin" time t_K is approximately equal to the time required for a star to contract gravitationally through a distance comparable to the radius of the star. Thus we have, to order of magnitude,

$$R \sim \frac{1}{2} a t_K^2 ,$$

assuming the star to contract with the constant acceleration a. For the sun we have $R \approx 7 \times 10^{10}$ cm, $t_K \approx 2 \times 10^7$ yrs $\approx 6 \times 10^{14}$ sec (*cf.* Sect. 17.4). These numbers give $a \sim 5 \times 10^{-19}$ cm/sec^2. However, the gravitational acceleration at a point, say, a distance $(1/2)R$ from the center of the sun is

$$a \sim \frac{M(r)/M}{(r/R)^2} g_{\text{surf}} \sim \frac{1}{(1/2)^2} g_{\text{surf}} \approx 4 \times 30{,}000 \sim 10^5 \text{ cm/sec}^2$$

(taking $M(r)/M \approx 1$). Thus the *fractional* departure of the sun from hydrostatic equilibrium during gravitational contraction is, on the average, ~ 1 part in 10^{24} (note that this fraction is essentially $(t_{ff}/t_K)^2$, *cf.* (0.8″)). For the actual, present-day sun the average departure is even very much less than this, because the present sun is also being maintained in *thermal* equilibrium (*cf.* Chap. 5), *i.e*, the "hydrogen burning" makes good the energy lost by radiation, so that the sun is not actually contracting gravitationally.

We may summarize by saying that the *pressure balance* in a star must be maintained essentially *moment by moment* if the star's condition is to change only slowly with time. As we shall see, the condition on the *energy balance* of the star is not nearly so stringent; upsets in the energy balance (of a reasonable order of magnitude, however) may persist for times up to some fraction of t_K, the "Kelvin" time, before the condition of the star will be affected appreciably. This fraction of a Kelvin time may amount to, say, $\sim 10^6$ years in the case of the sun.

Necessary conditions for the maintenance of hydrostatic equilibrium in a star are discussed in Sects. 17.5, 26.2b, and 27.3b.

CHAPTER 2

Radiation Theory

In our treatment of radiation theory it will be sufficient, for the most part, to adopt a purely phenomenological, or "macroscopic," description of a general radiation field, not taking explicit cognizance either of the wave (classical picture) or the "photonic" (quantum mechanical picture) character of the radiation. In this description the radiation is treated, in a sense, as a continuous fluid. Such a description is given by Chandrasekhar [Ch39, Chap. 5] as well as in many treatises on radiative transfer. Most of the work in Sects. 2.1 through 2.7, as well as part of that in Sect. 2.9, is based on this kind of description.

When we consider the interaction between matter and radiation, it will be necessary to adopt a specific picture, either classical or photonic, of the radiation field. In most situations involving the emission and absorption of radiation, for example, it is customary to adopt a quantum mechanical description, because of the great success which quantum theory has had in accounting for the interactions between matter and radiation (a basic reference on the quantum theory of radiation is Heitler [He54]). Most of the work in Sect. 2.8 and part of that in Sect. 2.9 will be based on such a quantum mechanical picture. However, this work will involve, for the most part, only *results* of quantum mechanical calculations; no detailed knowledge of quantum mechanics on the part of the reader is assumed, other than a general familiarity with certain basic quantum mechanical concepts such as the Bohr frequency condition.

Another kind of interaction between radiation and matter produces what are sometimes called "collective effects." An example of such a collective effect is a non-unity and frequency-dependent refractive index in a dispersive dielectric medium. Such collective effects have in the past been largely ignored in radiation transfer theory (except in radio astronomy; see, for example, Pawsey and Bracewell [Pa61]) and they are, in fact, probably not important under most conditions met with, at optical and higher frequencies, in stellar interiors and stellar atmospheres. They may, however, become

important under conditions of high density and low temperature; such conditions certainly exist in some stars. Moreover, radiation in stars interacts very strongly with the stellar matter, so a transfer theory for radiation in stars ought, properly, to include effects of dispersion. It is for these reasons, as well as simply for greater generality, that we have attempted to carry through, in this chapter as well as in Chaps. 3 through 8, the modifications of the usual results produced by dispersion (a basic reference for classical dispersion theory is Landau and Lifshitz [La60]; other references will be given in Sect. 2.10).

To facilitate this program, and also for the benefit of readers who may not be familiar with some of the less well known results of dispersion theory, we have summarized in Sect. 2.10 some of the more useful (for us) results of elementary dispersion theory. The picture on which this theory is based is largely classical, but some aspects of the theory have an interesting and useful "photonic" interpretation, which is pointed out in Sect. 2.10. The quantum mechanical modifications of classical dispersion theory are for the most part negligible (a basic reference for quantum mechanical dispersion theory is Van Vleck [Va32]). Although a number of basic results of elementary dispersion theory are summarized in this Sect. 2.10, and although this section is frequently referred to in this chapter (and the next few chapters), the section is almost self-contained and is, in effect, an appendix. This section may therefore be skipped without loss of continuity by the reader who is not interested in effects of dispersion.

It is with some trepidation that we have attempted to include these effects of dispersion, since many of these effects have not yet been adequately investigated. A complete and consistent treatment of dispersion effects in radiation transfer under astrophysically interesting conditions has, to our knowledge, not yet been carried out;* the whole subject of collective effects is, in fact, still in the research stage. Nevertheless, it is hoped that our attempts will prove useful in stimulating further work along these lines and will not turn out to be hopelessly misleading or incorrect. In any event it will be easy for the reader to ignore these modifications for dispersion effects if he so chooses.

Section 2.11 contains some purely formal results related to the work in Sect. 2.9 and used in Chaps. 5 and 8. This section plays, essentially, the role of an appendix and may be skipped with little or no loss of continuity by the reader.

Throughout this chapter we shall assume, unless we explicitly state

* A few attempts, however, have been made; see Sects. 2.9 and 2.10 for further details and references.

otherwise, that all material properties of a medium are *isotropic*, *i.e.*, the same in every direction at each point within the medium.

Finally, we remark that radiation theory can also be developed systematically and consistently from a purely kinetic, "particle" approach, in which the Boltzmann transport equation is applied to photons. Such a treatment has been carried out by Frank-Kamenetskii [Fr59, Chap. 5], Harris [Ha65], and others. A complete and elegant exposition of this approach has been published in a monograph by Sampson [Sa65a].

2.1 Specific Intensity

Consider an infinitesimal element of area $d\sigma$, with normal represented by the unit vector $\mathbf{n}$, and draw a line from some point P on $d\sigma$ in the direction specified by the unit vector $\mathbf{n}'$ (θ being the angle between $\mathbf{n}$ and $\mathbf{n}'$), in which direction the intensity is to be measured (see Fig. 2.1). We consider only

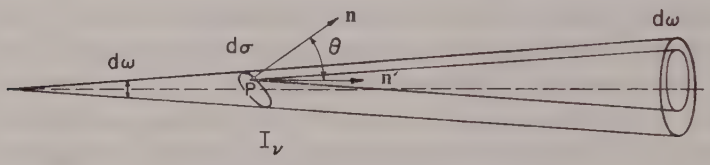


Fig. 2.1 A pencil of radiation.

rays whose directions of emergence at P are confined to the elemental solid angle $d\omega$ about $\mathbf{n}'$. We draw rays parallel to the original one from each point on $d\sigma$ and construct about each ray a cone of the same solid angle $d\omega$. The envelope of all these cones will then form a truncated cone having the solid angle $d\omega$. (In the limit of infinitesimal $d\sigma$, the "truncated" cone actually becomes a complete cone with apex at $d\sigma$.) Such a construction defines a *pencil* of radiation. If dE_ν is the amount of radiant energy having frequencies in the range $(\nu, \nu+d\nu)$ that flows across $d\sigma$ in time dt into directions confined to the solid angle $d\omega$ about the direction $\mathbf{n}'$, then the *specific intensity* I_ν is defined by the relation

$$dE_\nu \equiv I_\nu d\nu d\sigma \cos\theta \, d\omega dt \,.* \tag{2.1}$$

Thus I_ν is the amount of radiant energy of frequency ν, per unit frequency range, that flows per unit time across unit area of surface oriented normal to

* We must have $dt \gg 1/\nu$ for validity of this definition, so that fluctuations arising from the vibratory nature of the radiation will be averaged out. We must also have, for this "continuum" description to be valid, $(d\sigma)^{1/2} \gg \lambda$, where λ is the wavelength of the radiation.

the direction of flow, in directions confined to unit solid angle about the direction $\mathbf{n}'$. Hereafter we shall often refer to I_ν simply as the *intensity*.

In general, I_ν is a function of frequency ν, position $\mathbf{r}$, direction $\mathbf{n}'$, and time t:

$$I_\nu = I_\nu(\mathbf{r},\mathbf{n}',t)\,.$$

The continuum of values of I_ν over these four sets of variables constitutes a *general radiation field*. Some special cases are the following: *Isotropic* radiation field: I_ν does not depend on direction: $I_\nu = I_\nu(\mathbf{r},t)$. *Homogeneous* radiation field: I_ν does not depend on position: $I_\nu = I_\nu(\mathbf{n}',t)$. *Homogeneous and isotropic* radiation field: I_ν does not depend on either position or direction: $I_\nu = I_\nu(t)$. *Axially symmetric* radiation field: I_ν does not depend on the azimuthal angle ϕ (see Fig. 2.2): $I_\nu = I_\nu(\mathbf{r},\theta,t)$. *Time independent* radiation field: I_ν does not depend upon time: $I_\nu = I_\nu(\mathbf{r},\mathbf{n}')$.

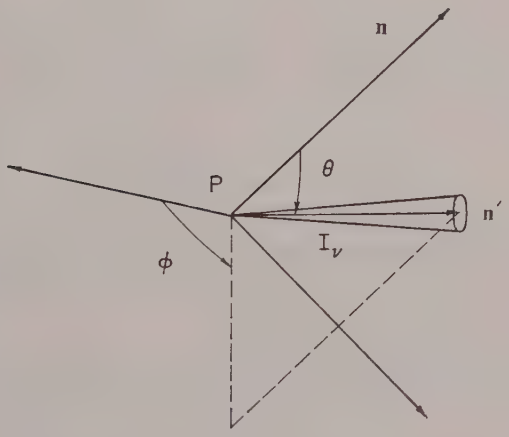


Fig. 2.2 Illustration of polar and azimuthal relations between some particular direction $\mathbf{n}$ and the direction of a pencil of radiation $\mathbf{n}'$.

Unless we specify otherwise, we shall hereafter consider only cases in which I_ν is independent of time.

2.1a *Integrated Intensity*

This is defined by the relation

$$I \equiv \int_0^\infty I_\nu d\nu\,. \tag{2.3}$$

2.1b Constancy of I_ν Along Every Ray Path in Free Space

Here we shall show that I_ν is constant along any ray path in free space, following Chandrasekhar's [Ch39, Chap. 5, p. 186] proof. Here and in the following we shall, unless we state otherwise, disregard effects such as pair production and photon-photon scattering, which can cause I_ν to vary along a ray path even in "empty" space.

We consider two infinitesimal elements of area, $d\sigma$ and $d\Sigma$, located at P and P', respectively, and a distance r apart. Let the normals to these elements of area subtend angles θ and Θ, respectively, to the line joining them. Then the energy in the frequency range $(\nu, \nu+d\nu)$ passing in time dt into the pencil

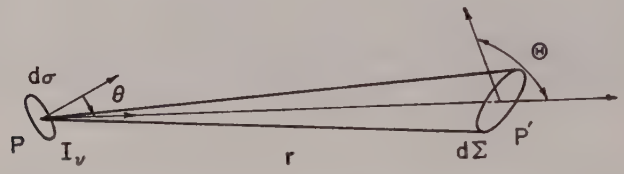


Fig. 2.3 Pencil of radiation at P having solid angle $d\omega = d\Sigma \cos\Theta/r^2$.

defined by $d\sigma$, the angle θ, and the solid angle subtended by $d\Sigma$ at $d\sigma$ is (see Fig. 2.3)

$$dE_\nu = I_\nu d\nu d\sigma \cos\theta \left(\frac{d\Sigma \cos\Theta}{r^2}\right) dt, \tag{2.4}$$

where I_ν is the specific intensity at the point P in the direction PP'. In free space (and in the limit of infinitesimal area $d\sigma$) this is the energy passing through $d\sigma$ that also passes through $d\Sigma$.

Now let I'_ν be the specific intensity at the point P' in the direction PP'. Consider the pencil defined by $d\Sigma$, the angle Θ, the direction PP', and the

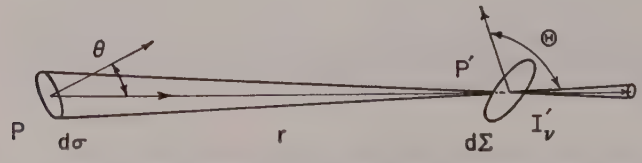


Fig. 2.4 Pencil of radiation at P' having solid angle $d\omega = d\sigma \cos\theta/r^2$.

solid angle subtended at P' by $d\sigma$. Then the energy dE'_ν within the frequency range $(\nu, \nu+d\nu)$ passing into this pencil in time dt is (see Fig. 2.4)

$$dE'_\nu = I'_\nu d\nu d\Sigma \cos\Theta \left(\frac{d\sigma \cos\theta}{r^2}\right) dt. \tag{2.5}$$

But in free space (and in the limit of infinitesimal area $d\Sigma$) this is also the energy passing through $d\sigma$ that also passes through $d\Sigma$. Thus we have

$$dE'_\nu = dE_\nu ,$$

or

$$I'_\nu = I_\nu . \tag{2.6}$$

Thus, in free space the specific intensity in the direction PP' is the same at P' as at P. Since the direction PP' is arbitrary, as well as the locations of P and P', it follows that *in free space* the specific intensity along any ray is constant.

The constancy of I_ν along any ray path in free space may be understood more intuitively, perhaps, by considering the *solar constant*, S, which is the amount of energy crossing unit area normal to the direction toward the sun in unit time at some distance r from the sun. At the distance r from the sun, the specific intensity of the solar radiation would be given by

$$I(r) = \frac{S(r)}{\Omega(r)} = S(r)\frac{r^2}{\pi R^2},$$

where $\Omega(r) = \pi R^2/r^2$ is the solid angle subtended by the sun, of radius R, at the distance r. At some other distance r', the specific intensity would be

$$I(r') = \frac{S(r')}{\Omega(r')} = [S(r)(r/r')^2][r'^2/\pi R^2] = S(r)\frac{r^2}{\pi R^2},$$

so that $I(r') = I(r)$. (We are here assuming that $\Omega(r)$ is so small that the sun is radiating almost as a point source, *i.e.*, the radiation intensity follows an inverse square law.)

2.1c *Constancy of I_ν/μ_ν^2 Along Every Ray Path for Variable Refractive Index*

Here we shall show the following: In any "transparent" medium (defined in Sect. 2.10) with a spatially variable (real) index of refraction μ_ν the quantity I_ν/μ_ν^2 is constant along every ray path in the medium, where I_ν is evaluated at every point in the direction of the tangent to the ray path at that point, provided that energy gains or losses through reflection, absorption, or emission are negligible. (We are here using the terms *absorption* and *emission* in the generalized sense of Sects. 2.6 and 2.7 below.) This result is also proved in Milne [Mi30] and in Woolley and Stibbs [Wo53, Chap. 11]. One condition that no reflection take place is that $|(\nabla\mu_\nu)/\mu_\nu| \ll 1/\lambda$, where λ is the wavelenght of the radiation in the medium (*cf.*, for example, Born and Wolf [Bo59, Chap. 3]); in other words, μ_ν must vary with position so slowly that the distance within which μ_ν changes by an appreciable fraction of itself must be large compared to the wavelength of the radiation.

Consider an infinitesimal surface element of area $d\sigma$ lying in the surface separating two media of refractive indices μ_ν and μ'_ν, and let $I_\nu \equiv I_\nu(\theta,\phi)$ be the intensity of a pencil of radiation flowing through $d\sigma$ from the medium with refractive index μ_ν into the medium with refractive index μ'_ν, the direction of the incident pencil being specified by the polar angles (θ,ϕ) (see Fig. 2.5).

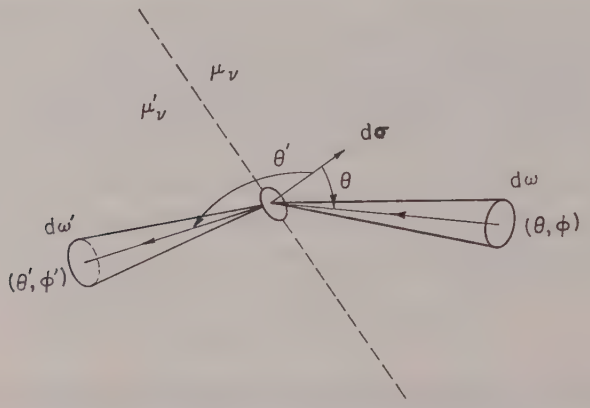


Fig. 2.5 Illustration for variable refractive index.

The refraction at the interface will cause the intensity to have the new value $I'_\nu \equiv I'_\nu(\theta',\phi')$, where the angles (θ',ϕ') specify the direction of the pencil after the refraction. Because of the differential bending of rays that accompanies refraction, the rays that defined the solid angle $d\omega$ about (θ,ϕ) will now define a new solid angle $d\omega'$ about (θ',ϕ').

The amount of energy having frequencies in the range $(\nu, \nu+d\nu)$ impinging on $d\sigma$ per unit time from directions confined to the solid angle $d\omega$ about (θ,ϕ) is $(I'_\nu d\nu d\sigma \cos\theta\, d\omega)$, and the amount of energy with frequencies $(\nu, \nu+d\nu)$ leaving $d\sigma$ per unit time and going into the solid angle $d\omega'$ about (θ',ϕ') is $(I'_\nu d\nu d\sigma \cos\theta'\, d\omega')$. If we assume only negligible losses or gains due to reflection at the interface or to absorption or emission, we will have

$$I_\nu \cos\theta\, d\omega = I'_\nu \cos\theta'\, d\omega' . \tag{2.7}$$

(Note that the assumption of negligible reflection losses at the interface implies that μ_ν must change *continuously* across the "interface," in accordance with the condition stated near the beginning of this subsection. The "interface" must accordingly be many wavelengths in thickness, but it may still be thin enough to be considered an "interface" from the standpoint of geometrical optics. Note also from (2.7) that $d\omega' < 0$ if $d\omega > 0$, since $\theta' > \pi/2$ (see Fig. 2.5) and since both sides must be positive. That this last result is

true follows also from (2.9) below.) Let $d\omega = \sin\theta\, d\theta d\phi$, $d\omega' = \sin\theta'\, d\theta' d\phi'$, so that (2.7) becomes

$$I_\nu \cos\theta \sin\theta\, d\theta d\phi = I'_\nu \cos\theta' \sin\theta'\, d\theta' d\phi' . \tag{2.8}$$

From Snell's laws of refraction we have

$$\left.\begin{aligned} \mu_\nu \sin\theta &= \mu'_\nu \sin\theta' , \\ \phi' &= \phi \pm \pi . \end{aligned}\right\} \tag{2.9}$$

Taking differentials with respect to angles, we obtain

$$\left.\begin{aligned} \mu_\nu \cos\theta\, d\theta &= \mu'_\nu \cos\theta'\, d\theta' , \\ d\phi' &= d\phi . \end{aligned}\right\} \tag{2.10}$$

Using (2.9) and (2.10) in (2.8), we obtain

$$\frac{I_\nu}{\mu_\nu^2} = \frac{I'_\nu}{\mu'^2_\nu} . \tag{2.11}$$

Since nothing has been assumed about how much μ'_ν differs from μ_ν, (2.11) is valid when μ'_ν differs only infinitesimally from μ_ν. From (2.11) we then conclude that (I_ν/μ_ν^2) remains constant along every ray path, where I_ν is evaluated at every point in the direction of the tangent to the ray path at that point, in a medium with a spatially varying refractive index, provided that there are only negligible energy losses or gains due to refraction, absorption, or emission, Q.E.D.

The result (2.11) is just a consequence of the change in the solid angle of a pencil of radiation as it enters a medium of different refractive index and of the definition of I_ν as energy flux *per unit solid angle*. Thus, using (2.9) and (2.10), we have

$$d\omega' = \left(\frac{\mu_\nu}{\mu'_\nu}\right)^2 \frac{\cos\theta}{\cos\theta'} d\omega . \tag{2.12}$$

In the case of normal incidence ($\theta = 0$, $\theta' = \pi$) we have $d\omega' = -(\mu_\nu/\mu'_\nu)^2\, d\omega$, which shows, for example, that $|d\omega'| < |d\omega|$ if $\mu'_\nu > \mu_\nu$: all rays are bent toward the normal if $\mu'_\nu > \mu_\nu$. Another interpretation of the quantity (I_ν/μ_ν^2) will be presented in Sects. 3.5 and 3.6.

2.2 Net Flux

The *net flux* F_ν is the amount of energy crossing unit area in *all* directions per unit time per unit frequency interval. Thus, the amount of radiant energy with frequencies in the range $(\nu, \nu + d\nu)$ crossing the element of area $d\sigma$ in time dt into all directions is

$$d\nu dt d\sigma \int_{4\pi} I_\nu \cos\theta\, d\omega \equiv F_\nu d\nu dt d\sigma .$$

The expression for F_ν in terms of I_ν is then

$$F_\nu = \int_{4\pi} I_\nu \cos\theta \, d\omega = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} I_\nu \cos\theta \sin\theta \, d\theta d\phi . \tag{2.13}$$

For a time independent I_ν, F_ν is a function of frequency, position, and the direction of the normal $\mathbf{n}$ to the surface element $d\sigma$: $F_\nu = F_\nu(\mathbf{r},\mathbf{n})$.

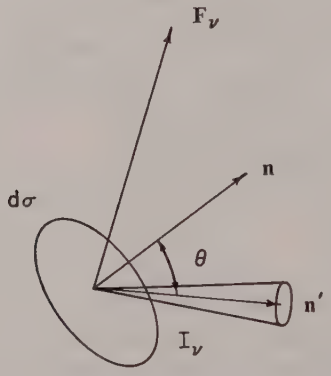


Fig. 2.6 Illustration for vector net flux.

The net flux is actually a vector. To develop the expression for the *vector* net flux, we consider a unit area whose normal is represented by the unit vector $\mathbf{n}$, and we denote the specific intensity in the direction of the unit vector $\mathbf{n}'$ by $I_\nu(\mathbf{n}')$ (see Fig. 2.6). If θ is the angle between $\mathbf{n}$ and $\mathbf{n}'$, then the net flux $F_\nu(\mathbf{n})$ across this unit area with normal $\mathbf{n}$ is

$$F_\nu(\mathbf{n}) = \int_{4\pi} I_\nu(\mathbf{n}') \cos\theta \, d\omega_{\mathbf{n}'} \tag{2.14}$$

or

$$F_\nu(\mathbf{n}) = \int_{4\pi} I_\nu(\mathbf{n}') \mathbf{n}\cdot\mathbf{n}' d\omega_{\mathbf{n}'} = \mathbf{n}\cdot\int_{4\pi} I_\nu(\mathbf{n}')\mathbf{n}' d\omega_{\mathbf{n}'} , \tag{2.15}$$

where $d\omega_{\mathbf{n}'}$ denotes an element of solid angle about the direction $\mathbf{n}'$. The integral in (2.15) is a vector, having some particular direction, independent of the direction $\mathbf{n}$ of the element of area. This integral, in fact, is the *vector net flux* $\mathbf{F}_\nu$:

$$\mathbf{F}_\nu = \int_{4\pi} I_\nu(\mathbf{n}')\mathbf{n}' d\omega_{\mathbf{n}'} . \tag{2.16}$$

The i^{th} component in a rectangular Cartesian coordinate system is

$$F_{\nu,i} = \int_{4\pi} I_\nu(\alpha_1', \alpha_2', \alpha_3')\, \alpha_i' d\omega_{\mathbf{n}'} , \tag{2.17}$$

where α_i' is the i^{th} direction cosine of the vector $\mathbf{n}'$. $F_{\nu,i}$ is the component of

$\mathbf{F}_\nu$ along the i axis, or the net radiant energy per unit frequency interval crossing unit area in the plane whose normal lies along the i axis, in unit time.

Thus $F_\nu(\mathbf{n})$ is actually the *component* of $\mathbf{F}_\nu$ in the direction $\mathbf{n}$; hence (2.14) may alternatively be written as

$$F_\nu(\mathbf{n}) = \mathbf{n} \cdot \mathbf{F}_\nu , \tag{2.18}$$

where $\mathbf{F}_\nu$ is defined by (2.16).

We may write (2.13) as

$$F_\nu = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi/2} I_\nu \cos\theta \sin\theta \, d\theta d\phi - \int_{0}^{2\pi} \int_{\pi}^{\pi/2} I_\nu \cos\theta \sin\theta \, d\theta d\phi = F_\nu^+ - F_\nu^- , \tag{2.19}$$

where F_ν^+ and F_ν^- are, respectively, the *outward* and the *inward* net flux. Thus F_ν^+ is the amount of energy per unit frequency range flowing "outward" across unit area in unit time, and F_ν^- is the (positive) amount of energy per unit frequency range flowing "inward" across unit area in unit time.

The integrated (or *total*) net flux is defined by the expression

$$F \equiv \int_0^\infty F_\nu d\nu . \tag{2.20}$$

An alternative notation is sometimes used. Let $\mu = \cos\theta$, $d\mu = -\sin\theta \, d\theta$; then we may write

$$F_\nu = \int_0^{2\pi} \int_0^{\pi} I_\nu \cos\theta \sin\theta \, d\theta d\phi \tag{2.21a}$$

$$= \int_0^{2\pi} \int_{-1}^{1} I_\nu(\mu, \phi) \mu d\mu d\phi . \tag{2.21b}$$

As an example, consider an *isotropic* radiation field, for which I_ν is independent of μ and ϕ. Thus

$$F_\nu = 2\pi I_\nu \int_{-1}^{1} \mu d\mu = 0 .$$

In particular,

$$F_\nu^+ = 2\pi I_\nu \int_0^1 \mu d\mu = \pi I_\nu ,$$

$$F_\nu^- = 2\pi I_\nu \int_0^{-1} \mu d\mu = \pi I_\nu .$$

2.3 Energy Density of Radiation

The *energy density* of radiation is defined, in a vacuum or in a non-dispersive medium of unity refractive index, as the amount of radiant energy per unit volume in the course of transit in all directions. We consider in particular the *monochromatic* energy density, or the amount of radiant energy per unit frequency interval in course of transit per unit volume.

Consider, for example, a beam of parallel rays impinging normally on a surface element of unit area in a vacuum. Let F be the amount of energy crossing unit area normal to the direction of flow per unit time. Then, in one second the radiation F having crossed the unit area will occupy a volume of c cm^3, where c is the velocity of light in vacuo. Hence the energy in the course of transit per unit volume is

$$u = \frac{F}{c}.$$

For a more general treatment, we consider an element of volume, v, so small that I_ν may be considered as constant throughout v, within a closed surface σ (see Fig. 2.7), and we assume the refractive index μ_ν of the medium

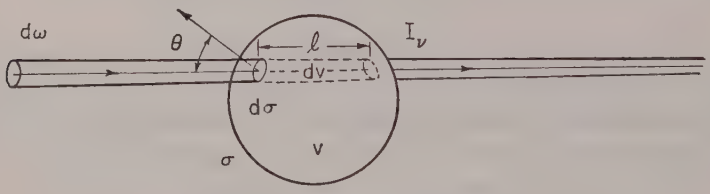


Fig. 2.7 Illustration for energy density of radiation.

in which the radiation field exists to be unity for all frequencies of interest. Let I_ν be the intensity in a given direction in the vicinity of the volume v. Now the amount of energy of frequency ν within dν flowing in time dt into the element of area dσ, whose outward normal forms an angle θ to the direction of the pencil, from within the solid angle dω about the direction considered, is

$$\mathrm{d}E_\nu = I_\nu \mathrm{d}\nu \mathrm{d}\sigma \cos\theta \, \mathrm{d}\omega \mathrm{d}t.$$

We take dt to be the time required for the light ray to travel across a section of length ℓ of v: d$t = \ell/c$. Thus

$$\mathrm{d}E_\nu = I_\nu \mathrm{d}\nu \mathrm{d}\sigma \cos\theta \, \mathrm{d}\omega \frac{\ell}{c} = I_\nu \mathrm{d}\nu \mathrm{d}\omega \frac{\mathrm{d}v}{c},$$

where d$v = \ell$ d$\sigma \cos\theta$. Integrating over the entire volume v (remembering that $I_\nu \approx$ const. over the small volume v) and over all solid angles gives the

total energy of frequency ν within $d\nu$ from all directions in the course of transit within v:

$$\frac{v}{c}d\nu \int_{4\pi} I_\nu d\omega \equiv u_\nu d\nu v .$$

The expression for the monochromatic energy density u_ν in terms of I_ν is then

$$u_\nu = \frac{1}{c}\int_{4\pi} I_\nu d\omega . \tag{2.22}$$

For time-independent radiation fields u_ν is a function of frequency and position, but not direction: $u_\nu = u_\nu(\mathbf{r})$.

The *integrated energy density* is defined by the relation

$$u = \int_0^\infty u_\nu d\nu = \frac{1}{c}\int_{4\pi} I d\omega \tag{2.24}$$

(for $\mu_\nu = 1$), where

$$I = \int_0^\infty I_\nu d\nu .$$

In the case of a dispersive medium the velocity of *energy* flow must be taken not as c but, as is pointed out in Sect. 2.10a, as v_g, the *group* velocity (see (2.129) and (2.131) below) of a wave packet. In terms of a "photonic" picture such a wave packet may be regarded (see Sect. 2.10a) as a *photon* in the dispersive medium or, in the language of Brittin and Chappell [Ch65a], as a "quasi-photon." The energy of a quasi-photon of frequency ν is $h\nu$ and its velocity is v_g.

Using v_g in place of c in the foregoing derivation, it is clear that the relation between u_ν and I_ν in an isotropic medium is

$$u_\nu = \frac{1}{v_g}\int_{4\pi} I_\nu d\omega = . \tag{2.24'}$$

As is pointed out in Sect. 2.10c, the energy density u_ν given by (2.24') is a measure of the *total* energy associated with the electromagnetic wave, and it includes not only the electromagnetic field energy itself but also the kinetic energy of ordered (coherent) motions of the charges associated with the electromagnetic wave in the dispersive medium.

If the medium is not dispersive (μ_ν independent of frequency ν), then (*cf.* Sect. 2.10a) $v_g = v_p = c/\mu_\nu$, where v_p is the *phase* velocity of the electromagnetic wave, always $\leq c$ in this case.

2.4 Average Intensity

This is defined by the relation

$$J_\nu = \frac{1}{4\pi} \int_{4\pi} I_\nu \, d\omega . \tag{2.25}$$

Thus, for time-independent radiation fields J_ν is a function only of frequency and position: $J_\nu = J_\nu(\mathbf{r})$. The general relation between u_ν and J_ν is thus, in a nondispersive medium with $\mu_\nu = 1$,

$$u_\nu = \frac{4\pi}{c} J_\nu . \tag{2.26}$$

For example, for an *isotropic* radiation field, for which I_ν is independent of μ and ϕ, we have

$$J_\nu = I_\nu ,$$

whence (for $\mu_\nu = 1$)

$$u_\nu = \frac{4\pi}{c} I_\nu . \tag{2.27}$$

In an isotropic dispersive medium c in (2.26) and (2.27) would be replaced by the group velocity v_g (see (2.129) and (2.131) below).

2.5 Radiation Pressure

The existence of radiation pressure follows from the electromagnetic theory of light, as well as from relativity theory and quantum theory; radiation pressure exists because momentum is carried by radiation.

To derive expressions relating the radiation pressure to the specific intensity I_ν, we temporarily adopt a "photonic" picture, in which photons are the carriers of radiant energy. A photon of frequency ν has energy $h\nu$ (h = Planck's constant), travels with speed (in vacuo) c, and has momentum (in a vacuum) $h\nu/c$. (This last expression may be "derived" from the following heuristic argument: If an amount of energy $h\nu$ is being carried in the form of radiation in a vacuum, then by relativity theory an amount of mass $m = h\nu/c^2$ must be associated with this energy. The momentum of the photon is then $mc = h\nu/c$. Equivalently, we may say that a photon is a particle of zero rest mass. It follows immediately from the special relativistic relation between energy and momentum (see (10.10) below) that $h\nu = pc$. Alternatively, the de Broglie wavelenght λ associated with the momentum p is given by $p = h/\lambda$, so that $p = h\nu/c$ for a photon in vacuo.)

We define the pressure of radiation at a given point in a general radiation field to be the net rate of transfer of momentum normal to an arbitrarily oriented surface of unit area at that point.

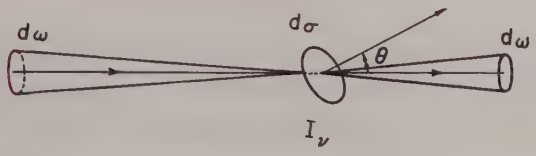


Fig. 2.8 Illustration for transport of momentum by radiation across an element of area $d\sigma$.

Consider a pencil of radiation of intensity I_ν impinging upon a surface element $d\sigma$ at an angle θ to the normal to $d\sigma$ (see Fig. 2.8). The radiant energy having frequencies in the range $(\nu, \nu+d\nu)$ transferred in time dt across $d\sigma$ from directions confined to the solid angle $d\omega$ is $(I_\nu d\nu d\sigma \cos\theta\, d\omega dt)$, and the corresponding number of photons is $(I_\nu d\nu d\sigma \cos\theta\, d\omega dt/h\nu)$. Multiplying this last expression by $(h\nu/c)$, the momentum per photon in a vacuum, we obtain $(1/c)\, I_\nu d\nu d\sigma \cos\theta\, d\omega dt$ for the amount of momentum transferred across $d\sigma$ by this pencil with frequency ν within $d\nu$ in time dt. The component of this momentum *normal* to $d\sigma$ is $(1/c)\, I_\nu d\nu d\sigma \cos^2\theta\, d\omega dt$. Dividing this expression by dt and $d\sigma$ and integrating it over all solid angles, we obtain for the *net* rate of transfer of this momentum normal to $d\sigma$

$$(1/c)d\nu \int_{4\pi} I_\nu \cos^2\theta\, d\omega .$$

But this is just the pressure of radiation having frequencies in the range $(\nu, \nu+d\nu)$: $p_{r,\nu}d\nu$. Hence the expression for the *monochromatic* radiation pressure in a vacuum or in a nondispersive medium with μ_ν (refractive index) = 1 is

$$p_{r,\nu} = (1/c) \int_{4\pi} I_\nu(\theta, \phi) \cos^2\theta\, d\omega , \tag{2.28a}$$

$$= (1/c) \int_0^{2\pi} \int_{-1}^{1} I_\nu(\mu, \phi) \mu^2 d\mu d\phi . \tag{2.28b}$$

Consider, for example, the case of an *isotropic* radiation field (I_ν independent of μ and ϕ). For $\mu_\nu = 1$, we have

$$p_{r,\nu} = \frac{2\pi}{c} I_\nu \int_{-1}^{1} \mu^2 d\mu = \frac{4\pi}{3c} I_\nu .$$

But we had in this case for an isotropic radiation field $u_\nu = (4\pi/c)I_\nu$. Hence, for an *isotropic* radiation field we have the important result (for $\mu_\nu = 1$)

$$p_{r,\nu} = \frac{1}{3}u_\nu . \tag{2.29}$$

In the case of a dispersive medium it has been shown by Britten and Chappell (*cf.* Ch65a) and by Harris [Ha65] that the momentum p of a quasi-photon (a photon in a dispersive medium) of energy $h\nu$ is

$$p = \mu_\nu(h\nu/c) = h\nu/v_p , \tag{2.30}$$

where μ_ν is the (real) refractive index of the material and $v_p = c/\mu_\nu$ is the *phase* velocity of the associated electromagnetic wave (*cf.* also Sect. 2.10a). Proceeding just as before, and retaining the definition of radiation pressure presented earlier in this section, we obtain the following expression for the monochromatic radiation pressure in a dispersive medium:

$$p_{r,\nu} = \frac{\mu_\nu}{c}\int_{4\pi} I_\nu(\theta,\phi)\cos^2\theta \, d\omega . \tag{2.28'}$$

It should be cautioned that $p_{r,\nu}$ as given by (2.28′), resulting solely from momentum transfer by quasi-photons, may not necessarily be the complete force per unit area exerted by the radiation on the matter in a dispersive medium. The general problem of the total stress exerted by radiation on a dispersive medium has apparently not been completely solved (*cf.* Landau and Lifshitz [La60, p. 256]; and Kaufmann [Ka65a]). In all later applications of (2.28′) and (2.34′) below, however, we shall assume that these expressions can be used to represent the total force per unit area and the total stress acting on the matter by the radiation.

In the case of an isotropic radiation field in an isotropic, dispersive medium we clearly have $p_{r,\nu} = (4\pi/3)I_\nu/v_p$. On the other hand, we also have in this case (see (2.24′)) $u_\nu = 4\pi I_\nu/v_g$ (v_g = group velocity) for the energy density of the radiation. Hence the relation between $p_{r,\nu}$ and u_ν in this case is

$$p_{r,\nu} = \frac{1}{3}\frac{v_g}{v_p}u_\nu , \tag{2.29'}$$

which reduces to (2.29) for a nondispersive medium ($v_p = v_g$), even when $\mu_\nu \neq 1$. According to Sect. 2.10a, $v_g/v_p \leq 1$ always, so that $p_{r,\nu} \leq (1/3)u_\nu$. Physically, this last inequality arises from the fact (see Sect. 2.10c) that a part of u_ν as defined by (2.24′) consists of the kinetic energy of ordered (coherent) motions of oscillating charges, and this part of u_ν does not contribute to momentum transfer.

2.5a Integrated Radiaton Pressure

This is defined by the relation

$$p_r = \int_0^\infty p_{r,\nu} \mathrm{d}\nu . \tag{2.31}$$

For an isotropic radiation field of energy density u we have, if $\mu_\nu = 1$,

$$p_r = \frac{1}{3} u . \tag{2.32}$$

(*Remark:* Physically, why does not the radiation pressure vanish for an isotropic radiation field? Here we have as much energy flowing from "left" to "right" across an element of surface as from "right" to "left". For radiation flowing from "left" to "right", we may consider the element of surface to be *gaining positive* momentum; thus a force to the right is experienced. Radiation flowing from right to left corresponds to a *loss* of *negative* momentum from the surface element. The *recoil* from this loss of momentum would also be a force toward the right. One could of course interchange the terms "left" and "right" in the above argument without affecting its validity. Thus the forces exerted by an isotropic radiation field on the two surfaces of a perfectly reflecting material disk, for example, would exactly balance and no net force would result.)

In the case of a dispersive medium the integrated radiation pressure would be given by

$$p_r = \frac{1}{3} \int_0^\infty \left(\frac{v_g}{v_p} \right) u_\nu \mathrm{d}\nu = \frac{1}{3} \langle v_g / v_p \rangle u , \tag{2.32'}$$

which defines the average $\langle v_g/v_p \rangle$. Note (see Sect. 2.10a) that $\langle v_g/v_p \rangle \leq 1$, whence $p_r \leq (1/3)u$.

2.5b Pressure Tensor

We shall consider here only the integrated radiation, for simplicity, although the following argument applies equally well to the monochromatic radiation. We shall also assume in this subsection, for simplicity, that the refractive index μ_ν is unity. In case this is not so, the formulae for the case $\mu_\nu \neq 1$ in an isotropic medium can be obtained by simply replacing c in all formulae in this subsection except (2.45), (2.47), and (2.48) by the phase velocity $v_p = c/\mu_\nu$.

We note that the radiation pressure as defined by (2.28) (with reference to ν omitted) depends on the orientation of the surface element under

consideration. This is also physically clear in the case of a non-isotropic radiation field. The radiation pressure as given by (2.28) is actually the *normal component* of the *stress*, which is the force (a vector) exerted on unit area of the surface element by the radiation.

To describe the radiation pressure of a general radiation field without reference to the orientation of the surface element used for its measurement, we must introduce the *pressure tensor*. We may obtain the expression for the pressure tensor from (2.28) by writing $\cos\theta = \mathbf{n}\cdot\mathbf{n}'$, where $\mathbf{n}$ is a unit vector in the direction of the *outward* normal (see next few sentences) to the element of area and $\mathbf{n}'$ is a unit vector in some arbitrary direction. (We are using here the *dyadic* notation; see, for example, Phillips [Ph33, Chap. 10].) In the case of a *mathematical* surface element immersed in a general radiation field, the choice of which of the two normals is to be regarded as the outward one is arbitrary. If, however, the surface element is regarded as a portion of the surface of a material body, the outward normal would be the one which extends *outward* from the interior of the body. Whichever of the two normals is chosen as the outward one, the *stress* is defined as the (vector) force per unit area which the "material" lying on the "outward" side of the surface element (the side containing the outward normal) exerts *on* the "material" lying on the opposite ("inward") side of the surface element. It is clear that the stress changes sign when the choice of the outward normal is changed.

We write $p(\mathbf{n})$ for the radiation pressure exerted on a surface element whose outward normal is in the direction $\mathbf{n}$, or, more accurately, for the normal component of the force per unit area exerted by the radiation on an element of area whose outward normal is $\mathbf{n}$. Then (2.28) may be written in the form

$$p(\mathbf{n}) = \frac{1}{c}\int_{4\pi} I(\mathbf{n}')\cos^2\theta\, d\omega_{\mathbf{n}'} = \mathbf{n}\cdot\left[\frac{1}{c}\int_{4\pi} I(\mathbf{n}')\mathbf{n}'\mathbf{n}'d\omega_{\mathbf{n}'}\right]\cdot\mathbf{n}\ . \tag{2.33}$$

This relation exhibits explicitly the dependence of $p(\mathbf{n})$ on $\mathbf{n}$, the direction of the outward normal to the surface element.

We now define the *pressure tensor* by the relation (for unity refractive index μ_ν)

$$\mathsf{p} \equiv \frac{1}{c}\int_{4\pi} I(\mathbf{n}')\mathbf{n}'\mathbf{n}'d\omega_{\mathbf{n}'}. \tag{2.34}$$

According to the developments near the beginning of this section, (2.34) would be replaced in the case of a dispersive medium by the equation

$$\mathsf{p} = \frac{\mu_\nu}{c}\int_{4\pi} I(\mathbf{n}')\mathbf{n}'\mathbf{n}'d\omega_{\mathbf{n}'}. \tag{2.34'}$$

The components of p in a rectangular Cartesian coordinate system are given by

$$p_{ij} = \frac{1}{c} \int_{4\pi} I(\alpha_1', \alpha_2', \alpha_3') \alpha_i' \alpha_j' d\omega_{\mathbf{n}'}, \tag{2.35}$$

where p is seen to be symmetric, and where the α_i' are the direction cosines of the unit vector $\mathbf{n}'$:

$$\mathbf{n}' = \alpha_1' \mathbf{e}_1 + \alpha_2' \mathbf{e}_2 + \alpha_3' \mathbf{e}_3, \tag{2.36}$$

where the $\mathbf{e}_i$ are unit vectors along the various coordinate axes (see Fig. 2.9).

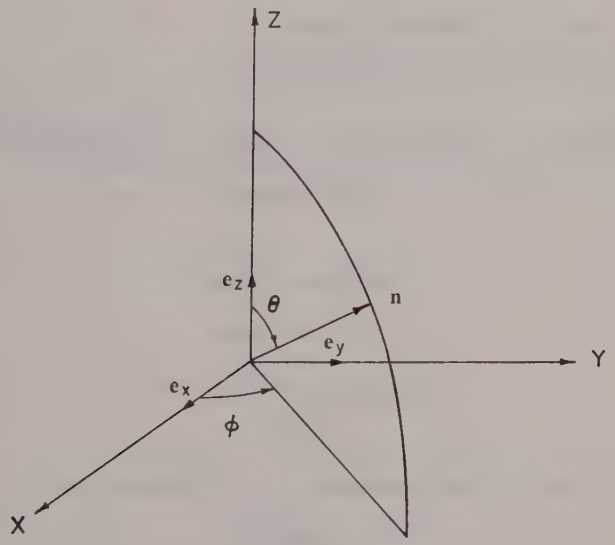


Fig. 2.9 Polar coordinate angles θ and ϕ.

Thus (2.33) may also be written in the form

$$p(\mathbf{n}) = \mathbf{n} \cdot \mathsf{p} \cdot \mathbf{n} = -\mathbf{T} \cdot \mathbf{n}, \tag{2.37}$$

where the *stress* T (force per unit area) acting on an element of area whose outward normal is **n** is related to the pressure tensor p as follows (*cf.*, for example, Phillips [Ph33, Chap. 10]):

$$\mathbf{T} = -\mathbf{n} \cdot \mathsf{p} = -\mathsf{p} \cdot \mathbf{n}, \tag{2.38}$$

where the second equality in (2.38) arises from the fact that p is symmetric. It is seen from (2.37) and (2.38) that $p(\mathbf{n})$ is the component of the force per unit area along the *inward* normal to the surface element. The pressure tensor is just the negative of the *stress* tensor.

It follows from the first equality in (2.38) and from the definition of stress that p_{ij} may be interpreted as follows. Define the outward normal of an arbitrarily oriented surface element to be the one which points into the hemisphere *away* from the origin of coordinates of a right-handed, rectangular Cartesian coordinate system. Then p_{ij} is the component along the *negative* j axis of the force per unit area exerted by the radiation field on the surface element when the outward normal of the surface element lies along the *positive* i axis. It is then clear that in a right-handed, rectangular coordinate system whose positive x_3 axis lies along the outward normal of a surface element, $p(\mathbf{n})$ as defined in (2.33) and (2.37) is actually the 3-3 component p_{33} of p.

Now consider the symmetric tensor

$$\mathsf{S} \equiv \int_{4\pi} \mathbf{n}\mathbf{n}\,d\omega_{\mathbf{n}} \tag{2.39}$$

with components in a right-handed, rectangular coordinate system

$$S_{ij} \equiv \int_{4\pi} \alpha_i \alpha_j\, d\omega_{\mathbf{n}}. \tag{2.40}$$

If we let

$$\begin{aligned} d\omega_{\mathbf{n}} &= \sin\theta\, d\theta d\phi, \\ \alpha_1 &= \sin\theta\cos\phi, \\ \alpha_2 &= \sin\theta\sin\phi, \\ \alpha_3 &= \cos\theta \end{aligned} \tag{2.40'}$$

(see Fig. 2.9), then by direct integration it can be shown that

$$S_{ij} = \frac{4\pi}{3}\delta_{ij}, \tag{2.41}$$

where δ_{ij} is the Kronecker delta:

$$\delta_{ij} = \begin{cases} 0 \text{ if } i \neq j \\ 1 \text{ if } i = j. \end{cases} \tag{2.42}$$

In dyadic notation (2.41) is

$$\mathsf{S} = \frac{4\pi}{3}\mathsf{I}, \tag{2.43}$$

where the unit tensor I is given by

$$\mathsf{I} = \sum_{ij} \delta_{ij}\mathbf{e}_i\mathbf{e}_j, \tag{2.44}$$

the e's being unit vectors along the various coordinate axes (see (2.36)). Thus the direction cosines α_i are orthogonal to one another with respect to the element of solid angle $d\omega_{\mathbf{n}}$.

It is clear from (2.43) that in the case of an *isotropic* radiation field the pressure tensor becomes, for $\mu_\nu = 1$,

$$\mathsf{p} = \frac{I}{c}\mathsf{S} = \frac{4\pi I}{3c}\mathsf{I} \equiv p\mathsf{I} = \frac{1}{3}u\mathsf{I} \tag{2.45}$$

(by (2.32)), where I is the (isotropic) specific intensity and p is the *hydrostatic* radiation pressure. Hence, in this case only the diagonal components of p are non-vanishing and these diagonal components are all the same and equal to $p = (1/3)u$. Using (2.45) in (2.38), we obtain

$$\mathbf{T} = -p\mathbf{n},$$

which shows that in this case the force exerted by the radiation on a surface element is always normal to the element and in a direction opposite to that of the outward normal $\mathbf{n}$.

We now define the *mean pressure* $\bar{p}$ by forming the average of $p(n)$ over all possible orientations of the surface element whose outward normal is $\mathbf{n}$:

$$\bar{p} = \frac{1}{4\pi}\int_{4\pi} p(\mathbf{n})\mathrm{d}\omega_{\mathbf{n}} \tag{2.46a}$$

$$= \frac{1}{4\pi}\int_{4\pi} \mathbf{n}\cdot\mathsf{p}\cdot\mathbf{n}\,\mathrm{d}\omega_{\mathbf{n}}$$

$$= \frac{1}{4\pi}\sum_{ij} p_{ij}\int_{4\pi}\alpha_i\alpha_j\mathrm{d}\omega_{\mathbf{n}}$$

$$= \frac{1}{4\pi}\sum_{ij} p_{ij}\cdot\frac{4\pi}{3}\delta_{ij}$$

or

$$\bar{p} = \frac{1}{3}\sum_i p_{ii} = \frac{1}{3}\mathrm{Tr}\,\mathsf{p}, \tag{2.46b}$$

where "Tr" stands for "Trace" or "Spur". In the third equality of (2.46) we have removed p_{ij} from under the integral sign because p_{ij} is a property of the *radiation field* and not of the orientation of the surface element. Now

$$\mathrm{Tr}\,\mathsf{p} = \sum_i p_{ii} = \frac{1}{c}\sum_i\int_{4\pi} I(\alpha_1', \alpha_2', \alpha_3')\,\alpha_i'\alpha_i'\mathrm{d}\omega_{\mathbf{n}'},$$

from (2.35). But $\sum_i \alpha_i' \alpha_i' = \mathbf{n}' \cdot \mathbf{n}' = \cos 0 = 1$; hence

$$\mathrm{Tr}\,\mathsf{p} = \frac{1}{c}\int_{4\pi} I \,\mathrm{d}\omega_{\mathbf{n}'} = u\,, \tag{2.47}$$

where u is the integrated energy density. Thus the mean pressure $\bar{p}$ is given for a general, non-isotropic radiation field with $\mu_\nu = 1$ by

$$\bar{p} = \frac{1}{3}u\,. \tag{2.48}$$

If the radiation field is isotropic, then $p(\mathbf{n})$ does not depend on $\mathbf{n}$ (orientation of surface element), whence it follows from (2.46) that $\bar{p} = p$, and so $p = (1/3)u$, just as we had obtained earlier.

In the case of a dispersive medium (2.48) would be replaced by

$$\bar{p} = \frac{1}{3}\langle v_g/v_p \rangle u\,, \tag{2.49}$$

where $\langle v_g/v_p \rangle$ is defined in (2.32′). Note (see Sect. 2.10a) that $\langle v_g/v_p \rangle \leq 1$, whence $\bar{p} \leq (1/3)u$.

2.6 Mass Emission Coefficient

We regard any addition of radiant energy to the energy of a pencil of radiation resulting from the interaction of radiation with matter (or of radiation with radiation, as in photon-photon scattering), as "emission" from that matter (or radiation). By "emission" we mean either "true" emission or radiation scattered into the pencil from other directions, or the sum of the two contributions. The *mass emission coefficient* j_ν is defined by the following statement: the amount of radiant energy of frequency ν within $\mathrm{d}\nu$ emitted in time $\mathrm{d}t$ by an element of mass $\mathrm{d}m$ into the solid angle $\mathrm{d}\omega$ about the direction $\mathbf{n}'$ is

$$\mathrm{d}E_\nu \equiv j_\nu \,\mathrm{d}\nu\,\mathrm{d}m\,\mathrm{d}\omega\,\mathrm{d}t\,. \tag{2.50}$$

(To cover cases of photon-photon scattering, for example, which may occur in the absence of matter, a *volume* emission coefficient would be more appropriate, defined just as above except with $\mathrm{d}m$ replaced by the volume element $\mathrm{d}V$. However, unless we state otherwise, j_ν will always refer to the *mass* emission coefficient.)

It should be noted that the value of j_ν as defined by (2.50) may be affected by effects of dispersion; *i.e.*, j_ν as computed assuming unity (real) refractive

index μ_ν will in general have a different value from that for $\mu_\nu \neq 1$. The relation between the two values is discussed in Sects. 2.9 and 3.6.

2.7 Mass Absorption Coefficient

If no emission of energy occurs within an element of matter, then a pencil of radiation traversing the element will in general be attenuated to some extent. This attenuation is often referred to as "absorption," by which is meant any process that removes energy from a pencil of radiation, whether the process be "true" absorption, scattering, "lossless" attenuation (such as total reflection) resulting from collective effects (see Sect. 2.10b), or some combination of these. Let the intensity in a given direction and at a given point s in matter of density ρ be I_ν. Then, at the point $s+\mathrm{d}s$ the intensity will be $I_\nu+\mathrm{d}I_\nu$. The change $\mathrm{d}I_\nu$ in intensity will be proportional (at least in cases of interest where I_ν is not enormously large) to the distance $\mathrm{d}s$, the density ρ of the material, and the incident intensity I_ν of the pencil:

$$\mathrm{d}I_\nu = -\kappa_\nu \rho I_\nu \mathrm{d}s\,, \tag{2.51}$$

where κ_ν, the factor of proportionality, is the *mass absorption coefficient* of the material. Equation (2.51) may also be written as

$$-\kappa_\nu = \frac{\mathrm{d}I_\nu}{I_\nu \rho \mathrm{d}s}.$$

Thus κ_ν is the *fractional* change in intensity of radiation traversing a unit amount of mass contained in a column of unit cross-sectional area and of height $\mathrm{d}s$ (assuming no emission occurs within this column). In an anisotropic medium κ_ν may be a function of direction as well as of position. (Sometimes a *volume* absorption coefficient is used, defined exactly as in (2.51) except with the factor ρ omitted; such an absorption coefficient would perhaps be more appropriate for absorption and scattering processes such as pair production and photon-photon scattering, which can operate in virtually "empty" space. However, unless otherwise stated, we shall always take κ_ν to mean the *mass* absorption coefficient.)

It should be noted that the value of κ_ν as defined by (2.51) may be affected by dispersive effects. The relation between the value of κ_ν in a medium having unity (real) refractive index μ_ν and that in a medium having $\mu_\nu \neq 1$ (other things being equal) is discussed in Sects. 2.9, 2.10b, and 3.6.

Integration of (2.51) gives

$$I_\nu(s) = I_\nu(s_0) \exp\left(-\int_{s_0}^{s} \kappa_\nu \rho \,\mathrm{d}s'\right) = I_\nu(s_0)\, e^{-\tau_\nu}, \tag{2.52}$$

where

$$\tau_\nu \equiv \int_{s_0}^{s} \kappa_\nu \rho \, ds' \tag{2.53}$$

is termed the *optical thickness* of the path segment between any two points in the medium, s_0 and s, or the optical depth to the point s_0 if s is at the "surface" of the absorbing region. Thus *unit* optical thickness is that optical thickness for which the intensity is reduced to $1/e$ of its initial value. For the case in which κ_ν and ρ are independent of position, we have $\tau_\nu = \kappa_\nu \rho \ell$, ℓ being the *geometrical* length of the path. Thus, for $\tau_\nu = 1$ we have

$$\rho \ell = \frac{1}{\kappa_\nu},$$

where $\rho\ell$ is the amount of material per unit cross sectional area which will reduce I_ν to e^{-1} of its initial value. For example, if $\kappa_\nu = 100$ cm^2/gm (a typical value for the stellar interior) $= 1$ cm^2/(1/100 gm), then 1/100 gm of material spread over an area of 1 cm^2 would reduce I_ν to $1/e$ of its initial value. The quantity $1/\kappa_\nu\rho$ may also be interpreted as the mean free path of a photon.

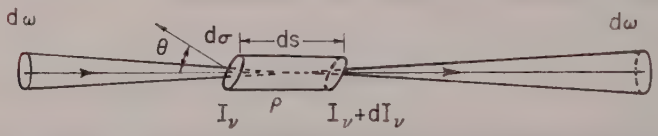


Fig. 2.10 Passage of a pencil of radiation through matter.

We now obtain an expression for the total change in energy of radiation as it traverses matter. Consider a pencil of radiation impinging upon the surface $d\sigma$ of an element of matter of density ρ and of length ds, the outward normal to $d\sigma$ forming an angle θ to the direction of the pencil (see Fig. 2.10), and travelling in a straight line across the element. If the intensity at the first surface is I_ν, the amount of radiant energy having frequencies in the range $(\nu, \nu + d\nu)$ flowing through the surface $d\sigma$ in time dt from directions within the solid angle $d\omega$ is $I_\nu d\nu d\sigma \cos\theta \, d\omega dt$. At the second surface, also of area $d\sigma$ with outward normal inclined at angle θ to the direction of the pencil, the intensity is $I_\nu + dI_\nu$. Thus the energy passing out of the element is $(I_\nu + dI_\nu)$ $d\nu d\sigma \cos\theta \, d\omega dt$. The total change in the energy of the pencil in passing through the element of matter is given by the difference between the above two expressions. If no emission of energy occurs, we have from (2.51)

$$dE_\nu = -I_\nu \kappa_\nu \rho ds d\nu d\sigma \cos\theta \, d\omega dt \, .$$

But the mass of the element is $dm = \rho d\sigma \cos\theta\, ds$, so that

$$dE_\nu = -\kappa_\nu I_\nu d\nu\, dm\, d\omega\, dt. \tag{2.54}$$

In the case of a medium with a spatially variable refractive index the ray paths followed by the radiation will generally be curved. When dealing with curved ray trajectories we shall always understand differentials such as dI_ν and dE_ν in the above formulas to mean infinitesimal differences between values of the corresponding quantities taken *along the curved ray trajectory*. It is easily seen that any differences in the values of θ and $d\omega$ between s and $s+ds$ resulting from any curvature of the ray path or from variation of μ_ν along the ray path (*cf.* Sect. 2.1c) are of first order in ds. Equation (2.54) is therefore valid, as it stands, to first order in small quantities (*i.e.*, in ds or dm) even for a curved ray trajectory.

2.8 Microscopic Picture of Emission and Absorption of Radiation

In this section we adopt a purely "microscopic" viewpoint, and we consider emission and absorption in terms of individual atomic processes. Our formalism will not exclude the possible presence of effects of dispersion, but we do not discuss these effects in detail in this section (see Sects. 2.9, 2.10b, and 3.6).

Consider two energy states 1 and 2 of an atom ($E_2 > E_1$), where by "atom" we mean either a neutral atom, ion, or molecule. We assume here, for definiteness, that the states are discrete, so that we are considering, explicitly, "bound-bound" transitions. Although states 1 and 2 are discrete, they will both under most circumstances behave as if they had a finite, but small, width, and the energies E_1 and E_2 refer to the "centers of gravity" of the two states (in case state 1 is a ground state, its width may be zero). The finite width is a consequence of the Heisenberg uncertainty principle (finite lifetimes of excited states) and possibly of external perturbations which may be present. By the Bohr frequency condition, a transition of the atom between state 1 and state 2 will be accompanied by the emission or absorption of (in nearly all cases) a single quantum $h\nu$ of radiation of frequency ν, where ν will always be very nearly equal to ν_0, the frequency of the "line center:"

$$h\nu_0 \equiv E_2 - E_1\,, \tag{2.55}$$

where h is Planck's constant.

In reality, other physical processes, such as "bound-free" and "free-free" transitions may contribute to the radiation having frequencies in the vicinity of ν_0. In these processes one or both of the states 1 and 2 may lie in the

continuum, in which case a "state" must be regarded as a group of closely spaced states confined to a narrow energy range. In these latter processes the definitions of the Einstein coefficients, to be given presently in this section for bound-bound transitions, would have to be suitably modified (see, for example, Menzel and Pekeris [Me35]; their treatment is summarized in Thomas and Athay [Th61a, Chap. 4]; see also Sampson [Sa65a, Chap. 1]). For a complete treatment, arguments paralleling the one to be presented in the remainder of this chapter would have to be carried through for each of these processes, and the combined effects of all three processes would have to be considered. Our treatment, in which we consider explicitly only one of these processes, as well as scattering, is therefore somewhat schematic, but it is illustrative of the kinds of arguments used. Our final results, when applied to the stellar interior, can be generalized in a simple and straightforward way so as to include the other processes (see Sect. 4.1).

2.8a Emission

Matter exposed to a general radiation field may emit energy in at least three ways: (1) spontaneous emission, (2) induced emission, and (3) scattering (recalling our definition of emission in Sect. 2.6). The nature of the scattering process we consider will be explained in Sect. 2.8a3. We consider each of these processes in turn, but first we define the notion of the probability per unit time of a transition. Suppose there are N_2 atoms or molecules of a given kind per unit volume in state 2 in an assemblage of such particles. In the time dt, suppose dN_2 of these N_2 particles per unit volume make transitions to the state 1. Then $(1/N_2)\, dN_2/dt \equiv$ probability per unit time that such a particle will undergo the transition 2 to 1.

2.8a1 Spontaneous Emission

An atom in an excited state 2 has a certain probability of making a downward transition to state 1 and thereby emitting a photon of frequency $\nu \approx \nu_0$. This probability consists of the sum of two probabilities, the first of which is *independent* of the nature or existence of the prevailing radiation field and represents *spontaneous emission*,* the second of which depends on the prevailing radiation field and represents *induced emission* (*cf.* Dirac [Di58, Chap. 10] and Heitler [He54, Chap. 5, Sect. 17]). The Einstein coefficient A_{21} for spontaneous emission is defined by the statement that the probability of a given atom in state 2 emitting in time dt radiation having frequencies in the range $(\nu, \nu+d\nu)$ (all ν's in the vicinity of ν_0) into the

* Actually, dispersive effects can influence even the spontaneous emission probability; see Sects. 2.9 and 3.6.

elementary solid angle $d\omega$ about some direction $\mathbf{n}'$, in the absence of an external radiation field, is equal to $A_{21}\psi_a^*(\nu)d\nu d\omega dt$, where A_{21} is assumed to be independent of frequency "in the line" (*i.e.*, for $\nu \approx \nu_0$) and where $\psi_a^*(\nu)$ is the "line shape function" for spontaneous emission in the absence of an external radiation field. It is a sharply peaked function differing appreciably from zero only for frequencies confined to a narrow range $\Delta\nu$ about ν_0 and is normalized to unity:

$$\int_{\Delta\nu} \psi_a^*(\nu)d\nu = 1, \tag{2.56}$$

where the integration is carried out over all frequencies within $\Delta\nu$. The subscript a signifies the absence of dispersive effects, *i.e.*, of an external radiation field in this case. If dispersive effects are present (*cf.* Sect. 2.10), we assume that they can be included in the line shape function and we shall then simply drop the subscript a. The normalization (2.56) applies only to $\psi_a^*(\nu)$ and not to $\psi^*(\nu)$ unless they are equal. Hence, A_{21} is the total spontaneous emission probability per unit time and unit solid angle for transitions from all sublevels of state 2 to all sublevels of state 1, in the absence of an external radiation field. It is important to note that, whereas A_{21} refers to the inherent tendency of an atom to make a particular downward transition, independently of the effects of an external radiation field, spontaneous *emissions* from an assemblage of atoms *may* be influenced by external radiation fields, partly because these fields may influence the population of the upper state 2, and partly because they may produce dispersive effects (see the footnote in this subsection and the remark following (2.56)).

The spontaneous emission of radiation arising from an individual atom is generally not isotropic relative to axes fixed in the atom. However, the spontaneous emission arising from an *assemblage* of *randomly oriented* atoms will henceforth, unless we state otherwise, be assumed isotropic. (Note, however, that the spontaneous emission of atoms whose orientations are aligned by an externally applied magnetic field, for example, may *not* be isotropic.)

2.8a2 Induced Emission

If radiation of frequency $\nu \approx \nu_0$ (*cf.* (2.55)) is impinging upon an atom already in the state 2, the probability of emission of radiation of frequency ν will be increased over that which would obtain in the absence of such radiation. This probability of induced emission is proportional to the intensity I_ν of the impinging radiation, and the photons in the resulting induced emission have the same direction, energy (*i.e*, frequency), and polarization as those in the inducing radiation.

It was shown many years ago by Einstein that it was necessary to include a term representing induced emission in order to be able to derive the Planck radiation law for equilibrium radiation from the principle of detailed balancing (*cf.* Einstein [Ei17]). Thus, since this principle is assumed to apply under equilibrium conditions, the presence of induced emission may be regarded as empirically demanded.

The necessity for and properties of induced radiation may also be deduced solely on the basis of the theory of quantum electrodynamics (see Dirac [Di58, Chap. 10] and Heitler [He54, Sect. 17]). From this theory we have the important result that *the probability for the emission of a photon into a particular unit cell in phase space* (*cf.* Sect. 3.3), *corresponding to a particular energy* (*i.e., frequency*), *direction, and polarization, is proportional to* $(1+n)$, *where n is the number of photons already occupying that cell in phase space.* Since the "1" represents the spontaneous emission and n the induced emission, then n is also the ratio of the induced to the spontaneous emission. Since, moreover, n is a measure of the intensity of the impinging radiation, it follows that the intensity of a pencil of induced radiation is proportional to that of the pencil of the impinging radiation, and the photons emitted by induced emission have the same frequency, direction, and polarization as the corresponding impinging photons.

Suppose, then, that we have radiation of frequency ν ($\approx \nu_0$) of intensity I_ν in the direction $\mathbf{n}'$, impinging upon an atom in the upper state 2. Then the Einstein coefficient B_{21} for *induced emission* is defined by the following statement: The probability of the atom emitting in time $\mathrm{d}t$ radiation having frequencies in the range $(\nu, \nu+\mathrm{d}\nu)$ into the solid angle $\mathrm{d}\omega$ about the direction $\mathbf{n}'$ due to the presence of radiation of frequency ν of intensity $I_\nu(\mathbf{n}')$, and in the absence of dispersive effects (*cf.* Sect. 2.10), is given by

$$B_{21} I_\nu \psi_a(\nu) \mathrm{d}\nu \mathrm{d}\omega \mathrm{d}t\,, \tag{2.57}$$

where B_{21} is assumed to be independent of frequency "in the line" (*i.e.* for $\nu \approx \nu_0$) and where $\psi_a(\nu)$ is the "line shape function" for induced emission in the absence of dispersive effects. It is a sharply peaked function differing appreciably from zero only in a narrow range of frequencies $\Delta\nu$ about ν_0 and is normalized to unity:

$$\int_{\Delta\nu} \psi_a(\nu)\mathrm{d}\nu = 1\,. \tag{2.58}$$

Just as in the case of spontaneous emission, we assume that any dispersive effects can be included in the line shape function. When such effects are present, we simply drop the subscript a. The normalization (2.58) applies only to $\psi_a(\nu)$ and not to $\psi(\nu)$ unless they are equal. The combined probability

of emission by an atom of radiation having frequencies in the range $(\nu, \nu+d\nu)$ in time dt into solid angle $d\omega$ from both spontaneous and induced $2\rightarrow1$ transitions is then given by the sum of the respective probabilities:

$$[A_{21}\psi^*(\nu)+B_{21}I_\nu\psi(\nu)]d\nu d\omega dt = A_{21}\psi^*(\nu)\left[1+\frac{B_{21}I_\nu\psi(\nu)}{A_{21}\psi^*(\nu)}\right]d\nu d\omega dt, \quad (2.59)$$

where the second term in square brackets on the right side is the ratio of the induced to the spontaneous emission. Note that we have dropped the subscripts a from the line shape functions, so that (2.59) is valid also when dispersive effects are present.

According to the general quantum mechanical principle stated near the beginning of this subsection, this second term is equal to n, the number of photons in the unit cell in phase space corresponding to the frequency, direction, and polarization of the inducing pencil of radiation. We shall show in Sect. 3.6 that $\psi^*(\nu) = \mu_\nu^2\psi(\nu)$ under very general conditions to be discussed there, where μ_ν is the (real) refractive index. We then have the general relation

$$n = \frac{B_{21}}{A_{21}}\frac{I_\nu}{\mu_\nu^2}, \quad (2.60)$$

where μ_ν would be set equal to unity if effects of dispersion were absent or not being considered. A statistical mechanical interpretation of (2.60) will be presented in Sect. 3.6, where it will also be shown that the ratio B_{21}/A_{21} is a "universal" function only of frequency (subject to certain restrictions to be stated there) and is therefore independent of the properties of the actual atoms present.

If there are N_2 atoms per unit volume in the upper state 2, the total number of transitions per unit volume from state 2 to state 1 in time dt that send radiation of frequency ν ($\approx\nu_0$) within $d\nu$ into the solid angle $d\omega$ about the direction $\mathbf{n}'$ is

$$N_2[A_{21}\psi^*(\nu)+B_{21}I_\nu\psi(\nu)]d\nu d\omega dt,$$

where we are neglecting Doppler shifts resulting from the motions of the atoms. The amount of energy associated with each transition is very nearly equal to $h\nu_0$; hence the total energy having frequencies in the range $(\nu, \nu+d\nu)$ $(\nu\approx\nu_0)$ emitted per unit volume in time dt into $d\omega$ is obtained by multiplying the above expression by $h\nu_0$, and the amount of energy of frequency ν ($\approx\nu_0$) within $d\nu$ radiated per unit mass in time dt into solid angle $d\omega$ is then

$$h\nu_0\frac{N_2}{\rho}[A_{21}\psi^*(\nu)+B_{21}I_\nu\psi(\nu)]d\nu d\omega dt,$$

where ρ is the mass density of the emitting substance. But this expression is just equal to $j_\nu^{(a)}\,d\nu\,d\omega\,dt$, $j_\nu^{(a)}$ being the mass emission coefficient for spontaneous and induced emission of radiation of frequency ν from atoms (we use the superscript to distinguish between "true" (spontaneous plus induced) emission, considered here, and emission resulting from scattering, to be considered in Sect. 2.8a3). Thus

$$j_\nu^{(a)} = h\nu_0 \frac{N_2}{\rho}\left[A_{21}\psi^*(\nu)+B_{21}I_\nu\psi(\nu)\right]. \tag{2.61}$$

We may note, finally, that induced emission may also be described in terms of an atomic cross section. The cross section may be defined as follows: Consider a "flux" of $\mathscr{F}(\Delta\omega_{\mathbf{n}})$ "particles" per unit area and time all having directions within the solid angle $\Delta\omega_{\mathbf{n}}$ about the direction specified by the unit vector $\mathbf{n}$, incident on a single "scattering" center, assumed to be at rest. If S denotes the total number of "particles" which "interact" with the "scattering" center in unit time (without regard to directions of ejection of possible reaction products), then the *total* cross section σ is defined by the general relation

$$\sigma \equiv S/\mathscr{F}(\Delta\omega_{\mathbf{n}})\,. \tag{2.62}$$

If the orientation of the "scattering" center is regarded as fixed relative to laboratory coordinates, then σ will in general depend on both the direction $\mathbf{n}$ of the incident flux of "particles" and on the size of the solid angle $\Delta\omega_{\mathbf{n}}$ (consider, for example, the scattering of light by a thin disk). In most contexts $\Delta\omega_{\mathbf{n}}$ is regarded as an infinitesimal solid angle, in which case σ is independent of the size of $\Delta\omega_{\mathbf{n}}$. If the orientation of the "scattering" center is unknown, however, then σ must be considered an average cross section over all possible orientations of the "scattering" center, corresponding to random orientations of many identical "scattering" centers; and in this case it is clear that σ will not depend on $\mathbf{n}$. It is this latter case that we shall always assume unless we explicitly state otherwise. (See Sect. 2.8a3 for a corresponding definition of the *differential* cross section.) Using the definition (2.62) and treating photons as "particles," it is easy to show that the cross section σ_ν, corresponding to induced emission of radiation of frequency ν ($\approx \nu_0$) per unit frequency interval is related to the Einstein coefficient B_{21} by

$$\sigma_\nu = h\nu_0 B_{21}\psi(\nu)\,, \tag{2.63}$$

where (*cf.* remark following (2.58)) $\psi(\nu)$ may include effects of dispersion.

2.8a3 Scattering

By "scattering" we shall normally mean "true" scattering in the sense of the quantum theory of radiation (see Heitler [He54, Chap. 5, Sects. 19, 20,

and 22]), *i.e.*, the essentially simultaneous interaction of just two photons (the "incident" photon and the "scattered" photon) with a given scattering center. We do not regard absorption of a photon by an atom, with subsequent re-emission, as true scattering. Except in a case of resonance fluorescence when the incident light is practically monochromatic, the absorption and subsequent re-emission by an atom may be treated as independent events (*cf.* Heitler [He54, Chap. 5, Sect. 20]), and the process cannot be considered true scattering in the sense stated above.

Perhaps the best example of such "true" scattering is Compton scattering by a free charge, and it is this kind of scattering that we will generally have in mind in later applications of the equations to be developed in this chapter. When we consider Compton scattering we shall always regard the scattering centers as free electrons, because Compton scattering from ions and nuclei is always small compared with that from free electrons. In fact, in the low frequency limit where Compton scattering goes over into Thomson scattering (see next paragraph), the ratio of the scattering cross sections for nuclei and electrons is $\sigma_0^{(n)}/\sigma_0^{(e)} = Z^4 m_e^2/H^2 A^2 \approx 3\times 10^{-7} Z^4/A^2$, where m_e, H, Z, and A are, respectively, electron rest mass, mass of one atomic mass unit (*cf.* Chap. 15), nuclear charge (in units of the electronic charge e), and nuclear mass in atomic mass units; see (16.36) below. Moreover, in later applications we shall generally consider only this low frequency limit.

In Compton scattering from an electron initially at rest we know that, in general, the frequency of the scattered photon differs from that of the incident photon, the difference depending on the scattering angle. In fact, the relation between the frequency ν_0 of the incident photon and the frequency ν of the scattered photon follows from energy and momentum conservation for the photon-electron system, and is (if the electron is initially at rest)

$$\nu = \nu_0 \left[1 - \frac{\alpha(1-\cos\theta)}{1+\alpha(1-\cos\theta)} \right], \tag{2.64}$$

where

$$\alpha \equiv h\nu_0/m_e c^2 \tag{2.65}$$

and θ is the scattering angle (measured from the direction of the incident photon) (*cf.* Heitler [He54, Chap. 1]). In the non-relativistic limit ($\alpha \ll 1$, or, roughly, $T \ll 5\times 10^9$ °K if we assume the average photon energy to be of the order of kT) we see that Compton scattering reduces to Thomson scattering, in which the recoil of the electron is negligible and the frequency of the scattered photon is practically the same as that of the incident photon. Such scattering without change in frequency is called *coherent* scattering.

Another example of "true" scattering is Raman scattering, which is non-coherent (see Heitler [He54, Chap. 5, Sect. 19]).

The scattering process must be treated carefully in order to arrive at the correct equation of transfer when scattering is taking place in addition to true emission and absorption. An error in the treatment of scattering was present in the astrophysical literature until 1947, when H. Mayer first demonstrated the error (see Sampson [Sa59a and Sa65a, Sect. 2.2]). We shall here formulate the problem fairly generally, so that even absorption of a photon by an atom and subsequent re-emission (not a "true" scattering process) can be treated in our formalism; but in most subsequent applications in later chapters we shall consider only perfectly coherent scattering. We shall neglect Doppler shifts caused by the motion of the scattering center. (See Sampson [Sa59a and Sa65a, Sect. 2.2] and Chin [Ch65b] for treatments of (non-coherent) Compton scattering in which these Doppler shifts arising from electron motions is considered. Some results of Sampson's and Chin's calculations are summarized in Sect. 16.6a. See also Oxenius [Ox64] for a general discussion of scattering by atoms.)

We first define the differential cross section for the (non-coherent) scattering of photons of initial frequency ν by a single scattering center. Consider a pencil of $I_\nu(\mathbf{n})d\nu d\omega_\mathbf{n}/h\nu$ photons having frequencies in the range $(\nu, \nu+d\nu)$, per unit area and time, having directions confined to the elementary solid angle $d\omega_\mathbf{n}$ about the direction $\mathbf{n}$, incident on a scattering center and scattered into the solid angle $d\omega_{\mathbf{n}'}$ about $\mathbf{n}'$ and into the frequency range $(\nu', \nu'+d\nu')$ (see Fig. 2.11). The differential cross section is then defined by the following statement: The number (out of the incident $I_\nu(\mathbf{n})d\nu\, d\omega_\mathbf{n}/h\nu$ photons per unit area and time) of photons scattered per unit time into $d\omega_{\mathbf{n}'}$ about $\mathbf{n}'$ and into $d\nu'$ about ν' is

$$\frac{I_\nu(\mathbf{n})d\nu d\omega_\mathbf{n}}{h\nu}\left[\frac{d^2\sigma(\nu,\nu';\mathbf{n},\mathbf{n}')}{d\nu' d\omega_{\mathbf{n}'}}\,d\nu' d\omega_{\mathbf{n}'}\right] = \frac{I_\nu(\mathbf{n})d\nu d\omega_\mathbf{n}}{h\nu}\left[\mathscr{S}(\nu,\nu';\mathbf{n},\mathbf{n}')d\nu' d\omega_{\mathbf{n}'}\right]. \tag{2.66}$$

This equation defines $\mathscr{S}(\nu,\nu';\mathbf{n},\mathbf{n}')$, which is the cross section for the scattering of photons of initial frequency ν from the direction $\mathbf{n}$ into *unit frequency interval* about the final frequency ν' and into *unit solid angle* about the final direction $\mathbf{n}'$. The quantity in square brackets in (2.66) is the differential scattering cross section for the process considered, and it is seen to have the dimensions of area and to be independent of the magnitudes of $d\nu$, $d\omega_\mathbf{n}$, and $I_\nu(\mathbf{n})$ in the limit as $d\nu\to 0$ and $d\omega_\mathbf{n}\to 0$. Any effects of dispersion (see Sect. 2.10) are assumed included in $\mathscr{S}(\nu,\nu';\mathbf{n},\mathbf{n}')$. The *total* cross section for the scattering of photons of inital frequency ν by a scattering center from direction $\mathbf{n}$ into *all* final frequencies and into *all* final directions is obtained by

integrating (2.66) over all relevant final frequencies and over all final directions of scattering:

$$\sigma_\nu(\mathbf{n}) \equiv \int_{(\nu')} \int_{4\pi} \mathscr{S}(\nu,\nu';\mathbf{n},\mathbf{n}')\,d\nu'\,d\omega_{\mathbf{n}'}, \tag{2.67}$$

where in general $\sigma_\nu(\mathbf{n})$ depends on the direction $\mathbf{n}$.

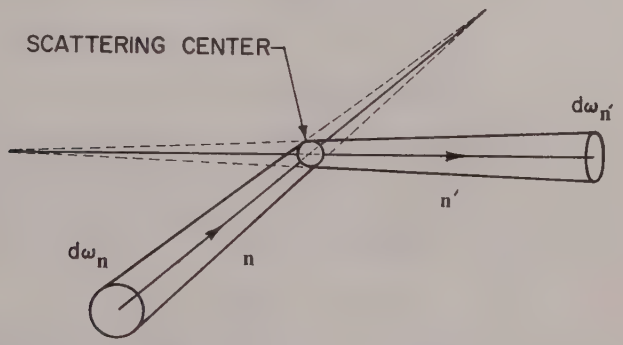


Fig. 2.11 Differential scattering from $d\omega_{\mathbf{n}}$ about $\mathbf{n}$ into $d\omega_{\mathbf{n}'}$, about $\mathbf{n}'$.

If Doppler shifts due to motions of the scattering center were taken into account, then $\mathscr{S}(\nu,\nu';\mathbf{n},\mathbf{n}')$ would also depend in general on the initial and final velocity of the scattering center, since ν and ν' are taken here to be frequencies in a fixed (laboratory) coordinate system, and the relations between ν and ν' and the frequencies as "seen" by the scattering center involve these velocities, through the Doppler effect. Equations (2.66) and (2.67) would then have to be considered as averaged over the initial and final velocity distributions of the scattering centers (see Sampson [Sa59a and Sa65a, Sect. 2.2] and Chin [Ch65b]).

If the scattering center is considered to have a fixed orientation (spin direction, say) relative to laboratory coordinates, then $\mathscr{S}(\nu,\nu';\mathbf{n},\mathbf{n}') \neq \mathscr{S}(\nu,\nu';\mathbf{n}',\mathbf{n})$ and $\sigma_\nu(\mathbf{n})$ will depend on $\mathbf{n}$ in general (*cf.* Heitler [He54, Chap. 5, Sect. 22]); this general case might be realized for electrons whose spin directions are aligned by a magnetic field, for example. If the scattering center is considered *not* to have a fixed orientation, however (as would be the case with an assemblage of randomly oriented scattering centers), then $\mathscr{S}(\nu,\nu';\mathbf{n},\mathbf{n}')$ depends only on the angle between $\mathbf{n}$ and $\mathbf{n}'$. Then we have

$$\mathscr{S}(\nu,\nu';\mathbf{n},\mathbf{n}') = \mathscr{S}(\nu,\nu';\mathbf{n}',\mathbf{n}) \tag{2.68}$$

and

$$\sigma_\nu(\mathbf{n}) = \sigma_\nu, \text{ independent of } \mathbf{n}. \tag{2.69}$$

Two special cases of $\mathscr{S}(\nu,\nu';\mathbf{n},\mathbf{n}')$ are as follows: (1) The relation between ν and ν' is independent of the scattering angle $\theta = \cos^{-1}(\mathbf{n}\cdot\mathbf{n}')$. In this case one could write

$$\mathscr{S}(\nu,\nu';\mathbf{n},\mathbf{n}') = \mathscr{S}_1(\nu,\nu')\cdot\mathscr{S}_2(\mathbf{n},\mathbf{n}')\,. \tag{2.70}$$

This case would correspond, for example, to absorption followed by re-emission in atoms (not "true" scattering). (2) The relation between ν and ν' is determined entirely by the scattering angle $\cos^{-1}(\mathbf{n}\cdot\mathbf{n}')$. In this case one could write

$$\mathscr{S}(\nu,\nu';\mathbf{n},\mathbf{n}') = S(\nu,\nu';\mathbf{n},\mathbf{n}')\cdot\delta(\nu'-\nu_f)\,, \tag{2.71}$$

where $\delta(\nu'-\nu_f)$ is the Dirac delta function (units of inverse frequency here); ν_f is a function of ν, $\mathbf{n}$, and $\mathbf{n}'$; and $S(\nu,\nu';\mathbf{n},\mathbf{n}')$ is the cross section for scattering into unit solid angle about $\mathbf{n}'$. Compton scattering is an example of this case. In Compton scattering we have

$$\nu_f = H(\nu,\,\mathbf{n}\cdot\mathbf{n}')\leq\nu\,, \tag{2.72}$$

where the function H is defined by (2.64). Integrating (2.71) over frequency, we have for this case (2)

$$\frac{d\sigma_\nu(\mathbf{n},\mathbf{n}')}{d\omega_{\mathbf{n}'}} = \int_{(\nu')}\mathscr{S}(\nu,\nu';\mathbf{n},\mathbf{n}')d\nu' = S(\nu,\nu_f;\mathbf{n},\mathbf{n}')\,. \tag{2.73}$$

The differential scattering cross section for Compton scattering (for unpolarized radiation and with effects of dispersion neglected), averaged over all initial and final electron spin directions, is (see Heitler [He54, Chap. 5, Sect. 22])

$$S(\nu,\nu_f;\,\mathbf{n},\mathbf{n}') = \frac{1}{2}a_0^2\left(\frac{\nu_f}{\nu}\right)^2\left(\frac{\nu}{\nu_f}+\frac{\nu_f}{\nu}-\sin^2\theta\right), \tag{2.74}$$

where

$$a_0 = \frac{e^2}{m_e c^2} \tag{2.75}$$

is the "classical radius of the electron" and where $\cos\theta = \mathbf{n}\cdot\mathbf{n}'$. Using (2.72) in (2.74), one obtains

$$\frac{d\sigma_\nu(\mathbf{n},\mathbf{n}')}{d\omega_{\mathbf{n}'}} = a_0^2\,\frac{1+\cos^2\theta}{2}\,\frac{1}{[1+\alpha(1-\cos\theta)]^2}$$
$$\left\{1+\frac{\alpha^2(1-\cos\theta)^2}{(1+\cos^2\theta)[1+\alpha(1-\cos\theta)]}\right\}, \tag{2.76}$$

where $\alpha\equiv h\nu/m_e c^2$.

In the special case of *coherent* scattering we have $\nu_f = \nu$, and

$$\frac{d\sigma_\nu(\mathbf{n},\mathbf{n}')}{d\omega_{\mathbf{n}'}} = S(\nu,\nu;\mathbf{n},\mathbf{n}'). \tag{2.77}$$

For example, for Thomson scattering ($\alpha \ll 1$) (2.76) reduces to

$$\frac{d\sigma(\mathbf{n},\mathbf{n}')}{d\omega_{\mathbf{n}'}} = \frac{1}{2} a_0^2(1+\cos^2\theta). \tag{2.78}$$

We now consider the scattering of photons *from all directions* into the final solid angle $d\omega_{\mathbf{n}}$ about $\mathbf{n}$ and from *all initial frequencies* into the final frequency ν within $d\nu$. Just as with "true" emission, this scattering will be made up of two contributions, *direct* and *induced* scattering. We consider direct scattering first.

From the definition of the cross section it is clear that, of $I_{\nu'}(\mathbf{n}')d\nu' d\omega_{\mathbf{n}'}/h\nu'$ photons of frequency ν' within $d\nu'$ and direction $\mathbf{n}'$ within $d\omega_{\mathbf{n}'}$, per unit time and area, incident on a scattering center, the number scattered per unit time into frequency ν within $d\nu$ and into direction $\mathbf{n}$ within $d\omega_{\mathbf{n}}$ is

$$[I_{\nu'}(\mathbf{n}')d\nu' d\omega_{\mathbf{n}'}/h\nu'] \cdot \mathcal{S}(\nu',\nu;\mathbf{n}',\mathbf{n})d\nu d\omega_{\mathbf{n}}\,.$$

Integrating this expression over all initial frequencies ν' and directions $\mathbf{n}'$, we obtain

$$d\nu d\omega_{\mathbf{n}} \int_{(\nu')} \int_{4\pi} \frac{I_{\nu'}(\mathbf{n}')}{h\nu'} \mathcal{S}(\nu',\nu;\mathbf{n}',\mathbf{n}) d\nu' d\omega_{\mathbf{n}'} \tag{2.79}$$

as the number of photons of all inital frequencies ν' and of all initial directions $\mathbf{n}'$ scattered in unit time into final frequency ν within $d\nu$ and into final direction $\mathbf{n}$ within $d\omega_{\mathbf{n}}$.

In the special case in which the relation between ν' and ν is determined entirely by the scattering angle $\cos^{-1}(\mathbf{n}\cdot\mathbf{n}')$ (case (2) above as, for example, in Compton scattering), (2.79) takes a special form since the integrations over ν' and $\omega_{\mathbf{n}'}$ cannot now be carried out independently. Let ν_i denote the frequency of the incident pencil of radiation (flowing in direction $\mathbf{n}'$). Since the photons, after the scattering, are to be of frequency ν and to be moving in the direction $\mathbf{n}$ within $d\omega_{\mathbf{n}}$, we must formally replace the equation immediately preceding (2.79) by

$$[I_{\nu_i}(\mathbf{n}')d\nu_i d\omega_{\mathbf{n}'}/h\nu_i] \cdot S(\nu_i,\nu'';\mathbf{n}',\mathbf{n})\,\delta(\nu''-\nu)d\nu'' d\omega_{\mathbf{n}}\,,$$

in which we are temporarily using the "dummy" variable ν'' to denote the final frequency, which in this formalism must be regarded as capable of having any value. We write the equation this way because the scattering cross section for inital frequency ν_i and initial and final directions $\mathbf{n}'$ and $\mathbf{n}$

is zero unless ν'' is equal to the *actual* final frequency ν, which is an explicit function of ν_i and $(\mathbf{n} \cdot \mathbf{n}')$. (In the case of Compton scattering we have $\nu = H(\nu_i, \mathbf{n} \cdot \mathbf{n}') \leq \nu_i$, where the function H is defined by (2.64).) Integrating the above expression over all frequencies ν'' for the given ν_i, $\mathbf{n}'$, and $\mathbf{n}$, we obtain for the number of photons having initial frequencies in the range $(\nu_i, \nu_i + d\nu_i)$ and initial directions within $d\omega_{\mathbf{n}'}$ about $\mathbf{n}'$, scattered per unit time into $d\omega_{\mathbf{n}}$ about $\mathbf{n}$

$$[I_{\nu_i}(\mathbf{n}')d\nu_i d\omega_{\mathbf{n}'}/h\nu_i] \cdot S(\nu_i, \nu; \mathbf{n}', \mathbf{n})d\omega_{\mathbf{n}}.$$

Because of the change in frequency accompanying the scattering process, photons in the frequency range $(\nu_i, \nu_i + d\nu_i)$ will, after the scattering, lie in the range $(\nu, \nu + d\nu)$, where ν is an explicit function of ν_i and $\mathbf{n} \cdot \mathbf{n}'$. Hence, the number of photons scattered per unit time from all initial directions into $d\omega_{\mathbf{n}}$ about the final direction $\mathbf{n}$ and into the final frequency range $(\nu, \nu + d\nu)$ is

$$d\nu d\omega_{\mathbf{n}} \int_{4\pi} \frac{I_{\nu_i}(\mathbf{n}')}{h\nu_i} \frac{d\nu_i}{d\nu} S(\nu_i, \nu; \mathbf{n}', \mathbf{n}) d\omega_{\mathbf{n}'}. \tag{2.79'}$$

Equation (2.79') is the explicit form of (2.79) for the special case of Compton scattering, for example.

We consider now the induced scattering. According to the general quantum mechanical principle stated in Sect. 2.8a2, the probability of induced emission in the direction $\mathbf{n}$ is equal to n times the probability of spontaneous emission, where n is the number of photons in a unit cell in phase space corresponding to the frequency, direction, and state of polarization of the radiation described by $I_\nu(\mathbf{n})$. It will be shown in Sect. 3.6 that the general relation $n = (c^2/2h\nu^3)\,(I_\nu/\mu_\nu^2)$ (μ_ν = (real) refractive index) is valid for unpolarized radiation. Accordingly, in the present case of scattering the number of photons n in the unit cell in phase space corresponding to the frequency and direction of the (unpolarized) radiation field described by $I_\nu(\mathbf{n})$ is just $(c^2/2h\nu^3)I_\nu(\mathbf{n})/\mu_\nu^2$. Moreover, just as in the case of induced emission, scattering into the direction $\mathbf{n}$ of radiation with final frequency ν will be enhanced if radiation flowing in the direction $\mathbf{n}$ and having frequency ν is already present; and the factor giving the enhancement is again equal to $(1+n)$. In the present case the direct scattering, given by (2.79), is analogous to the spontaneous emission in "true" emission. We can then write the expression for the number of photons with final frequency ν within $d\nu$ scattered by induced scattering per unit time into $d\omega_{\mathbf{n}}$ about $\mathbf{n}$ as

$$\frac{c^2}{2h\nu^3} \frac{I_\nu(\mathbf{n})}{\mu_\nu^2} d\omega_{\mathbf{n}} d\nu \int_{(\nu')} \int_{4\pi} \frac{I_{\nu'}(\mathbf{n}')}{h\nu'} \mathscr{S}(\nu', \nu; \mathbf{n}', \mathbf{n}) d\nu' d\omega_{\mathbf{n}'}. \tag{2.80}$$

Multiplying (2.79) and (2.80) by the factor $(h\nu/\rho)\,(N_e/d\nu d\omega_{\mathbf{n}})$, where ρ and N_e are mass density and electron density, and adding, we obtain the mass emission coefficient for direct and induced scattering of (unpolarized) photons of frequency ν:

$$j_\nu^{(s)} = \frac{N_e}{\rho}\left[1+\frac{c^2}{2h\nu^3}\frac{I_\nu(\mathbf{n})}{\mu_\nu{}^2}\right]\int_{(\nu')}\int_{4\pi}\left(\frac{\nu}{\nu'}\right) I_{\nu'}(\mathbf{n}')\mathscr{S}(\nu',\nu;\mathbf{n}',\mathbf{n})d\nu' d\omega_{\mathbf{n}'}. \quad (2.81)$$

In the special case of Compton scattering it is clear from the derivation of (2.79′) that the explicit form of (2.81) is

$$j_{\nu,\,\text{Comp.}}^{(s)} = \frac{N_e}{\rho}\left[1+\frac{c^2}{2h\nu^3}\frac{I_\nu(\mathbf{n})}{\mu_\nu{}^2}\right]\int_{4\pi}\frac{\nu}{\nu_i}\frac{d\nu_i}{d\nu} I_{\nu_i}(\mathbf{n}')S(\nu_i,\nu;\mathbf{n}',\mathbf{n})d\omega_{\mathbf{n}'}, \quad (2.81')$$

where $\nu = H(\nu_i, \mathbf{n}\cdot\mathbf{n}') \leq \nu_i$ (see (2.64)).

The *total* mass emission coefficient, for (unpolarized) radiation of frequency ν, including both "true" emission and scattering (perhaps as modified by dispersive effects, *cf.* Sect. 2.9), is then

$$j_\nu = j_\nu^{(a)} + j_\nu^{(s)}, \quad (2.82)$$

where the two terms on the right side of (2.82) are given by (2.61) and (2.81).

2.8b Absorption

Photons can be removed from a beam of radiation by at least three processes: (1) "true" absorption; (2) scattering, both direct and induced; and (3) "lossless" attenuation (such as total reflection) resulting from collective effects in a dispersive medium. Process (3), which can be effective simultaneously with processes (1) and (2) in a dispersive medium, is discussed in Sect. 2.10b, where a general relation between process (3) and processes (1) and (2) is derived. In this subsection we discuss processes (1) and (2), but we do not exclude the presence of process (3).

2.8b1 "True" absorption

This process consists of an atom (or a free electron in the field on an atom or ion) absorbing a photon and thereby being excited to a state of higher energy than before the absorption.

Consider again two atomic energy levels whose "centers of gravity" are E_2 and E_1 $(E_2 > E_1)$, assumed discrete. Such an atom in state 1 may absorb radiant energy of frequency $\nu \approx \nu_0$ by being excited to the state 2, where $h\nu_0 = E_2 - E_1$, if radiation of frequency ν and intensity I_ν is impinging upon

the atom. The amount of energy absorbed will be proportional to the intensity I_ν of the impinging radiation, at least as long as the intensity is not enormously large.

Suppose we have radiation of frequency ν ($\approx \nu_0$) of intensity I_ν, impinging upon an atom in the state 1. Then the Einstein coefficient B_{12} for *true absorption* is defined by the following statement: The probability of the atom absorbing in the time dt radiation having frequencies in the range $(\nu,\nu+\mathrm{d}\nu)$ and having directions within the solid angle dω from the impinging radiation of intensity I_ν, in the absence of dispersive effects (*cf.* Sect. 2.10), is

$$B_{12} I_\nu \phi_a(\nu) \mathrm{d}\nu \mathrm{d}\omega \mathrm{d}t,$$

where B_{12} is assumed to be independent of ν "within the line" (*i.e.*, for $\nu \approx \nu_0$) and where $\phi_a(\nu)$ is the "line shape function" for absorption in the absence of dispersive effects. It is a sharply peaked function differing appreciably from zero only in a narrow range of frequencies $\Delta\nu$ about ν_0 and is normalized to unity:

$$\int_{\Delta\nu} \phi_a(\nu) \mathrm{d}\nu = 1. \tag{2.83}$$

Just as in the case of spontaneous and induced emission, we assume that any dispersive effects can be included in the line shape function. When such effects are present, we simply drop the subscript a. The normalization (2.83) applies only to $\phi_a(\nu)$ and not to $\phi(\nu)$ unless they are equal. If we multiply the above expression by $h\nu_0 N_1/\rho$, where N_1 is the number of atoms per unit volume in state 1 and ρ is the mass density of the absorbing matter, we will obtain for the energy of frequency ν within dν absorbed per unit mass from directions included in solid angle dω in time dt (referring to (2.54)):

$$h\nu_0 (N_1/\rho) B_{12} I_\nu \phi(\nu) \mathrm{d}\nu \mathrm{d}\omega \mathrm{d}t = \kappa_\nu^{(a)} I_\nu \mathrm{d}\nu \mathrm{d}\omega \mathrm{d}t,$$

where the superscript a stands for "true" absorption as distinguished from scattering and not as used in Sect. 2.10b. Thus the mass absorption coefficient $\kappa_\nu^{(a)}$ for the frequency ν ($\approx \nu_0$) is

$$\kappa_\nu^{(a)} = h\nu_0 \frac{N_1 B_{12} \phi(\nu)}{\rho}, \tag{2.84}$$

which may include dispersive effects.

2.8b2 Scattering

We again consider scattering of unpolarized radiation and use the same notation and terminology as in our treatment of emission resulting from

scattering. Again both direct and induced scattering act to remove photons from the pencil of radiation. We consider direct scattering first.

We consider a pencil of radiation of frequency ν within $d\nu$ and intensity $I_\nu(\mathbf{n})$ having initial directions $\mathbf{n}$ within $d\omega_\mathbf{n}$, impinging on a single scatterer. Then, from our definition of cross section it follows that the number of photons having initial frequencies in the range $(\nu,\nu+d\nu)$ and initial directions within $d\omega_\mathbf{n}$ about $\mathbf{n}$, directly scattered into *all* directions in time dt and shifted into *all* other frequencies, is

$$\frac{I_\nu(\mathbf{n})d\nu d\omega_\mathbf{n} dt}{h\nu}\int_{(\nu')}\int_{4\pi}\mathscr{S}(\nu,\nu';\,\mathbf{n},\mathbf{n}')d\nu' d\omega_{\mathbf{n}'} = (h\nu)^{-1}I_\nu(\mathbf{n})d\nu d\omega_\mathbf{n}\sigma_\nu(\mathbf{n})dt \qquad (2.85)$$

from (2.67), where $\sigma_\nu(\mathbf{n})$, the total cross section, is in general a function of $\mathbf{n}$ (*cf.* Sect. 2.8a3). Any effects of dispersion (*cf.* Sect. 2.10) are assumed included in $\mathscr{S}(\nu,\nu';\mathbf{n},\mathbf{n}')$.

We consider now the effects of induced scattering. As we have seen, scattering of radiation of initial frequency ν from direction $\mathbf{n}$ into direction $\mathbf{n}'$ and with final frequency ν' will be enhanced by the radiation of frequency ν' already present flowing in direction $\mathbf{n}'$. Following the same line of reasoning which led to (2.80), we obtain for the number of (unpolarized) photons having initial frequencies in the range $(\nu,\ \nu+d\nu)$ and initial directions within $d\omega_\mathbf{n}$ about $\mathbf{n}$, scattered by induced scattering into *all* directions and shifted into *all* other frequencies in time dt,

$$\frac{c^2}{2h\nu^3}\frac{I_\nu(\mathbf{n})d\nu d\omega_\mathbf{n} dt}{h\nu}\int_{(\nu')}\int_{4\pi}\left(\frac{\nu}{\nu'}\right)^3\frac{I_{\nu'}(\mathbf{n}')}{\mu_{\nu'}{}^2}\mathscr{S}(\nu,\nu';\mathbf{n},\mathbf{n}')d\nu' d\omega_{\mathbf{n}'}. \qquad (2.86)$$

This induced scattering is seen to amount to a removal of photons from the incident pencil and thus, in accordance with our definition (*cf.* Sect. 2.7), to act as an additional source of "absorption." Note that, in general, the integral in (2.86) will be a function of $\mathbf{n}$ (*cf.* Sect. 2.8a3). Multiplying (2.85) and (2.86) by $(N_e/\rho)(h\nu/I_\nu(\mathbf{n})d\nu d\omega_\mathbf{n} dt)$ and adding, we obtain the mass absorption coefficient for scattering of (unpolarized) radiation:

$$k_\nu^{(s)} = \frac{N_e}{\rho}\left[\sigma_\nu(\mathbf{n})+\frac{c^2}{2h\nu^3}\int_{(\nu')}\int_{4\pi}\left(\frac{\nu}{\nu'}\right)^3\frac{I_{\nu'}(\mathbf{n}')}{\mu_{\nu'}{}^2}\mathscr{S}(\nu,\nu';\mathbf{n},\mathbf{n}')d\nu' d\omega_{\mathbf{n}'}\right], \qquad (2.87)$$

where $k_\nu^{(s)}$ is seen to be a function of $\mathbf{n}$ in general since both terms of the right side of (2.87) in general depend on $\mathbf{n}$. Thus, for example, an assemblage of electrons with their spins aligned would be expected to behave, as far as the scattering absorption coefficient is concerned, like an anisotropic medium.

In the special case of Compton scattering, for example, we have $\mathscr{S}(\nu,\nu';\mathbf{n},\mathbf{n}') = S(\nu,\nu';\mathbf{n},\mathbf{n}')\cdot\delta(\nu'-\nu_f)$, where $\nu_f = H(\nu,\mathbf{n}\cdot\mathbf{n}')\leq\nu$ (see (2.64)), and (2.87) becomes

$$\overline{k^{(s)}_{\nu,\text{Comp.}}} = \frac{N_e}{\rho}\left[\sigma_\nu(\mathbf{n}) + \frac{c^2}{2h\nu^3}\int_{4\pi}\left(\frac{\nu}{\nu_f}\right)^3 \frac{I_{\nu_f}(\mathbf{n}')}{\mu_{\nu_f}^{\,2}}\, S(\nu,\nu_f;\mathbf{n},\mathbf{n}')\mathrm{d}\omega_{\mathbf{n}'}\right]. \qquad (2.87')$$

The total mass absorption coefficient for both "true" absorption and scattering (perhaps as modified by dispersive effects, *cf.* Sects. 2.9 and 2.10b) for (unpolarized) radiation of frequency ν is then

$$\kappa_\nu^{(\text{total})} = \kappa_\nu^{(a)} + k_\nu^{(s)}, \qquad (2.88)$$

where $\kappa_\nu^{(a)}$ and $k_\nu^{(s)}$ are given by (2.84) and (2.87).

2.9 Equation of Transfer

We saw in Sect. 2.7 that the total change in energy of a pencil of radiation traversing an element of mass $\mathrm{d}m = \rho\mathrm{d}s\mathrm{d}\sigma\cos\theta$ is given by

$$\mathrm{d}E_\nu = \mathrm{d}I_\nu\mathrm{d}\nu\mathrm{d}\sigma\cos\theta\,\mathrm{d}\omega\mathrm{d}t,$$

where $\mathrm{d}E_\nu$ and $\mathrm{d}I_\nu$ denote differences taken along a (possibly curved) ray path. This change in energy consists, in general, of the contributions from emission, absorption, and refraction within $\mathrm{d}m$. The first two of these contributions are given respectively by (2.50) and (2.54). Hence, adding up the gains and losses of energy, including a term representing effects of refraction, we must have $\mathrm{d}I_\nu\mathrm{d}\nu\mathrm{d}\sigma\cos\theta\,\mathrm{d}\omega\mathrm{d}t = j_\nu\mathrm{d}\nu\rho\mathrm{d}\sigma\cos\theta\,\mathrm{d}s\mathrm{d}\omega\mathrm{d}t - \kappa_\nu\rho\mathrm{d}sI_\nu\mathrm{d}\nu\mathrm{d}\sigma\cos\theta\,\mathrm{d}\omega\mathrm{d}t + (\mathrm{d}I_\nu)_{\text{refr}}\mathrm{d}\nu\mathrm{d}\sigma\cos\theta\,\mathrm{d}\omega\mathrm{d}t$, or

$$\frac{\mathrm{d}I_\nu}{\rho\mathrm{d}s} = j_\nu - \kappa_\nu I_\nu + \left(\frac{\mathrm{d}I_\nu}{\rho\mathrm{d}s}\right)_{\text{refr}}, \qquad (2.89)$$

where $(\mathrm{d}I_\nu/\rho\mathrm{d}s)_{\text{refr}}$ is the contribution to $(\mathrm{d}I_\nu/\rho\mathrm{d}s)$ resulting from spatial variation of the (real) refractive index μ_ν.

This last contribution is obtained as follows: We recall that it was shown in Sect. 2.1c that the quantity (I_ν/μ_ν^2) was constant along every ray path in a medium of variable refractive index, provided that there are only negligible energy losses or gains due to absorption, emission, or refraction. In the (assumed) absence of such losses or gains of energy, then, we have $\mathrm{d}(I_\nu/\mu_\nu^2)/\rho\mathrm{d}s = 0$, where the differentiation must be carried out along the ray path. This term gives a contribution $(\mathrm{d}I_\nu/\rho\mathrm{d}s)_{\text{refr}} = (2I_\nu/\mu_\nu)\mathrm{d}\mu_\nu/\rho\mathrm{d}s$ to the

rate of change of I_ν due to spatial variation of the refractive index alone. Using this expression in (2.89) and rearranging, we obtain

$$\frac{\mathrm{d}}{\rho \mathrm{d}s}\left(\frac{I_\nu}{\mu_\nu^2}\right)=\frac{j_\nu-\kappa_\nu I_\nu}{\mu_\nu^2}. \tag{2.90}$$

Equation (2.90) (see also (2.90′) below) is known as the *equation of transfer* and is fundamental to the study of the transfer of radiation through matter. Equation (2.90) is derived by Harris [Ha65] directly from the Boltzmann transport equation for photons. In case μ_ν is unity or constant in space, (2.90) reduces to the more usual form of the equation of transfer:

$$\frac{\mathrm{d}I_\nu}{\rho \mathrm{d}s}=j_\nu-\kappa_\nu I_\nu. \tag{2.91}$$

Two important features of (2.90) should be emphasized. First, the differentiation implied by the operator d/ds must be taken *along the ray path*. The necessity of this requirement is clear from the derivation of (2.90). It becomes still more clear if (2.90) is regarded as a "conservation equation" for photons, with the two terms on the right side representing source and sink terms for photons travelling along the ray path. If the ray path is curved, then $\mathrm{d}(I_\nu/\mu_\nu^2)/\mathrm{d}s$ must include contributions arising not only from the spatial variation of (I_ν/μ_ν^2) (in a fixed direction) along the ray path, but also from the variation of (I_ν/μ_ν^2) (at a fixed point on the ray path) with the direction of the (curved) ray path. Such an explicit interpretation of the operator d/ds has been pointed out by Harris [Ha65].

Let us select an arbitrary, but fixed, point $\mathbf{r}$ in a general radiation field in a dispersive medium, and an arbitrary, but fixed, direction, specified by the unit vector $\mathbf{n}$. By arbitrarily selecting both $\mathbf{r}$ and $\mathbf{n}$, we have thereby picked out *that* particular pencil of radiation which happens to be passing through the point $\mathbf{r}$ in the direction $\mathbf{n}$. It is obvious that $\mathbf{n}$ is also the unit tangent vector at $\mathbf{r}$ of the (curved) ray path along which the photons in the pencil thus selected are travelling. As was pointed out above, in order for the equation of transfer to describe the radiation at point $\mathbf{r}$ flowing in the direction $\mathbf{n}$, the differential operator d/ds must be taken *along* the (curved) ray trajectory of that particular pencil of radiation specified by $\mathbf{r}$ and $\mathbf{n}$. In moving a distance $\mathrm{d}\mathbf{s} = \mathbf{n}\,\mathrm{d}s$ along the ray path, the total change in (I_ν/μ_ν^2) will consist of one part representing the change in (I_ν/μ_ν^2) due to the *change in position* $\mathbf{r}$ (at fixed $\mathbf{n}$), and another part representing the change in (I_ν/μ_ν^2) due to the *change in direction* $\mathbf{n}$ (at fixed $\mathbf{r}$) of the ray path. Hence d/ds, written out explicitly, is

$$\frac{\mathrm{d}}{\mathrm{d}s}=\mathbf{n}\cdot\nabla+\frac{\mathrm{d}\mathbf{n}}{\mathrm{d}s}\cdot\nabla_n, \tag{2.92}$$

where ∇ is the ordinary spatial gradient vector operator and ∇_n is a "directional" gradient vector operator lying in the plane perpendicular to $\mathbf{n}$ and representing differentiation with respect to *direction* at a given point.

If we let (θ,ϕ) denote the polar and azimuthal angles of $\mathbf{n}$ in a fixed spherical coordinate system, then in this coordinate system we have (see Sect. 2.11)

$$\nabla_n = \mathbf{e}_\theta \frac{\partial}{\partial\theta} + \mathbf{e}_\phi \frac{1}{\sin\theta}\frac{\partial}{\partial\phi}, \tag{2.93}$$

where $\mathbf{e}_\theta$ and $\mathbf{e}_\phi$ are unit vectors forming a right-handed orthogonal coordinate system with $\mathbf{n}$; *i.e.*, $\mathbf{e}_\theta$ is in the direction of increasing θ (at fixed ϕ) and $\mathbf{e}_\phi$ is in the direction of increasing ϕ (at fixed θ) (see Fig. 2.12).

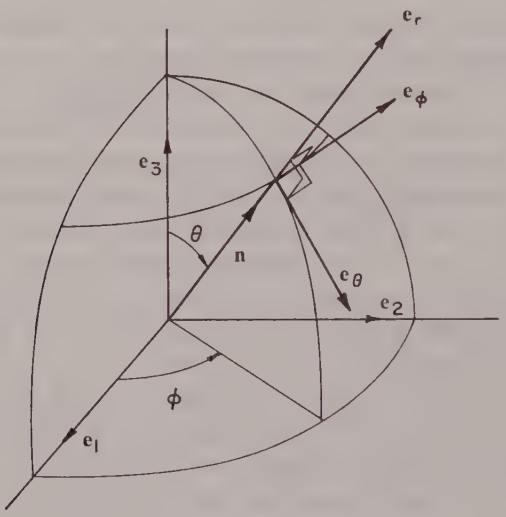


Fig. 2.12 Polar coordinate system for $\mathbf{n}$ and ∇_n.

It is clear, also, that $d\mathbf{n}/ds$ is a vector lying in the plane perpendicular to $\mathbf{n}$. In fact, if $\mathbf{m}$ denotes a unit vector in this plane in the direction of $d\mathbf{n}/ds$, then

$$\frac{d\mathbf{n}}{ds} = \mathbf{e}_\theta \frac{d\theta}{ds} + \mathbf{e}_\phi \sin\theta \frac{d\phi}{ds} = \mathbf{m}\frac{1}{\lambda_\mu}, \tag{2.94}$$

where

$$\lambda_\mu = \left[\left(\frac{d\theta}{ds}\right)^2 + \sin^2\theta\left(\frac{d\phi}{ds}\right)^2\right]^{-1/2} \tag{2.94'}$$

is the magnitude of the local radius of curvature of the ray path at the point under consideration.

The rate of change $\mathrm{d}\mathbf{n}/\mathrm{d}s$ of $\mathbf{n}$ along the ray path is related to the gradient of the (real) refractive index μ_ν. It is shown by Born and Wolf [Bo59, p. 121] that this relation is

$$\frac{\mathrm{d}\mathbf{n}}{\mathrm{d}s} = \frac{1}{\mu_\nu}[\nabla\mu_\nu - \mathbf{n}(\mathbf{n}\cdot\nabla\mu_\nu)], \tag{2.95}$$

which shows that $\mathrm{d}\mathbf{n}/\mathrm{d}s = 0$ (*i.e.*, $\lambda_\mu = \infty$) whenever $\mathbf{n}$ is in the direction of $\pm\nabla\mu_\nu$, or whenever $\nabla\mu_\nu = 0$, as expected. Also, $\mathrm{d}\mathbf{n}/\mathrm{d}s = \mu_\nu^{-1}\nabla\mu_\nu$ if $\mathbf{n}\cdot\nabla\mu_\nu = 0$ (*i.e.*, if $\mathbf{n}$ is perpendicular to $\nabla\mu_\nu$). Comparison of (2.94) and (2.95) shows, moreover, that

$$\frac{1}{\lambda_\mu} \sim \frac{|\nabla\mu_\nu|}{\mu_\nu},$$

or that the local radius of curvature of the ray paths is of the order of magnitude of the characteristic length for significant changes in μ_ν.

Hence (2.90), written out explicitly, is

$$\frac{1}{\rho}\mathbf{n}\cdot\nabla\left(\frac{I_\nu}{\mu_\nu^2}\right) + \frac{1}{\rho}\frac{\mathrm{d}\mathbf{n}}{\mathrm{d}s}\cdot\nabla_n\left(\frac{I_\nu}{\mu_\nu^2}\right) = \left(\frac{j_\nu}{\mu_\nu^2}\right) - \kappa_\nu\left(\frac{I_\nu}{\mu_\nu^2}\right). \tag{2.90'}$$

The second term on the left side vanishes if $\nabla\mu_\nu = 0$, *i.e.*, if μ_ν is constant in space, and (2.90′) then assumes the usual form of the equation of transfer:

$$\frac{1}{\rho}\mathbf{n}\cdot\nabla I_\nu = j_\nu - \kappa_\nu I_\nu. \tag{2.90''}$$

The second important feature of (2.90) and (2.90′) to be emphasized is that effects of a non-unity (and frequency dependent) refractive index appear not only *explicitly* as shown in (2.90) and (2.90′), but also *implicitly* in j_ν and κ_ν. Letting j_ν° and κ_ν° denote values of j_ν and κ_ν computed as if dispersive effects were not present (*i.e.*, with $\mu_\nu = 1$), the general argument presented in Sect. 2.10b would imply that

$$\kappa_\nu = \kappa_\nu^\circ/\mu_\nu. \tag{2.97}$$

It follows from Kirchhoff's law (3.7) (below) that (granting (2.97)) *in thermodynamic equilibrium* (see Chap. 3)

$$j_\nu = \mu_\nu j_\nu^\circ. \tag{2.98}$$

If the emission properties of the system do not depend on the thermodynamic equilibrium assumption, then of course (2.98) would be valid even if the system were not in thermodynamic equilibrium (see Sects. 3.1 and 3.6).

It must be cautioned that (2.97) and (2.98) may not be generally valid (even in thermodynamic equilibrium); indeed, the whole matter of effects of dispersion on emission and absorption is still in the research stage. Equations (2.97) and (2.98) are in agreement with results of direct (classical) calculations of emission and absorption in a plasma, with collective effects taken into account (see, for example, Dawson and Oberman [Da62]); Mercier [Me64]). These equations, however, do not agree with those of Oster [Os63b].*

2.9a Inclusion of the Time Derivative

We may include the time derivative into (2.90) or (2.90′) and (2.90″) by taking $ds = v_g dt$ as the element of distance traversed by the light beam in time dt, where the group velocity v_g is the velocity of *energy* propagation in a dispersive medium (v_g is also the velocity of quasiphotons, *cf.* Sect. 2.10a). Clearly, we would have $v_g = c$ for unity refractive index μ_ν. We consider that we are "following" the radiation, much as we "follow" the elements of a material fluid in the Lagrangian description in hydrodynamics. We may then use the hydrodynamical relation

$$\frac{d}{dt} = \frac{\partial}{\partial t} + v_g \frac{\partial}{\partial s}, \tag{2.99}$$

since v_g is the velocity of our "fluid" of radiation. We then obtain

$$\frac{\partial}{v_g \rho \partial t}\left(\frac{I_\nu}{\mu_\nu^2}\right) + \frac{\partial}{\rho \partial s}\left(\frac{I_\nu}{\mu_\nu^2}\right) = \frac{j_\nu - \kappa_\nu I_\nu}{\mu_\nu^2}, \tag{2.100}$$

where the meaning of the operator $\partial/\partial s$ was explained in the discussion following (2.91). Writing out the operator $\partial/\partial s$ explicitly, we may express the equation of transfer in the form

$$\frac{1}{\rho}\frac{1}{v_g}\frac{\partial}{\partial t}\left(\frac{I_\nu}{\mu_\nu^2}\right) + \frac{1}{\rho}\mathbf{n}\cdot\nabla\left(\frac{I_\nu}{\mu_\nu^2}\right) + \frac{1}{\rho}\frac{d\mathbf{n}}{ds}\cdot\nabla_n\left(\frac{I_\nu}{\mu_\nu^2}\right) = \left(\frac{j_\nu}{\mu_\nu^2}\right) - \kappa_\nu\left(\frac{I_\nu}{\mu_\nu^2}\right). \tag{2.90‴}$$

If we divide (2.100) by κ_ν, we see that we isolate a quantity $(\kappa_\nu \rho v_g)^{-1}$, which has the dimension of time. This time may be regarded as a "characteristic" time for "relaxation" of the radiation field, and is given, to order of magnitude, by

$$t_r \sim (\kappa_\nu \rho v_g)^{-1}. \tag{2.101}$$

* Oster's [Os63b] results, however, appear to contradict the second law of thermodynamics, as has been pointed out by Cronyn [Cr66]. See also Oster [Os66].

The time t_r is of the order of magnitude of the time required for radiation moving with velocity v_g to traverse unit optical thickness; t_r is also of the order of the mean time between successive interactions of a photon with material particles (*e.g.*, absorption, scattering). For typical conditions in stellar interiors we have, taking $v_g = c$, $t_r \sim 10^{-10} - 10^{-12}$ sec. We may then neglect the time derivative in (2.100) if I_ν changes significantly only in times long compared with t_r and if all material velocities are small compared with c. Since significant changes in I_ν nearly always, in real stars, require times much longer than $\sim 10^{-10}$ sec, then in practically all cases we can neglect the time derivative in (2.100).

2.9b *Equation of Transfer in Terms of Atomic Parameters*

We consider first the contribution to the right side of the equation of transfer (2.90) by the scattering terms. We have (see (2.81) and (2.87))

$$j_\nu^{(s)}(\mathbf{n}) - k_\nu^{(s)}(\mathbf{n}) I_\nu(\mathbf{n}) = \frac{N_e}{\rho} \left\{ \int_{(\nu')} \int_{4\pi} \left(\frac{\nu}{\nu'}\right) I_{\nu'}(\mathbf{n}') \mathscr{S}(\nu', \nu; \mathbf{n}', \mathbf{n}) \mathrm{d}\nu' \mathrm{d}\omega_{\mathbf{n}'} \right.$$

$$- \sigma_\nu(\mathbf{n}) I_\nu(\mathbf{n}) + \frac{c^2}{2h\nu^3} I_\nu(\mathbf{n}) \left[\frac{1}{\mu_\nu^{\,2}} \int_{(\nu')} \int_{4\pi} \left(\frac{\nu}{\nu'}\right) I_{\nu'}(\mathbf{n}') \mathscr{S}(\nu', \nu; \mathbf{n}', \mathbf{n}) \mathrm{d}\nu' \mathrm{d}\omega_{\mathbf{n}'} \right.$$

$$\left. \left. - \int_{(\nu')} \int_{4\pi} \left(\frac{\nu}{\nu'}\right)^3 \frac{I_{\nu'}(\mathbf{n}')}{\mu_{\nu'}^{\,2}} \mathscr{S}(\nu, \nu'; \mathbf{n}, \mathbf{n}') \mathrm{d}\nu' \mathrm{d}\omega_{\mathbf{n}'} \right] \right\}, \tag{2.102}$$

where the effects of induced scattering are entirely contained in the term involving the square brackets. It is seen that this term does not vanish in general, so that effects of induced scattering would have to be taken into account in a general case.

In the case of perfectly coherent scattering, however ($\mathscr{S}(\nu,\nu';\mathbf{n},\mathbf{n}') = S(\nu,\nu';\mathbf{n},\mathbf{n}')\,\delta(\nu'-\nu)$, etc.), (2.102) becomes

$$j_\nu^{(s)}(\mathbf{n}) - k_\nu^{(s)}(\mathbf{n}) I_\nu(\mathbf{n}) = \frac{N_e}{\rho} \left\{ \int_{4\pi} I_\nu(\mathbf{n}') \frac{\mathrm{d}\sigma_\nu(\mathbf{n}', \mathbf{n})}{\mathrm{d}\omega_{\mathbf{n}}} \mathrm{d}\omega_{\mathbf{n}'} - \sigma_\nu(\mathbf{n}) I_\nu(\mathbf{n}) \right.$$

$$\left. + \frac{c^2}{2h\nu^3} \frac{I_\nu(\mathbf{n})}{\mu_\nu^{\,2}} \int_{4\pi} I_\nu(\mathbf{n}') \left[\frac{\mathrm{d}\sigma_\nu(\mathbf{n}', \mathbf{n})}{\mathrm{d}\omega_{\mathbf{n}}} - \frac{\mu_\nu^{\,2}(\mathbf{n})}{\mu_\nu^{\,2}(\mathbf{n}')} \frac{\mathrm{d}\sigma_\nu(\mathbf{n}, \mathbf{n}')}{\mathrm{d}\omega_{\mathbf{n}'}} \right] \mathrm{d}\omega_{\mathbf{n}'} \right\}, \tag{2.103}$$

where we have set $S(\nu,\nu;\mathbf{n},\mathbf{n}') \equiv \mathrm{d}\sigma_\nu(\mathbf{n},\mathbf{n}')/\mathrm{d}\omega_{\mathbf{n}'}$, etc., and where we would have $\mu_\nu^{\,2}(\mathbf{n})/\mu_\nu^{\,2}(\mathbf{n}') = 1$ in an isotropic medium. It is seen that even in the case of perfectly coherent scattering, the effects of induced scattering may be non-zero if the scattering centers are preferentially oriented (as by the action

of a magnetic field on electron spins, for example); in this case, also, $\sigma_\nu(\mathbf{n})$ might depend on the direction $\mathbf{n}$ of propagation of the radiation. If the scattering centers are randomly oriented, however, the quantity in square brackets in (2.103) vanishes in an isotropic medium, since then the differential scattering cross section depends only on the angle between $\mathbf{n}$ and $\mathbf{n}'$; moreover, $\sigma_\nu(\mathbf{n}) = \sigma_\nu$, independent of $\mathbf{n}$. We then have the important result that, *if the scattering centers are randomly oriented in an isotropic medium, the effects of induced emission and absorption in coherent scattering cancel out exactly in the equation of transfer and consequently do not have to be taken into account.* This exact cancelling is possible for scattering because there is both induced *absorption* and induced *emission* in the scattering process. We shall henceforth assume, unless we specifically state otherwise, that the scattering centers are randomly oriented in an isotropic medium, so that the effects of induced (coherent) scattering can be omitted from the equation of transfer (*i.e.*, we assume (2.68) and (2.69) to be valid); this assumption is permissible in most cases of astrophysical interest.* We recall again that the assumption of coherence in the scattering is probably permissible with good accuracy as long as the electron gas is non-relativistic (*i.e.*, if $T \ll 5 \times 10^9$°K, say, *cf.* Sect. 2.8a3).

In the case of random orientations of the scattering centers and for coherent scattering it is convenient to define a *phase function* $p_\nu(\mathbf{n},\mathbf{n}')$ by the following relations:

$$d\sigma_\nu(\mathbf{n},\mathbf{n}') = \sigma_\nu p_\nu(\mathbf{n},\mathbf{n}')d\omega_{\mathbf{n}'}/4\pi = \sigma_\nu p_\nu(\mathbf{n}',\mathbf{n})d\omega_{\mathbf{n}}/4\pi\,, \tag{2.104}$$

where σ_ν (total cross section) is given by (2.67). The second equality in (2.104) arises because $p_\nu(\mathbf{n},\mathbf{n}')$ depends only on the angle between $\mathbf{n}$ and $\mathbf{n}'$. From (2.67) it follows, also, that

$$\int_{4\pi} p_\nu(\mathbf{n},\mathbf{n}')\frac{d\omega_{\mathbf{n}'}}{4\pi} = \int_{4\pi} p_\nu(\mathbf{n}',\mathbf{n})\frac{d\omega_{\mathbf{n}}}{4\pi} = 1\,. \tag{2.105}$$

* It can be shown, by a simple application of Fermi statistics, in fact, that in the presence of a magnetic field H, the electron spins will be randomly oriented if $kT \gg 2\,(e\hbar/2m_ec)H$ if the electron gas is partially degenerate or non-degenerate. The lower limit of T is much smaller than this if the electron gas is highly degenerate (*cf.* Chap. 24). Numerically, we have that random orientations of the electron spins will obtain if $T \gg 1.34 \times 10^{-4}\,\alpha H_{\text{gauss}}$ °K, where H is expressed in gauss and $\alpha = 1$ for non-degenerate cases and $\alpha \ll 1$ for highly degenerate cases. For $H = 10^8$ gauss (of the order of the strongest permissible field strengths in equilibrium stars with masses and radii of solar order (*cf.* Sect. 17.5)) and $\alpha = 1$, we have that $T \gg 13{,}400$ °K for random orientations of spin. Since this condition is fulfilled throughout most of the mass of a magnetic star, it is probably safe to assume random orientations for the electron spins.

For *isotropic* coherent scattering we have $p_\nu(\mathbf{n},\mathbf{n}') = p_\nu(\mathbf{n}',\mathbf{n}) = \text{const.} = 1$, from (2.105).

For random orientations of the scattering centers we then have left in the scattering terms only the direct emission and absorption, and the equation of transfer becomes for coherent scattering and for bound-bound transitions

$$\mu_\nu^2 \frac{\mathrm{d}}{\rho \mathrm{d}s}\left(\frac{I_\nu}{\mu_\nu^2}\right) = \frac{1}{\rho}\Bigg[h\nu_0 N_2 A_{21}\psi^*(\nu) + h\nu_0 N_2 B_{21} I_\nu \psi(\nu) - h\nu_0 N_1 B_{12} I_\nu \phi(\nu)$$

$$+ N_e \sigma_\nu \int_{4\pi} I_\nu(\mathbf{n}') p_\nu(\mathbf{n}', \mathbf{n}) \frac{\mathrm{d}\omega_{\mathbf{n}'}}{4\pi} - N_e \sigma_\nu I_\nu \Bigg]$$

$$= \frac{h\nu_0}{\rho} N_1 B_{12} \phi(\nu) \left\{ \frac{N_2 A_{21}\psi^*(\nu)}{N_1 B_{12}\phi(\nu)} + \left[\frac{N_2 B_{21}\psi(\nu)}{N_1 B_{12}\phi(\nu)} - 1\right] I_\nu \right\}$$

$$+ \frac{N_e \sigma_\nu}{\rho} \left\{ \int_{4\pi} I_\nu(\mathbf{n}') p_\nu(\mathbf{n}', \mathbf{n}) \frac{\mathrm{d}\omega_{\mathbf{n}'}}{4\pi} - I_\nu \right\}.$$

This equation may also be written in the form

$$\mu_\nu^2 \frac{\mathrm{d}}{\rho \mathrm{d}s}\left(\frac{I_\nu}{\mu_\nu^2}\right) = \kappa_\nu^{(a)} \left[1 - \frac{N_2 B_{21}\psi(\nu)}{N_1 B_{12}\phi(\nu)}\right] \left\{ \frac{A_{21}\psi^*(\nu)}{B_{21}\psi(\nu)} \left[\frac{N_1 B_{12}\phi(\nu)}{N_2 B_{21}\psi(\nu)} - 1\right]^{-1} - I_\nu \right\}$$

$$+ \kappa_\nu^{(s)} \left\{ \int_{4\pi} I_\nu(\mathbf{n}') p_\nu(\mathbf{n}', \mathbf{n}) \frac{\mathrm{d}\omega_{\mathbf{n}'}}{4\pi} - I_\nu \right\}, \tag{2.106}$$

where we have

$$\kappa_\nu^{(a)} \equiv \frac{h\nu_0}{\rho} N_1 B_{12} \phi(\nu) \quad \text{and} \quad \kappa_\nu^{(s)} \equiv \frac{N_e \sigma_\nu}{\rho}, \tag{2.107}$$

representing the respective mass absorption coefficients for "true" absorption uncorrected for induced emission (*cf.* (2.84)) and for direct, coherent scattering (*cf.* (2.87)). We note that $N_2 B_{21}\psi(\nu)/N_1 B_{12}\phi(\nu)$ gives the ratio of the induced emission to the true absorption. Recall that effects of dispersion, if present, are contained in $\kappa_\nu^{(a)}$, $\psi^*(\nu)$, $\psi(\nu)$, $\phi(\nu)$, and σ_ν (*cf.* Sect. 3.6).

We wish to remind the reader once again that (2.106) is not a complete equation of transfer, as only bound-bound transitions have been explicitly included in the "true" emission and absorption terms (see the discussion at the beginning of Sect. 2.8). Two other terms, representing bound-free and free-free transitions, would appear on the right side of (2.106) in a complete equation of transfer.

Moreover, we have neglected in (2.106) the effects of sophisticated processes such as neutrino production and transfer. These processes will be studied separately in Sect. 17.20 but will never be included in the equation of transfer.

2.10 Elementary Theory of Dispersion

In this section we shall describe and summarize certain aspects of the elementary theory of the properties of electromagnetic waves in material, dispersive media in the absence of a steady magnetic field and some results of this theory. More general and complete treatments may be found in such works as Landau and Lifshitz [La60], Stix [St62], Brandstatter [Br63], Ginsburg [Gi62], Van Vleck [Va32], Born and Wolf [Bo59], and Brittin and Chappell [Br62 and Ch65a]. As was pointed out in the introduction to this chapter, such effects of dispersion are probably not important in applications to the stellar interior except possibly under conditions of high density and low temperature. This section is included largely for completeness; it is essentially an appendix and may be skipped with little or no loss of continuity by the reader who is not interested in effects of dispersion.

Our treatment will be essentially classical and will be based on the Maxwell equations. Quantum mechanical corrections to classically derived results in dispersion theory are usually negligible (*cf.* Van Vleck [Va32]).

We shall always assume the macroscopic material properties of the medium to be *isotropic*. We shall also assume that they are spatially constant at least over distances comparable to the wavelength λ of the radiation in the medium, and temporally constant at least over times long compared to $1/\omega$, where ω is the angular frequency of the radiation. Our "continuum" treatment therefore requires, for example, that λ be large compared to the average interparticle separation in the medium, a condition not satisfied at sufficiently high frequencies. It will be shown at the end of Sect. 2.10d, however, that effects of dispersion are generally negligible at such high frequencies, at least for matter densities smaller than some $10^9 - 10^{10}$ gm/cm^3.

Since the medium is assumed to be *dispersive*, the dielectric and magnetic permeabilities ε and μ are in general functions of frequency, where ε and μ are defined by the relations*

$$\mathbf{D} = \varepsilon \mathbf{E}, \quad \mathbf{B} = \mu \mathbf{H}. \tag{2.108}$$

Here **D**, **E**, **B**, and **H** are, respectively, the electric displacement, electric vector, magnetic induction, and magnetic vector. In Gaussian units (see, for

* At optical and higher frequencies ε and μ must be regarded as *operators* instead of as algebraic factors, *cf.* Landau and Lifshitz [La60, Sect. 58].

example, Stratton [St41]), which we use throughout, ε and μ are unity in vacuo.

In Sect. 2.10a we shall set up the wave equations for **E** and **H** and consider some properties of plane wave solutions. In Sect. 2.10b we shall consider the real and imaginary parts of the complex index of refraction and the relation between the attenuation of an electromagnetic wave in a material medium and "true" absorption of energy from the wave. In Sect. 2.10c we shall consider the electromagnetic field energy in dispersive media. All equations presented in these three subsections are general in the sense that they are independent of any specific (microscopic) model of the dispersive medium. Finally, the properties of a simple, classical model of a dispersive medium (the "Lorentz-Lorenz" model) are examined in Sect. 2.10d.

2.10a Plane Electromagnetic Waves

In the classical theory of electromagnetic radiation in matter the electric vector **E** of the wave is assumed to induce collective oscillations of the electric charges in the medium, and these oscillations constitute an oscillating electric current or, equivalently, an oscillating "polarization" of the medium. These collectively oscillating charges, in turn, radiate electromagnetic waves which interfere with and so modify the original wave. This modification is described in terms of the *refractive index* p_ν, which is in general complex. We write

$$p_\nu = \mu_\nu + ig_\nu\,, \tag{2.109'}$$

where μ_ν and g_ν are the real and imaginary parts of p_ν. The real part μ_ν is the ordinary refractive index and is defined by $\mu_\nu = c/v_p$, where c is the velocity of light in vacuo and v_p is the *phase velocity* of the electromagnetic wave in the medium (*cf.* (2.124) below). The imaginary part g_ν is related to the "absorption" coefficient for the radiation in matter (see Sect. 2.10b). In most cases of interest in this book (specifically, cases in which an equation of transfer is valid, see Harris [Ha65]) we will have $|g_\nu| \ll \mu_\nu$, so that p_ν will be almost purely real in these cases.

Media for which $|g_\nu| \ll \mu_\nu$ in the relevant frequency ranges are said to be "transparent" for these frequencies (see Landau and Lifshitz [La60, Sect. 64]; Harris [Ha65]). Assuming that $\mu_\nu \approx 1$, then, we must have $|g_\nu| \ll 1$ for "transparent" media. We can see from (2.137) and (2.141) below that $|g_\nu| \ll 1$ implies that $1/(\kappa_\nu^\circ \rho) \ll \lambda/4\pi$, or that the photon mean free path for "true" absorption (*cf.* Sect. 2.10b) must be large compared to the wavelength λ of the electromagnetic wave in the medium. This condition is well satisfied for wavelengths shortward of, at most, the near infra-red under all stellar conditions except at very high densities (say except for $\rho \gtrsim 10^4 - 10^6$ gm/cm^3).

At such high densities (and for sufficiently low temperatures, say $T \lesssim 10^8\,^\circ\mathrm{K}$), μ_ν may approach zero (*cf.* Sect. 2.10d) and g_ν may become of order unity or larger at important frequencies such as those near the maximum of the Planck function. Under these conditions propagation of electromagnetic energy may be seriously impeded and the material could not be considered "transparent." The quantitative importance of this effect in dense stars has not yet been fully investigated. A competing mechanism of energy transfer under these conditions is electron thermal conduction (see Sect. 16.7).

The oscillating electric current, of current density $\mathbf{J}$, generated by the varying electromagnetic field in the medium can be related to $\dot{\mathbf{E}} \equiv \partial \mathbf{E}/\partial t$ by adopting an appropriate model for the oscillating charges, setting up the equation of motion for the charges, and obtaining the steady-state solution. Using Gaussian units, we can write this relation formally as

$$\mathbf{J} = -(1/4\pi)(1-p_\nu^2)\dot{\mathbf{E}}\,, \tag{2.109}$$

where the functional dependence of the factor multiplying $\dot{\mathbf{E}}$ on frequency ν, particle density, etc., will be determined by the details of the model of the oscillating charges (see Sect. 2.10d for an example of such a model).

The Maxwell equations are (see, for example, Born and Wolf [Bo59, Chap. 1])

$$\nabla \times \mathbf{E} = -\dot{\mathbf{B}}/c\,, \qquad \text{(a)}$$

$$\nabla \times \mathbf{H} = \dot{\mathbf{D}}/c + 4\pi\mathbf{J}/c \qquad \text{(b)}$$

$$\nabla \cdot \mathbf{D} = 4\pi\rho_E\,, \qquad \text{(c)}$$

$$\nabla \cdot \mathbf{B} = 0\,, \qquad \text{(d)} \tag{2.110}$$

where ρ_E denotes electric charge density. We assume that $(\mathbf{D},\mathbf{E})$ and $(\mathbf{B},\mathbf{H})$ are related by (2.108). Proceeding as usual by taking the curl of the left sides of (2.110a) and (2.110b) and using the vector identity $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}$ (and making use of the assumed spatial and temporal constancy of ε and μ over the relevant distances and times, *cf.* the introduction to this section), we obtain the wave equations for $\mathbf{E}$ and $\mathbf{H}$:

$$\nabla^2\mathbf{E} - \frac{\varepsilon\mu}{c^2}\ddot{\mathbf{E}} = \frac{4\pi}{\varepsilon}\nabla\rho_E + \frac{4\pi}{c^2}\dot{\mathbf{J}}\,, \tag{2.111}$$

$$\nabla^2\mathbf{H} - \frac{\varepsilon\mu}{c^2}\ddot{\mathbf{H}} = -\frac{4\pi}{c}\nabla \times \mathbf{J}\,. \tag{2.112}$$

Adopting first a "microscopic" picture, in which the material medium consists of discrete charges separated by vacuum, we must have $\varepsilon = \mu = 1$.

Setting $\rho_E = 0$ (*i.e.*, assuming electrical neutrality) and using (2.109) for $\mathbf{J}$, we obtain for (2.111) and (2.112)

$$\nabla^2\mathbf{E} - \frac{p_v^2}{c^2}\ddot{\mathbf{E}} = 0, \tag{2.113}$$

$$\nabla^2\mathbf{H} - \frac{p_v^2}{c^2}\ddot{\mathbf{H}} = 0. \tag{2.114}$$

We now adopt a "macroscopic" picture, in which we assume that the effects of the oscillating charges are entirely contained in ε and μ. Setting, accordingly, $\mathbf{J} = 0$ and assuming, again, that $\rho_E = 0$ in (2.111) and (2.112), we obtain by comparison with (2.113) and (2.114) the well-known result

$$p_v^2 = \varepsilon\mu \tag{2.115}$$

relating the complex refractive index p_v to ε and μ. In case $\mu \approx 1$ (as is the case for most dielectrics), (2.115) is

$$p_v^2 = \varepsilon. \tag{2.116}$$

The last case (2.116) is the only one that we shall consider, unless we specify otherwise, in the remainder of this section. If, moreover, p_v is almost purely real ($p_v \approx \mu_v$), we have

$$\mu_v^2 \approx \varepsilon. \tag{2.117}$$

We consider now plane wave solutions of (2.113) and (2.114):

$$\mathbf{E} = \mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}, \tag{2.118}$$

$$\mathbf{H} = \mathbf{H}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}, \tag{2.119}$$

where

$$\mathbf{k} = p_v\mathbf{k}_0 = (\mu_v + ig_v)\mathbf{k}_0 = (k_r + ik_i)\mathbf{e} = k\mathbf{e} \tag{2.120}$$

is the propagation vector (in general complex) of the electromagnetic wave in the medium, $\mathbf{k}_0$ is the corresponding propagation vector of a wave of the same frequency in vacuo, $\mathbf{e} = \mathbf{k}_0/k_0 = \mathbf{k}/k$ is a unit vector in the direction of $\mathbf{k}_0$ (or $\mathbf{k}$), and k_r and k_i are the real and imaginary parts of k. Note that $k_0 = 2\pi/\lambda_0 = \omega/c$, where λ_0 is the wavelength of the wave in vacuo. Hence

$$k_r = \mu_v\omega/c = \mu_v k_0, \tag{2.121}$$

$$k_i = g_v\omega/c = g_v k_0. \tag{2.122}$$

Moreover, since $k_r = 2\pi/\lambda$, λ being the wavelength of the wave in the medium, it follows that

$$\lambda = \lambda_0/\mu_v. \tag{2.123}$$

The actual (physical) **E** and **H** vectors are of course given by the real parts of (2.118) and (2.119).

The *phase velocity* (velocity of constant phase) of the wave is clearly

$$v_p = \omega/k_r = \omega/(\mu_v k_0) = c/\mu_v . \tag{2.124}$$

Consider now the relations between **E** and **H** in the wave. We note first from (2.118) and (2.119) that $\nabla \times \mathbf{E} = i\mathbf{k} \times \mathbf{E}$, $\nabla \cdot \mathbf{E} = i\mathbf{E} \cdot \mathbf{k}$, and $\dot{\mathbf{E}} = -i\omega\mathbf{E}$, with similar relations for **H**. It then follows from the Maxwell equations (2.110c) and (2.110d) (with $\rho_E = 0$ and $\mathbf{J} = 0$ in (2.110)) that

$$\mathbf{E} \cdot \mathbf{k} = 0, \quad \mathbf{H} \cdot \mathbf{k} = 0, \tag{2.125}$$

so that both **E** and **H** are perpendicular to **k**. Also, from (2.110a) we have

$$\mathbf{H} = (p_v/\mu)\mathbf{e} \times \mathbf{E} \tag{2.126}$$

$$= p_v \mathbf{e} \times \mathbf{E} \tag{2.127}$$

if $\mu = 1$. If p_v is complex, $p_v = |p_v| e^{i\alpha}$, then $\mathbf{H} = |p_v| e^{i\alpha} \mathbf{e} \times \mathbf{E}$, which shows that, in general, **E** and **H** are of different magnitudes and they do not oscillate in phase with each other. If p_v is real, however ($\alpha = 0$, $|p_v| = \mu_v$), then **E** and **H** *do* oscillate in phase with each other, and

$$|\mathbf{H}| = \mu_v |\mathbf{E}| . \tag{2.128}$$

The relation $\mathbf{k} = \mathbf{k}(\omega)$ is called the *dispersion relation* for the dispersive medium. Waves of different frequency propagate with different phase velocities in such a medium. If the wave under consideration consists not of a single frequency ω but, rather, a continuous distribution of frequencies in some range about ω (as must be the case with any real wave), then these Fourier components of the wave will form a *wave packet*. It can be shown (see, for example, Born and Wolf [Bo59, Chap. 1]) that such a wave packet propagates with the *group velocity* v_g:

$$v_g = \partial\omega/\partial k_r , \tag{2.129}$$

where v_g is the velocity of the *constructive interference maximum* of the packet. It can also be shown (see, for example, Stix [St62, Chap. 3]) that v_g is also the velocity of *energy propagation* of the wave in a dispersive medium.

The connection between the *wave* and *photon* pictures is obtained by associating a *photon* (rather, a *quasi-photon* in the language of Brittin and Chappell [Ch65a]) with a *wave packet* propagating with the velocity v_g (*cf.* Harris [Ha65]). The energy of a quasi-photon of frequency ν is $h\nu$ and the momentum of such a quasi-photon is (see Harris [Ha65]; Brittin and Chappell [Ch65a])

$$p = \hbar k_r = \mu_v \hbar k_0 = \mu_v h\nu/c \tag{2.130}$$

($2\pi\hbar$ = Planck's constant), where $\hbar k_0 = h\nu/c$ is the momentum of a "bare" photon of frequency ν (a photon in vacuo). It can also be shown (see, for example, Harris [Ha65]) that ν has the same value when the wave is in a dispersive medium as when it is in vacuo. The dispersion relation provides the relation between the energy $E=h\nu$ and the momentum p of the quasi-photon. For a "bare" photon, for example, this relation is just $E = pc$. Another example is provided in Sect. 2.10d.

The group velocity v_g can also be written in the form

$$v_g = \frac{c}{\partial(\mu_\nu \nu)/\partial \nu}. \tag{2.131}$$

We have, moreover, the relation

$$\frac{v_p}{v_g} = \frac{1}{\mu_\nu}\frac{\partial(\mu_\nu \nu)}{\partial \nu}, \tag{2.132}$$

which shows that $v_g = v_p$ in a nondispersive medium (μ_ν independent of ν).

The following inequalities can be shown (see Landau and Lifshitz [La60, Sect. 64]) to be generally valid in "transparent" media:

$$\frac{\partial(\mu_\nu \nu)}{\partial \nu} \geq \mu_\nu, \quad \frac{\partial(\mu_\nu \nu)}{\partial \nu} \geq \frac{1}{\mu_\nu}. \tag{2.133}$$

It then follows that

$$v_g \leq c/\mu_\nu, \quad v_g \leq c\mu_\nu; \tag{2.134}$$

these relations shows that $v_g \leq c$ whether $\mu_\nu > 1$ or $\mu_\nu < 1$ (although $v_p > c$ if $\mu_\nu < 1$). We also have the relations

$$\frac{v_p}{v_g} \geq 1, \quad \frac{v_p}{v_g} \geq \frac{1}{\mu_\nu{}^2}, \tag{2.135}$$

which shows that $v_p/v_g \geq 1$ regardless of the value of μ_ν. We note from (2.132) and (2.134) that $v_p \leq c$ always in a *non*dispersive medium.

2.10b Relation Between Attenuation and "True" Absorption

It is well known that collective effects in a dispersive medium can produce spatial attenuation of an electromagnetic wave even when "true" absorption is completely negligible, *i.e.*, even in the virtual absence of "atomic" mechanisms which can remove energy from an electromagnetic wave (in this section we include scattering in "true" absorption, in contrast to our terminology in all the other sections in this chapter). We shall in this subsection discuss the relation between the mass absorption coefficient κ_ν (which

includes collective effects), defined in Sect. 2.7, and the "true" mass absorption coefficient κ_ν° (which does *not* include collective effects). In other words, κ_ν° is the mass absorption coefficient computed as if the (real) refractive index μ_ν were unity, *i.e.*, solely from atomic cross sections.

Strong spatial attenuation (with only negligible "true" absorption) of an electromagnetic wave in a dispersive medium can occur, for example, when μ_ν vanishes (see Sect. 2.10d for an example of this case). The spatial attenuation can in this case be described as a kind of "total reflection." Equation (2.124) shows that the phase velocity $v_p = \infty$ when $\mu_\nu = 0$. In the wave picture the electromagnetic wave is reduced to a (non-propagating) standing wave which may be strongly damped spatially (as can be seen from (2.136) below, with $k_r = 0$ and $k_i > 0$, *i.e.*, g_ν (imaginary part of the complex refractive index p_ν)>0). Equation (2.127) shows that, since p_ν is purely imaginary in this case, the **H** vector oscillates just 90° out of phase with the **E** vector. Since the time average (over one period) $\bar{\mathbf{S}}$ of the Poynting vector (see (2.143) below) accordingly vanishes in this case, no energy is propagated by the wave.

According to (2.134), the group velocity $v_g = 0$ when $\mu_\nu = 0$. Hence, in the photon picture one says that the photons (rather, the quasi-photons) do not propagate in this case. Such non-propagating quasi-photons have zero momentum (see (2.130)), but may nevertheless possess energy, just as a standing wave possesses energy but does not propagate. Quasi-photons therefore behave, in this respect, in the same way as do material particles, which possess rest-mass energy even when they are not in motion (*cf.* Sects. 2.10d and 17.20c).

The relation between the monochromatic mass absorption coefficient κ_ν and the imaginary part g_ν of the complex refractive index p_ν is immediately obtained. Using the third equation in (2.120) and considering a plane wave propagating in the x direction, we obtain the expression for the **E** vector:

$$\mathbf{E} = \mathbf{E}_0 e^{i(k_r x - \omega t)} \cdot e^{-k_i x}. \tag{2.136}$$

Since the specific intensity I_ν (see Sect. 2.1) is proportional to the time average $|\bar{\mathbf{S}}|$ of the Poynting vector (see (2.143) below), which is in turn proportional to $\overline{|\mathbf{E}^2|}$, then I_ν is damped according as exp $(-2k_i x)$. The relation between κ_ν and g_ν is then (*cf.* Sect. 2.7)

$$\kappa_\nu \rho = 2k_i = 2g_\nu \omega / c, \tag{2.137}$$

where ρ is the mass density of the medium.

To establish the relation between κ_ν (which includes collective effects) and κ_ν° (which does not), we make use of a general result concerning the

imaginary part ε'' of the complex dielectric permeability $\varepsilon = \varepsilon' + i\varepsilon''$. As is shown by Landau and Lifshitz [La60, Sect. 62], ε'' is a measure of the dissipation of energy in a dispersive dielectric medium. More specifically, they show that ε'' is proportional to the "oscillator strength" for the material, *i.e.*, to the cross section for whatever "true" absorption mechanisms (including scattering) are effective. We shall therefore make the general assumption that ε'' is proportional to κ_ν°.

Setting the magnetic permeability $\mu = 1$, we have (see (2.109))

$$\varepsilon = \varepsilon' + i\varepsilon'' = (\mu_\nu + ig_\nu)^2, \tag{2.138}$$

whence

$$\varepsilon' = \mu_\nu{}^2 - g_\nu{}^2, \tag{2.139}$$

$$\varepsilon'' = 2\mu_\nu g_\nu = \mu_\nu \kappa_\nu \rho c/\omega, \tag{2.140}$$

where we have used (2.137) in the second equality in (2.140). If ε'' is a measure of energy dissipation, then it should be independent of μ_ν. In fact, ε'' should, according to the above discussion, be related to κ_ν° by a relation similar to the second equality in (2.140), except with μ_ν set equal to unity and with κ_ν replaced by κ_ν°. We then conclude that the relation between the mass absorption coefficient κ_ν (with collective effects included) and the "true" mass absorption coefficient κ_ν° (with collective effects ignored) is

$$\kappa_\nu = \kappa_\nu^\circ/\mu_\nu. \tag{2.141}$$

We shall assume throughout this book that (2.141) applies to all "true" absorption mechanisms (including scattering) of interest to us, or any combination of them (in which case κ_ν° would be a sum over the relevant mechanisms). Although direct calculations (with collective effects included) have confirmed the validity of (2.141) in a number of special cases (see, for example, Dawson and Oberman [Da62]; Mercier [Me64]; and Sect. 2.10d), it has not been proved by direct calculations that (2.141) is actually valid for all known "true" absorption mechanisms.* Hence it is only the generality of (2.141) that may be open to question.† In any case the validity of most of the final results that we shall obtain in this book will not depend on (2.141).

* Equation (2.141) does not agree with results obtained by Oster [Os63b]; however, see the first footnote in Sect. 2.9.

† We are indebted to Dr. Gabor Kalman (private communication, 1966) for pointing out that (2.141) is certainly not generally valid if one includes the whole range of plasma phenomena, including such things as dissipative effects of collisions, Landau damping, presence of steady magnetic fields, etc.

2.10c Electromagnetic Field Energy in Dispersive Media

An important and well-known identity can be derived from the Maxwell equations (2.110) (with ρ_E and $\mathbf{J}$ set equal to zero). We take the scalar product of (2.110a) with $\mathbf{H}$ and of (2.110b) with $\mathbf{E}$, add the resulting equations, and make use of the vector identity $\nabla \cdot (\mathbf{E}\times\mathbf{H}) \equiv -\mathbf{E}\cdot\nabla\times\mathbf{H} + \mathbf{H}\cdot\nabla\times\mathbf{E}$. We obtain

$$-\nabla\cdot\mathbf{S} = \frac{1}{4\pi}(\mathbf{H}\cdot\dot{\mathbf{B}} + \mathbf{E}\cdot\dot{\mathbf{D}}), \tag{2.142}$$

where

$$\mathbf{S} \equiv \frac{c}{4\pi}\mathbf{E}\times\mathbf{H} \tag{2.143}$$

is the *Poynting vector*, equal in magnitude to the energy flux (energy per unit area per unit time) carried by the electromagnetic field. Although (2.142), sometimes called "Poynting's Theorem," is valid at each instant of time, in applications to near-monochromatic waves it is more convenient to regard (2.142) as averaged over a complete period of the wave; such a time average will be denoted by a bar over the appropriate symbol(s). The specific intensity I_ν (*cf.* Sect. 2.1) is proportional to $\bar{S} \equiv |\bar{\mathbf{S}}|$. The quantity on the right side of (2.142) (averaged over a period) is interpreted as the time rate of change of $\bar{u}$, the (time averaged) energy density of the electromagnetic field:

$$\frac{\partial\bar{u}}{\partial t} = \frac{1}{4\pi}\overline{(\mathbf{H}\cdot\dot{\mathbf{B}} + \mathbf{E}\cdot\dot{\mathbf{D}})}. \tag{2.144}$$

For a near-monochromatic wave in a *non*dispersive medium (ε and μ in (2.108) independent of frequency ν) use of (2.108) leads to the usual expression for $\bar{u}$:

$$\bar{u} = (\mu\overline{H^2} + \varepsilon\overline{E^2})/8\pi. \tag{2.145}$$

In a *dispersive* medium (ε and μ functions of ν), however, it can be shown (see Landau and Lifshitz [La60, Sect. 61]; Stix [St62, Chap. 3]) that (2.145) must be replaced by the equation

$$\bar{u} = \frac{1}{8\pi}\left[\overline{H^2}\frac{\partial(\nu\mu)}{\partial\nu} + \overline{E^2}\frac{\partial(\nu\varepsilon)}{\partial\nu}\right], \tag{2.146}$$

which obviously reduces to (2.145) in the case of a nondispersive medium. The energy density $\bar{u}$ as given by (2.146) is a measure of the *total* energy associated with the electromagnetic field in the medium; *i.e.*, it includes not only the electromagnetic field energy itself (given by (2.145)), but also the

kinetic energy of collective motions of the charges associated with the field in a dispersive medium.

Physically, the expression (2.146) for $\bar{u}$ arises because of the dissipation (however small) that is inherently associated with the dispersion phenomenon. As a result of the presence of dissipation, it is incorrect to regard an actual electromagnetic wave in a dispersive medium as strictly monochromatic: There must be a continuous distribution of Fourier components in a small range about the main frequency ν of the wave. These Fourier components allow for the possibility of a slow time change in the amplitude of the wave. Inclusion of these Fourier components then results in the appearance of the derivatives in (2.146).

As we shall generally be interested only in the case where $\mu = 1$, we may use the relation $\varepsilon = \mu_\nu^2$ in (2.146), in which ε is assumed almost purely real, to obtain

$$\bar{u} = \frac{1}{8\pi}\left[\overline{H^2} + \overline{E^2}\frac{\partial(\nu\mu_\nu^2)}{\partial\nu}\right]. \tag{2.147}$$

It is shown by Landau and Lifshitz [La60, Sect. 61] that (2.143) for the Poynting vector $\mathbf{S}$ is valid not only in vacuo but also in a dispersive medium.

Consider now a near-monochromatic electromagnetic wave propagating with negligible attenuation in some direction. For such a wave we have (see (2.128)) $|\mathbf{H}| = \mu_\nu|\mathbf{E}|$, whence the energy density is

$$\bar{u} = \frac{\overline{E^2}}{8\pi}\left[\mu_\nu^2 + \frac{\partial(\nu\mu_\nu^2)}{\partial\nu}\right] = \frac{\overline{E^2}}{4\pi}\mu_\nu\frac{\partial(\nu\mu_\nu)}{\partial\nu} = \frac{\overline{E^2}}{4\pi}\frac{\mu_\nu c}{v_g}, \tag{2.148}$$

where v_g, the group velocity, is given by (2.131). On the other hand, the magnitude of the (time averaged) Poynting vector (see (2.143)) is

$$\bar{S} = \frac{\overline{E^2}}{4\pi}\mu_\nu c. \tag{2.149}$$

Comparison of (2.148) with (2.149) gives the physically plausible result that

$$\bar{u} = S/v_g, \tag{2.150}$$

or that, if $\bar{u}$ is interpreted as the total energy associated with the electromagnetic wave, this energy is propagated with the group velocity v_g of the wave. In the quasi-photon picture, then, the quasi-photons must be considered as the carriers not only of electromagnetic field energy itself, but of the total energy associated with the wave. Equation (2.150) provides the justification for the relation (2.24′) between the monochromatic radiant energy density u_ν and the specific intensity I_ν for radiation in a dispersive medium.

If $\bar{u}$ as given by (2.146) includes the kinetic energy of collective motions of the charges, as well as the electromagnetic field energy $\bar{u}_{em} = (\overline{E^2}/4\pi)\mu_\nu^2$ (see (2.145) and (2.128)) itself, then the ratio $\bar{u}/\bar{u}_{em}$ should never be smaller than unity. We have, in fact, using (2.124) for the phase velocity v_p,

$$\frac{\bar{u}}{\bar{u}_{em}} = \frac{v_p}{v_g} \geq 1 \tag{2.151}$$

(see (2.135)), which confirms our expectations. In a nondispersive medium we have $v_g = v_p$, whence $\bar{u}/\bar{u}_{em} = 1$.

2.10d Lorentz-Lorenz Model for a Dielectric

This model (see, for example, Born and Wolf [Bo59, Sect. 2.3]) is a purely classical and not very realistic one. It is, nevertheless, useful for orientation, and results based on it are used in a number of places in this book. Moreover, the model does result in expressions for the real and imaginary parts μ_ν and g_ν of the complex refractive index p_ν which have the correct functional dependence on frequency ν in many cases, at least in the non-relativistic limit. (By "correct" we mean as computed quantum mechanically and as observed empirically.) In the limit of very high frequencies (see (2.163) below) even the constants have the correct values (see Van Vleck [Va32, Sect. 82]).

This model consists of a collection of many identical oscillators, each made up of an electron bound to an atom. The electron has a "resonance" angular frequency ω_0 and is in the presence of an oscillating electromagnetic field whose electric vector is of the form $\mathbf{E} = \mathbf{E}_0 e^{i\omega t}$, where $\mathbf{E}_0$ is assumed constant over a region containing many oscillators. Hence the wavelength of the electromagnetic wave is assumed very large compared to the mean distance between oscillators. The (non-relativistic) equation of motion of the electron is (neglecting the Lorentz force)

$$\ddot{\mathbf{r}} = -\Gamma\dot{\mathbf{r}} - \omega_0^2\,\mathbf{r} - (e/m_e)\mathbf{E}, \tag{2.152}$$

where $\mathbf{r}$ is the displacement of the electron from its "equilibrium" position, $(-e)$ and m_e are electron charge and mass, and Γ is a "damping constant." If the damping is provided by collision effects, then Γ is twice the mean collision frequency per oscillator. If the damping is "natural damping," resulting from the "self-force" of the electron, then the classical expression for Γ is (see Heitler [He64, Chap.1])

$$\Gamma = \frac{2}{3}\frac{e^2\omega^2}{m_e c^3}, \tag{2.153}$$

where $\omega = 2\pi\nu$. The steady-state solution of (2.152) is

$$\mathbf{r} = \frac{e/m_e}{(\omega^2-\omega_0{}^2)-i\omega\Gamma}\mathbf{E}. \tag{2.154}$$

If there are N_e such oscillating electrons per unit volume, uniformly distributed, then the current density due to the oscillating charges is $\mathbf{J} = -N_e e\dot{\mathbf{r}}$ (assuming the relatively massive ions to remain stationary), or

$$\mathbf{J} = -\frac{N_e e^2/m_e}{(\omega^2-\omega_0{}^2)-i\omega\Gamma}\dot{\mathbf{E}}. \tag{2.155}$$

Comparing this result with (2.109), we obtain the desired expression for p_ν:

$$p_\nu{}^2-1 = -\frac{4\pi N_e e^2/m_e}{(\omega^2-\omega_0{}^2)-i\omega\Gamma} = -\frac{\omega_P{}^2}{(\omega^2-\omega_0{}^2)^2+\omega^2\Gamma^2}[(\omega^2-\omega_0{}^2)+i\omega\Gamma], \tag{2.156}$$

where

$$\omega_P{}^2 \equiv \frac{4\pi N_e e^2}{m_e} \tag{2.157}$$

is the square of the *plasma frequency* (non-relativistic). (An alternative derivation of (2.157) is given in Sect. 17.20c, where relativistic expressions for ω_P are also presented.) Using the relation $p_\nu = \mu_\nu + ig_\nu$ (μ_ν, g_ν real) and equating real and imaginary parts of (2.156), we obtain

$$\mu_\nu{}^2-g_\nu{}^2 = 1-\frac{\omega_P{}^2(\omega^2-\omega_0{}^2)}{(\omega^2-\omega_0{}^2)^2+(\omega\Gamma)^2}, \tag{2.158}$$

$$2\mu_\nu g_\nu = \frac{\omega_P{}^2(\omega\Gamma)}{(\omega^2-\omega_0{}^2)^2+(\omega\Gamma)^2}. \tag{2.159}$$

Equation (2.159) can also be written (see (2.137)) as

$$\mu_\nu\kappa_\nu\rho = \frac{\omega_P{}^2\omega^2\Gamma/c}{(\omega^2-\omega_0{}^2)^2+(\omega\Gamma)^2}, \tag{2.160}$$

where κ_ν is the mass absorption coefficient (*cf.* Sect. 2.7) for the process under consideration and ρ is the mass density of the material. Using (2.157) for $\omega_P{}^2$ and (2.153) for Γ in (2.160) and comparing the resulting equation with (16.143) below, we see that in this case the right side of (2.160) is equal to $N_e\sigma = \kappa_\nu^\circ\rho$, where σ, the cross section for the process under consideration, is given by (16.143) and where κ_ν° is the mass absorption coefficient computed in the absence of collective effects. Hence the general relation (see (2.141))

$\kappa_\nu = \kappa_\nu^\circ/\mu_\nu$ is seen to be valid for the present case of the Lorentz-Lorenz model with "natural" damping. Since in the limit of very high frequencies ($\omega^2 \gg \omega_0^2$, $\omega^2 \gg \Gamma^2$), σ becomes the Thomson scattering cross section (see (16.143)), we may conclude that (2.141) is valid also for Thomson scattering by free electrons.

Solving (2.158) and (2.159) for μ_ν^2, we obtain

$$\mu_\nu^2 = \left[1 - \frac{\omega_P^2(\omega^2 - \omega_0^2)}{(\omega^2 - \omega_0^2)^2 + (\omega\Gamma)^2}\right] \left[\frac{1}{2} + \frac{1}{2}\left\{1 + \frac{\omega_P^4(\omega\Gamma)^2}{[(\omega^2 - \omega_0^2)^2 + (\omega\Gamma)^2 - \omega_P^2(\omega^2 - \omega_0^2)]^2}\right\}^{1/2}\right]. \quad (2.161)$$

For applications to the stellar interior the material is often almost fully ionized. When this is the case there is no resonance frequency ($\omega_0 = 0$) and (2.161) becomes

$$\mu_\nu^2 = \left(1 - \frac{\omega_P^2}{\omega^2 + \Gamma^2}\right)\left\{\frac{1}{2} + \frac{1}{2}\left[1 + \frac{\omega_P^4\Gamma^2}{\omega^2(\omega^2 - \omega_P^2 + \Gamma^2)^2}\right]^{1/2}\right\}, \quad (2.162)$$

which is also the limiting form of (2.161) for very high frequencies ($\omega^2 \gg \omega_0^2$). In most cases, moreover, $\Gamma \ll \omega$ for the frequencies of interest (see last paragraph in this subsection). For frequencies ω greater than and not too close to ω_P, or in the approximation where dissipation is entirely neglected ($\Gamma = 0$), (2.162) reduces to the usual well-known form:

$$\mu_\nu = \sqrt{1 - \omega_P^2/\omega^2} \qquad (\omega \geq \omega_P). \quad (2.163)$$

The corresponding expression for the mass absorption coefficient κ_ν is

$$\mu_\nu \kappa_\nu \rho = \frac{\omega_P^2 \Gamma/c}{\omega^2} \qquad (\omega \geq \omega_P). \quad (2.164)$$

In the case of "natural" damping (see (2.153)) the right side of (2.164) reduces to $N_e\sigma_0$, where σ_0 is the Thomson scattering cross section.

In the approximations (2.162) and (2.163) the dispersion is produced entirely by the free electrons. Equation (2.163) is also obtained, under similar approximations, from a quantum mechanical calculation (see Van Vleck [Va32, Sect. 82]).

According to (2.162), μ_ν becomes very small at the plasma frequency ω_P; in fact, for $\Gamma/\omega_P \ll 1$, we have

$$\mu_\nu \approx \frac{1}{\sqrt{2}}\left(\frac{\Gamma}{\omega_P}\right)^{1/2} \approx g_\nu^{\frac{1}{2}} \qquad (\omega = \omega_P) \quad (2.165)$$

from (2.159). The mass absorption coefficient in this case is given by

$$\kappa_\nu \rho \approx \frac{\sqrt{2}}{c}(\omega_P \Gamma)^{1/2} \qquad (\omega = \omega_P). \tag{2.166}$$

In the absence of dissipation we would therefore have $\mu_\nu = g_\nu = 0$ at $\omega = \omega_P$. Hence the **H** vector in the wave would vanish identically in this case (see (2.127)), and the wave would be undamped spatially but non-propagating.

At frequencies below and not too close to ω_P we have, again for $\Gamma/\omega_P \ll 1$,

$$\mu_\nu \approx \frac{1}{2}\frac{\Gamma}{\omega_P}\left(\frac{\omega_P}{\omega}\right)^3 \left(\frac{\omega_P^2}{\omega^2}-1\right)^{-1/2} \ll 1 \qquad (\omega < \omega_P), \tag{2.167}$$

so that μ_ν would be exactly zero for $\omega < \omega_P$ if there were no dissipation. On the other hand, the corresponding expressions for g_ν and $\kappa_\nu \rho$ are

$$g_\nu \approx \sqrt{\frac{\omega_P^2}{\omega^2}-1} \qquad (\omega < \omega_P), \tag{2.168}$$

$$\kappa_\nu \rho \approx \frac{2\omega}{c}\sqrt{\frac{\omega_P^2}{\omega^2}-1} \qquad (\omega < \omega_P). \tag{2.169}$$

These expressions show that strong spatial attenuation of an electromagnetic wave can be present ($g_\nu \gtrsim 1$) even in the virtually complete absence of "true" absorption. It is clear that in this case of $\omega < \omega_P$, the material cannot be considered "transparent" (*cf.* Sect. 2.10a).

We note that the group velocity v_g for a dispersion relation of the form (2.163) is given simply by

$$v_g = \mu_\nu c. \tag{2.170}$$

Equation (2.163) can also be written in an alternative form which provides an interesting interpretation in terms of quasi-photons (*cf.* Brittin and Chappell [Ch65a]). Using the relation $\mu_\nu = ck/\omega$, we may write (2.163) in the form

$$\omega^2 = c^2 k^2 + \omega_P^2. \tag{2.163'}$$

Multiplying (2.163') through by $\hbar^2$ and using the quantum mechanical relations E(energy) $= \hbar\omega$, p (momentum) $= \hbar k$, we obtain

$$E^2 = p^2 c^2 + (\hbar\omega_P)^2. \tag{2.171}$$

This equation shows (*cf.* (10.10) below) that electromagnetic waves in a plasma with a dispersion relation of the form (2.163) or (2.163') behave in some respects as material relativistic particles ("quasi-photons") each

having a rest mass equal to $\hbar\omega_P/c^2$. Further discussion of quasi-photons is given in Sects. 10.2, 17.20c, and elsewhere.

We conclude this subsection by examining some of the limitations of (2.163). First, we recall that the "continuum" treatment on which the derivation of (2.163) is based (see the introduction to this section) is valid only if the wavelength of the radiation in the medium is long compared to the mean interparticle separation, which we shall approximate by $N_e^{-1/3}$. We use the approximations $\lambda \approx 2\pi c/\omega$ (*i.e.*, $\mu_\nu \approx 1$), (2.157) for ω_P, and (15.24) below (which assumes complete ionization and that all electrons present are "ionization" electrons (see Chap. 15)) for N_e:

$$N_e = \frac{1}{2} N_0 \rho (1+X) \tag{2.172}$$

(N_0 = Avogadro's number, X = relative mass abundance of hydrogen). The condition

$$\lambda > N_e^{-1/3} \tag{2.173}$$

may then be expressed in the form

$$\omega_P^2/\omega^2 > 6\times 10^{-6}\rho^{1/3}(1+X), \tag{2.174}$$

where ρ is in gm/cm^3. Since the right side of (2.174) is small compared to unity for all $\rho < \sim 10^9 - 10^{10}$ gm/cm^3, it follows (*cf.* (2.163)) that μ_ν is very nearly unity for wavelengths small enough to be comparable to the mean interparticle distance. Hence collective effects are not important anyway for such short wavelengths.

A more serious limitation of (2.163) arises from the fact that it is a non-relativistic formula, and therefore it is valid (see Griem [Gr64, p. 43]) only for

$$(\hbar\omega)^2 \ll (m_e c^2)^2 \qquad \text{or} \qquad \hbar\omega < m_e c^2 \tag{2.175}$$

($\hbar$ = Planck's constant/2π). The condition (2.175) may be written in the form

$$\omega_P^2/\omega^2 > 5\times 10^{-6}\rho(1+X), \tag{2.176}$$

where ρ is in gm/cm^3. The right side of (2.176) is comparable to unity or larger for $\rho > \sim 10^5$ gm/cm^3. For such densities, then, relativistic effects could be important while collective effects are still significant.

Finally, we note that the approximation $\Gamma^2 \ll \omega^2$ is always justified in the case of "natural" damping (*i.e.*, for all frequencies for which (2.163) is valid). Using (2.153), we have

$$\frac{\Gamma}{\omega} = \frac{2}{3}\left(\frac{e^2}{\hbar c}\right)\left(\frac{\hbar\omega}{m_e c^2}\right) \approx \frac{2}{3}\left(\frac{1}{137}\right)\left(\frac{\hbar\omega}{m_e c^2}\right), \tag{2.177}$$

where $(e^2/\hbar c) \approx 1/137$ is the fine structure constant. Since $\hbar\omega < m_e c^2$ for validity of (2.163), the above statement follows.

2.11 The "Directional Gradient" and Some of Its Properties

The "directional gradient" (*cf.* Sect. 2.9) arises in connection with differentiation of a function $F(\theta,\phi)$ of the polar angles θ and ϕ in a fixed coordinate system, with respect to direction (see Fig. 2.12). In this section we discuss this gradient and some of its properties. Since this section serves, essentially, as a mathematical appendix, it may be skipped entirely without any loss of continuity.

We use the unit vector $\mathbf{n}$ to specify the direction (θ,ϕ), so that

$$\mathbf{n} = \alpha_1 \mathbf{e}_1 + \alpha_2 \mathbf{e}_2 + \alpha_3 \mathbf{e}_3 , \tag{2.178}$$

where $\mathbf{e}_i$ $(i = 1,2,3)$ are unit vectors in the fixed, right-handed, orthogonal coordinate system (see Fig. 2.12) and α_i $(i = 1,2,3)$ are the direction cosines of $\mathbf{n}$ in this coordinate system (see (2.40′)).* For our present purposes, however, we shall find it more convenient to work in terms of a right-handed, orthogonal coordinate system which is not fixed, but which is defined by the unit vector $\mathbf{n}$ and the surfaces of constant θ and constant ϕ in the fixed coordinate system. In this new coordinate system the three unit vectors are $\mathbf{e}_r$, $\mathbf{e}_\theta$, and $\mathbf{e}_\phi$ (see Fig. 2.12), where (by definition)

$$\mathbf{n} = \mathbf{e}_r ; \tag{2.179}$$

also,

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij} \quad (i,j = r,\theta,\phi), \tag{2.180}$$

and

$$\mathbf{e}_i \times \mathbf{e}_j = \varepsilon_{ijk} \mathbf{e}_k \quad (i,j,k = r,\theta,\phi), \tag{2.181}$$

where δ_{ij} is the Kronecker delta ($\delta_{ij} = 0$ or 1 according as $i \neq j$ or $i = j$, respectively) and ε_{ijk} is the "Levi-Civita tensor density" ($\varepsilon_{ijk} = 0$ if any 2 of i, j, k are equal; $\varepsilon_{ijk} = +1$ if i, j, k form a cyclic permutation of r,θ,ϕ; and ε_{ijk} changes sign upon interchange of any pair of i,j,k).

Consider now an infinitesimal change $(\mathrm{d}\theta,\mathrm{d}\phi)$ in direction, which can be represented by the infinitesimal vector $\mathrm{d}\mathbf{n}$:

$$\mathrm{d}\mathbf{n} = \mathbf{e}_\theta \mathrm{d}\theta + \mathbf{e}_\phi \sin\theta \, \mathrm{d}\phi , \tag{2.182}$$

* In "flat" space any vector (such as $\mathbf{n}$) which depends on position can always be transferred parallel to itself to the origin of a fixed coordinate system.

where $\mathbf{dn}$ is clearly perpendicular to $\mathbf{n}$: $\mathbf{n} \cdot \mathbf{dn} = 0$. The corresponding change in $F(\theta,\phi)$ is

$$\begin{aligned} dF &= \left(\frac{\partial F}{\partial \theta}\right) d\theta + \left(\frac{\partial F}{\partial \phi}\right) d\phi \\ &= \left(\frac{\partial F}{\partial \theta}\right)(\mathbf{e}_\theta \cdot \mathbf{e}_\theta) d\theta + \left(\frac{\partial F}{\sin\theta \partial \phi}\right)(\mathbf{e}_\phi \cdot \mathbf{e}_\phi) \sin\theta \, d\phi \\ &= \left(\mathbf{e}_\theta \frac{\partial F}{\partial \theta} + \mathbf{e}_\phi \frac{\partial F}{\sin\theta \partial \phi}\right) \cdot (\mathbf{e}_\theta d\theta + \mathbf{e}_\phi \sin\theta \, d\phi) \qquad (2.183) \\ &= (\nabla_n F) \cdot \mathbf{dn}, \qquad (2.184) \end{aligned}$$

where $\mathbf{dn}$ is given by (2.182) and where it is seen by comparison of (2.183) and (2.184) that the "directional gradient" ∇_n may be expressed in the present coordinate system by the operator relation

$$\nabla_n \equiv \mathbf{e}_\theta \frac{\partial}{\partial \theta} + \mathbf{e}_\phi \frac{\partial}{\sin\theta \partial \phi}. \qquad (2.185)$$

Hence $\nabla_n F$ is a vector lying in the plane perpendicular to $\mathbf{n}$.

Note that, as θ and ϕ change, the unit vectors $\mathbf{e}_i$ $(i = r,\theta,\phi)$ all (in general) change direction. It is easily seen that the following relations, which will be needed later in this section, are valid:

$$\begin{aligned} \frac{\partial \mathbf{e}_r}{\partial \theta} &= \mathbf{e}_\theta, \quad & \frac{\partial \mathbf{e}_r}{\sin\theta \partial \phi} &= \mathbf{e}_\phi, \\ \frac{\partial \mathbf{e}_\theta}{\partial \theta} &= -\mathbf{e}_r, \quad & \frac{\partial \mathbf{e}_\theta}{\sin\theta \partial \phi} &= \mathbf{e}_\phi \cot\theta, \qquad (2.186) \\ \frac{\partial \mathbf{e}_\phi}{\partial \theta} &= 0, \quad & \frac{\partial \mathbf{e}_\phi}{\sin\theta \partial \phi} &= -\mathbf{e}_r - \mathbf{e}_\theta \cot\theta. \end{aligned}$$

We shall now establish four relations, some of which will be used in Chaps. 5 and 8, involving the unit vector $\mathbf{n}$ and any vector $\mathbf{v}$ which is assumed to be always perpendicular to $\mathbf{n}$ $(\mathbf{n} \cdot \mathbf{v} = 0)$:

$$\mathbf{v} = \mathbf{e}_\theta v_\theta + \mathbf{e}_\phi v_\phi, \qquad (2.187)$$

where v_θ and v_ϕ are in general functions of θ and ϕ. We assume that v_θ and v_ϕ are finite and single valued functions of direction.

The four desired relations are the following:

$$\nabla_n \cdot \mathbf{n} \equiv 2, \tag{2.188}$$

$$\nabla_n \cdot (\mathbf{nn}) \equiv 2\mathbf{n}, \tag{2.189}$$

$$\nabla_n \cdot (\mathbf{nv}) \equiv 2\mathbf{v}, \tag{2.190}$$

$$\int_{4\pi} \nabla_n \cdot \mathbf{v} \, d\omega_{\mathbf{n}} \equiv 0, \tag{2.191}$$

where in (2.191) $d\omega_{\mathbf{n}}$ is an element of solid angle about $\mathbf{n}$ and the integration is extended over all solid angles. Since these are vector relations, they are valid in *all* coordinate systems, not just in the one used (below) to establish them.

We now proceed to prove these results.

(2.188): We have from (2.179) and (2.185)

$$\begin{aligned}\nabla_n \cdot \mathbf{n} &= \mathbf{e}_\theta \cdot \frac{\partial \mathbf{e}_r}{\partial \theta} + \mathbf{e}_\phi \cdot \frac{\partial \mathbf{e}_r}{\sin\theta \partial \phi} \\ &= \mathbf{e}_\theta \cdot \mathbf{e}_\theta + \mathbf{e}_\phi \cdot \mathbf{e}_\phi \quad \text{(from (2.186))} \\ &= 2, \quad \text{Q.E.D.}\end{aligned}$$

(2.189): We have*

$$\begin{aligned}\nabla_n \cdot (\mathbf{nn}) &= \nabla_n \cdot (\mathbf{e}_r \mathbf{e}_r) \\ &= \mathbf{e}_\theta \cdot \frac{\partial (\mathbf{e}_r \mathbf{e}_r)}{\partial \theta} + \mathbf{e}_\phi \cdot \frac{\partial (\mathbf{e}_r \mathbf{e}_r)}{\sin\theta \partial \phi} \\ &= \left(\mathbf{e}_\theta \cdot \frac{\partial \mathbf{e}_r}{\partial \theta}\right) \mathbf{e}_r + \left(\mathbf{e}_\phi \cdot \frac{\partial \mathbf{e}_r}{\sin\theta \partial \phi}\right) \mathbf{e}_r,\end{aligned}$$

since, *e.g.*, $\mathbf{e}_\theta \cdot \mathbf{e}_r \, \partial \mathbf{e}_r / \partial \theta = 0$ (see (2.180)). Using (2.186), we obtain

$$\begin{aligned}\nabla_n \cdot (\mathbf{nn}) &= (\mathbf{e}_\theta \cdot \mathbf{e}_\theta)\mathbf{e}_r + (\mathbf{e}_\phi \cdot \mathbf{e}_\phi)\mathbf{e}_r = 2\mathbf{e}_r \\ &\equiv 2\mathbf{n}, \text{ Q.E.D.}\end{aligned}$$

(2.190): We have for the second rank tensor $\mathbf{nv}$:

$$\mathbf{nv} = \mathbf{e}_r \mathbf{e}_\theta v_\theta + \mathbf{e}_r \mathbf{e}_\phi v_\phi,$$

* We are using in this section the dyadic notation; *cf., e.g.*, Phillips [Ph33].

so that

$$\nabla_n \cdot (\mathbf{nv}) = \mathbf{e}_\theta \cdot \frac{\partial(\mathbf{e}_r \mathbf{e}_\theta v_\theta)}{\partial\theta} + \mathbf{e}_\theta \cdot \frac{\partial(\mathbf{e}_r \mathbf{e}_\phi v_\phi)}{\partial\theta} + \mathbf{e}_\phi \cdot \frac{\partial(\mathbf{e}_r \mathbf{e}_\theta v_\theta)}{\sin\theta\,\partial\phi} + \mathbf{e}_\phi \cdot \frac{\partial(\mathbf{e}_r \mathbf{e}_\phi v_\phi)}{\sin\theta\,\partial\phi}$$

$$= v_\theta \left(\mathbf{e}_\theta \cdot \frac{\partial \mathbf{e}_r}{\partial\theta} \right) \mathbf{e}_\theta + v_\phi \left(\mathbf{e}_\theta \cdot \frac{\partial \mathbf{e}_r}{\partial\theta} \right) \mathbf{e}_\phi + v_\theta \left(\mathbf{e}_\phi \cdot \frac{\partial \mathbf{e}_r}{\sin\theta\,\partial\phi} \right) \mathbf{e}_\theta + v_\phi \left(\mathbf{e}_\phi \cdot \frac{\partial \mathbf{e}_r}{\sin\theta\,\partial\phi} \right) \mathbf{e}_\phi$$

$$= v_\theta \mathbf{e}_\theta + v_\phi \mathbf{e}_\phi + v_\theta \mathbf{e}_\theta + v_\phi \mathbf{e}_\phi$$

$$\equiv 2\mathbf{v}, \quad \text{Q.E.D.}$$

(2.191): We have, first,

$$\nabla_n \cdot \mathbf{v} = \mathbf{e}_\theta \cdot \frac{\partial(\mathbf{e}_\theta v_\theta)}{\partial\theta} + \mathbf{e}_\theta \cdot \frac{\partial(\mathbf{e}_\phi v_\phi)}{\partial\theta} + \mathbf{e}_\phi \cdot \frac{\partial(\mathbf{e}_\theta v_\theta)}{\sin\theta\,\partial\phi} + \mathbf{e}_\phi \cdot \frac{\partial(\mathbf{e}_\phi v_\phi)}{\sin\theta\,\partial\phi}$$

$$= v_\theta \mathbf{e}_\theta \cdot \frac{\partial \mathbf{e}_\theta}{\partial\theta} + \frac{\partial v_\theta}{\partial\theta} + v_\phi \mathbf{e}_\theta \cdot \frac{\partial \mathbf{e}_\phi}{\partial\theta} + v_\theta \mathbf{e}_\phi \cdot \frac{\partial \mathbf{e}_\theta}{\sin\theta\,\partial\phi} + v_\phi \mathbf{e}_\phi \cdot \frac{\partial \mathbf{e}_\phi}{\sin\theta\,\partial\phi} + \frac{\partial v_\phi}{\sin\theta\,\partial\phi}$$

$$= \frac{\partial v_\theta}{\partial\theta} + v_\theta \cot\theta + \frac{\partial v_\phi}{\sin\theta\,\partial\phi}$$

$$= \frac{1}{\sin\theta} \frac{\partial(v_\theta \sin\theta)}{\partial\theta} + \frac{\partial v_\phi}{\sin\theta\,\partial\phi}.$$

Hence, recalling that $d\omega_n = \sin\theta d\theta d\phi$, we have

$$\int_{4\pi} \nabla_n \cdot \mathbf{v}\, d\omega_n = \int_0^{2\pi} d\phi \int_0^{\pi} \frac{\partial(v_\theta \sin\theta)}{\partial\theta} d\theta + \int_0^{\pi} d\theta \int_0^{2\pi} \frac{\partial v_\phi}{\partial\phi} d\phi.$$

The first integral on the right side vanishes because $\sin\theta = 0$ at both limits of integration and because we have assumed that v_θ is finite. The second integral on the right side vanishes because, in accordance with our assumption that v_ϕ was a single valued function of direction, v_ϕ has the same value when $\phi = 2\pi$ as when $\phi = 0$. Hence the right side of the above equation is identically zero, Q.E.D.

CHAPTER 3

Thermodynamic Equilibrium

The concept of thermodynamic equilibrium is one of the most important concepts in all of theoretical astrophysics, particularly in the study of stellar interiors. In this chapter we shall discuss this concept and derive a number of useful relations which apply under the ideal condition of thermodynamic equilibrium.

A general discussion of thermodynamic equilibrium is presented in Sect. 3.1. In Sect. 3.2 we present and discuss the basic distribution law (derived from statistical mechanics) for a system in statistical (thermodynamic) equilibrium. Statistical weight is discussed in Sect. 3.3. In Sects. 3.4 and 3.5 the concepts developed in Sects. 3.2 and 3.3 are applied to obtain the thermodynamic equilibrium distribution laws for, respectively, matter and radiation. Relations among the Einstein coefficients (see Chap. 2) are derived and discussed in Sect. 3.6. Finally, various properties of black body radiation are discussed in Sect. 3.7. A number of frequently used equations are derived there.

For generality, we have in this chapter retained a general value of the (real) index of refraction μ_ν. The case $\mu_\nu = 1$ of usual interest in stellar interiors is easily obtained as a special case of the formulae we shall present.

3.1 General Discussion of Thermodynamic Equilibrium

Consider an adiabatically sealed enclosure (see Chap. 9) with rigid walls (one which is thermally and mechanically isolated from the rest of the universe) containing radiation and various (arbitrary) types of matter. At first the walls of the enclosure and the matter within may have different temperatures and the temperature may not be uniform throughout the enclosure; moreover, irregulatiries in the pressure may result in unbalanced pressure forces. After some time (say the "mechanical relaxation time") has elapsed, however, the unbalanced pressure forces will have resulted in a redistribution of the matter such that the contents of the enclosure are now

in *mechanical* (or *hydrostatic*) *equilibrium*, with the forces arising from any force fields which may be present just balanced by the pressure gradients (see Chap. 1); if the enclosure is not in the presence of any force fields, the pressure will be constant throughout the enclosure. After a time, t_{therm} (the "thermal relaxation time"), generally much longer than the mechanical relaxation time, the walls and the entire contents of the enclosure will also have relaxed "thermally" to a condition characterized by a common, uniform temperature T; this condition of mechanical equilibrium characterized by a uniform temperature T is referred to in thermodynamics as *thermal equilibrium*.* If we do not admit the possibility of the occurrence of "chemical" reactions (used here in the general sense of meaning, for example, changes of phase; true chemical reactions involving atoms and molecules; excitation and ionization processes involving molecules, atoms, ions, electrons, and photons; thermonuclear reactions involving nuclei; etc.), then the system will also be in a perfectly steady *equilibrium* condition at the temperature T. This temperature T will remain constant (aside from minor statistical fluctuations arising from the atomicity of matter) for all time under the assumed conditions (rigid walls, thermal isolation from the rest of the universe, no "chemical" reactions) and, in fact, all other macroscopic properties of the system will remain forever unchanged.†

Such an enclosure whose contents have come to a perfectly steady state at a uniform temperature T is said to be in *thermodynamic equilibrium* (or, for short, *t.e.*; whenever the abbreviation *t.e.* appears in this book, it shall *always* refer to *thermodynamic* equilibrium and not, for example, to *thermal* equilibrium).‡ When we admit the possibility of the occurrence of "chemical" reactions within the enclosure, the concept of *t.e.* is no longer absolute, but is related to the particular physical properties of the system of interest and to the time scales which are relevant for the system. This "relativity" of the concept of *t.e.* owes its origin to the fact that different types of "chemical" reactions proceed at vastly differing rates and depend (generally) strongly on temperature. Consider, for example, the ideal case where only one type of "chemical" reaction can occur among the constituents of the enclosure. The relative concentrations of the reactants, if initially arbitrarily chosen, will

* Note (see Chap. 5) that this term has a different meaning in the theory of stellar structure. Unfortunately, a certain inconsistency in terminology seems inevitable when many different physical disciplines must be considered within a single general context such as stellar interiors.

† We assume that any force fields which may be present are constant in time.

‡ We neglect here and throughout this book general relativistic effects which can prevent the attainment of thermal or thermodynamic equilibrium (see Tolman [To34, Chap. 9] and Landau and Lifshitz [La58, p. 29]).

then change with time because of the occurrence of the reaction and will eventually approach equilibrium values. These equilibrium concentrations will not change with time, statistically, if the density ρ and temperature T remain constant; such a statistically steady state is often referred to as *chemical equilibrium*. The characteristic time required for the attainment of chemical equilibrium, at *constant* ρ and T, will be referred to here as t_{react} (the "reaction relaxation time"), which is generally a function of ρ and T. Consider now the sequence of events in our adiabatically sealed enclosure following the establishment of mechanical equilibrium. If $t_{\text{react}} < t_{\text{therm}}$, where t_{therm} is the "thermal relaxation time" (see first paragraph), then *chemical* equilibrium will become established, locally, before *thermal* equilibrium. After a time larger than t_{therm}, however, thermal equilibrium will also have become established and the system will then be in both thermal *and* chemical equilibrium (uniform temperature and equilibrium concentration of the reactants). It is clear that under the assumed conditions no further change could ever occur in the system, and we would again have a case of *t.e.* We see, then, that in the present example *strict t.e.* implies *both* thermal *and* chemical equilibrium.

If, however, $t_{\text{react}} > t_{\text{therm}}$, then thermal equilibrium (uniform temperature) will become established before chemical equilibrium. If, moreover, $t_{\text{react}} \gg t_{\text{therm}}$, then there may exist a range of times, long compared with t_{therm} but short compared with t_{react}, within which the state of the system could have changed only negligibly because of insufficient time for the reaction to proceed appreciably, and we might then have a very close approximation to *t.e. for this range of times.* Clearly, the condition of *t.e.* will be the more closely realized, the smaller are the times compared with t_{react}, consistent with the requirement that these times be large compared with t_{therm}. If the times of interest for the system are comparable to t_{react}, however, the relative concentrations of the reactants will, in general, change significantly *in these times* (resulting, possibly, in corresponding changes in temperature T), and we would then not be justified in saying that the system is even approximately in *t.e.* In times long compared with t_{react} we would of course again have both thermal and chemical equilibrium, and so strict *t.e.* Since in nature numerous kinds of "chemical" reactions are possible, each kind having its own reaction time t_{react} (generally a function of ρ and T), it is clear that *strict t.e.* is an ideal condition, possibly not realized anywhere in the real universe.* In the real world, then, systems can only be considered (generally) as being in *approximate t.e.* and references to *t.e.* are meaningful only in the context of the particular physical system under consideration and of the time scales of interest. We shall always assume in applications of the

* See footnote page 94.

concept of *t.e.* to real systems that this additional information needed to specify the precise meaning of *t.e.* for the particular system of interest has been supplied.

In applications to stars, the times of interest (usually > 1 sec) are generally long (say for $T > 10^3$°K) compared with t_{react} for chemical reactions involving molecules and for excitation and ionization reactions involving molecules, atoms, ions, electrons, and photons. On the other hand, these times of interest (say $< 10^{10}$ years, the approximate age of the Galaxy) are generally short under most stellar conditions (say for $T < 10^9$ °K) compared with the time required for the establishment of chemical equilibrium among nuclei. Only for temperatures larger than, say, 10^9 °K may this latter time become comparable with or smaller than the times of interest. Hence, in stellar work a suitable and convenient interpretation of *t.e.* would be that it implies, in addition to *thermal* equilibrium, *chemical* equilibrium among molecules, atoms, ions, electrons, and photons, but *not necessarily* among nuclei. Thus, for example, a system in thermal equilibrium and in excitation-ionization equilibrium, but in which irreversible nuclear reactions were occurring at a significant rate, would be regarded, according to this interpretation, as being in *t.e.* in spite of the non-equilibrium character of the system from the strict thermodynamic standpoint; if the times of interest are short compared with t_{react} for thermonuclear reactions, then this way of regarding the system could, of course, be highly accurate. In our applications of the concept of *t.e.* to stars in this book we shall always, unless we specifically indicate otherwise, mean *t.e.* in this sense.

A system in the ideal state of *t.e.* exhibits an interesting kind of "balance". It is clear, for example, that in the case of our adiabatically sealed enclosure the total rate of *emission* of electromagnetic energy by any object within the enclosure must exactly equal the total rate of *absorption* by that object; for, if this were not so, temperature differences could be set up within the enclosure which could be made to do work, without any simultaneous changes occurring in the system or its environment, in violation of the second law of thermodynamics (see Sect. 9.9). Furthermore, this exact equality of emission and absorption of radiation must hold at *every* frequency ν; for, if this were not so for some body within the enclosure, the body could effect

* In nature, strict *t.e.* might obtain if the "fundamental particles" of matter (in particular, the "heavy" fundamental particles, or *baryons*) were in chemical equilibrium with one another. Something approaching this condition may actually be approximately realized in the central regions of certain highly evolved red giant stars, where temperatures in excess of 10^9 °K are expected; this is the condition necessary for the operation of the "e-process" in the theory of element synthesis in stars (see Burbidge, Burbidge, Fowler and Hoyle [Bu57] and Clifford and Tayler [C165]).

temperature changes in the surrounding objects by appropriate exchanges of radiant energy, thus again violating the second law of thermodynamics (for details of arguments of this kind, see, for example, Eddington [Ed26, Chap. 3]).

Indeed, a *principle of detailed balancing* appears to apply under conditions of *t.e.*: *In any closed system in t.e. at a certain temperature, every detailed process is statistically balanced by its precise reciprocal process.*

The primary justification for believing that this principle is realized under conditions *t.e.* is that it is very hard to understand how the experimentally observed properties of *t.e.* could come about if the principle were not true. More specifically, the distribution laws for matter and radiation in *t.e.* are *entirely* independent of the reaction rates of the detailed atomic processes; for example, if the refractive index is unity, then the nature of the radiation field is quite independent of the kinds of objects in the enclosure and of the kind of material of which the walls are constructed: all objects look alike inside a read-hot furnace. This remarkable property of *t.e.* can be accounted for if the principle of detailed balancing is assumed, as we shall see. Thus, validity of this principle is certainly a *sufficient* condition for *t.e.* The validity of the principle is also generally assumed to be *necessary* for *t.e.*, although it is very difficult to provide a completely general and rigorous argument that the principle is actually necessarily valid in *t.e.* (see Eddington [Ed26, Chap. 3] for a good discussion of this point; also Heitler [He54, Sect. 5 of Appendix]). Nevertheless, we shall always assume in our work that detailed balancing occurs in *t.e.*

The nature of the *radiation field* in an enclosure in *t.e.* can be determined from the equation of transfer, which is (see (2.90))

$$\mu_\nu^2 \frac{\mathrm{d}}{\rho \mathrm{d}s}\left(\frac{I_\nu}{\mu_\nu^2}\right) = j_\nu - \kappa_\nu I_\nu, \tag{3.1}$$

where μ_ν is the (real) index of refraction, in conjunction with the principle of detailed balancing. It follows from this principle and from the interpretation of the mass emission and absorption coefficients j_ν and κ_ν that

$$j_\nu = \kappa_\nu I_\nu. \tag{3.2}$$

Hence, it follows from (3.1) that

$$I_\nu/\mu_\nu^2 = \text{const. along every ray path in the enclosure.*} \tag{3.3}$$

* Equation (3.3) does not follow from (3.1) in case $\mu_\nu = 0$. However, it is clear from the arguments immediately below and from considerations of continuity that (3.3) must nevertheless be valid in this case also.

It follows also that the quantity (I_ν/μ_ν^2) must not only be constant *along* every ray path passing through an arbitrary, given point, but must also have the same (constant) value for *all* ray paths passing through the given point. Hence (I_ν/μ_ν^2) must have the same value in every direction at every point in the enclosure. This must be true arbitrarily near the surface of every object in the enclosure, regardless of the nature of the object. Hence (I_ν/μ_ν^2) must be equal to a universal function only of ν and T, not of direction nor of the nature of the walls of the enclosure or of the objects therein. This universal function is the *Planck function* $B_\nu(T)$. Hence, for an enclosure in *t.e.* we have

$$I_\nu/\mu_\nu^2 = B_\nu(T), \tag{3.4}$$

where (see Sect. 3.5, in which (3.4) and (3.5) are derived from statistical mechanical considerations)

$$B_\nu(T) = \frac{2h\nu^3}{c^2}\frac{1}{e^{h\nu/kT}-1}. \tag{3.5}$$

The radiation field which is described by the Planck function has the following properties: It is isotropic, homogeneous, independent of the nature of the walls or the objects within the enclosure, and randomly polarized. Such a radiation field is frequently referred to as "black body radiation."

The state of the *matter* in an enclosure in *t.e.* is such that all reactants in all relevant "chemical" reactions are in complete *chemical equilibrium* (see the foregoing remarks in this chapter). We shall often express this state of affairs equivalently by the statement that the basic distribution laws for matter have their *t.e.* forms. Thus, for two discrete states 2 and 1 of an atom, we have, if the density of the matter is not enormously high, the Boltzmann equation

$$\frac{N_2}{N_1} = \frac{g_2}{g_1}e^{-h\nu/kT}, \tag{3.6}$$

where the g's are the statistical weights (=degrees of degeneracy) of the two levels and $h\nu = E_2 - E_1$. For energy states in the continuum, (3.6) is equivalent to the Maxwell-Boltzmann distribution law for velocities, which is therefore also valid in *t.e.* (assuming the gas to be non-relativistic and non-degenerate). The Saha ionization equation may also be considered as a generalized form of (3.6), so that this equation also holds in *t.e.* (These relations will all be derived in Sects. 3.4 and 3.5 from a basic formula of statistical mechanics to be presented in the next section.)

The *interaction* between matter and radiation in an enclosure in *t.e.* is described by Kirchhoff's law (3.2). Thus, using (3.4), we have

$$j_\nu = \mu_\nu^2 \kappa_\nu B_\nu(T). \tag{3.7}$$

In the special case of unity refractive index (3.7) becomes

$$j_\nu = \kappa_\nu B_\nu(T)\,. \tag{3.7'}$$

Although Kirchhoff's law in the form (3.7) or (3.7′) was derived specifically for the case of *t.e.*, the law sometimes applies even when the system is not in *t.e.* This will be the case, for example, if the emission properties (in particular, j_ν) of the system are independent of the prevailing radiation field or of whether or not the system is in *t.e.* In this case either j_ν or κ_ν can be computed entirely from knowledge of the other (assuming that $\mu_\nu = 1$), since $B_\nu(T)$, the factor of proportionality in (3.7′), is a universal function of ν and T, not of the nature of the emitter. Thus, it is often said that a "good emitter" is also a "good absorber." A special case of Kirchhoff's law is the so-called "reciprocity relation" of antenna theory (see Pawsey and Bracewell [Pa61, Chap. 2]).

3.2 Basic Distribution Law for a System in Statistical Equilibrium

From Statistical Mechanics (see, for example, Tolman [To38, Chap. 10]; Landau and Lifshitz [La58, Chap. 5]; or Sects. 10.4 and 10.5 of this book) the number of "elements" of a given kind (assumed non-interacting or only weakly interacting) in an assembly (a collection of elements) in statistical (*i.e*, thermodynamic) equilibrium having total energies per element in the range $(\varepsilon, \varepsilon + d\varepsilon)$ is

$$dN = \frac{dg}{e^{-\eta+\varepsilon/kT} \pm 1}, \tag{3.8}$$

where dg is the number of possible quantum states which a single, representative element of the assembly may have in the energy range $d\varepsilon$ about ε.* The assumption that the elements are non-interacting or only weakly interacting implies that one may compute these possible quantum states as though only one element were present in the assembly. The factor $[e^{-\eta+\varepsilon/kT} \pm 1]^{-1}$ may be thought of as the "occupation" of single quantum state, *i.e.*, the number of elements in the assembly in the given quantum state when the assembly is in statistical equilibrium. (Note that this factor depends only on the *energy* of the quantum state under consideration. If a number of separate quantum states all have the same energy ("degeneracy"), then,

* The "elements" that we will deal with mostly in this book will be "particles," such as atoms, electrons, and photons. In statistical mechanics books the term "system" is frequently used to describe the individual components (which we have called "elements") of an assembly.

according to (3.8), each of these degenerate states of given energy is "occupied" by the same number of elements.) The factor dg is sometimes called the "statistical weight" of the element, corresponding to the energy range $d\varepsilon$, since, aside from the factor $[e^{-\eta+\varepsilon/kT}\pm 1]^{-1}$, the probability of finding a particular element in the energy range $d\varepsilon$ is proportional to dg. The quantity η (sometimes called the "degeneracy parameter") may be found formally by integrating (3.8) over all the elements in the assembly:

$$N = \int \frac{dg}{e^{-\eta+\varepsilon/kT}\pm 1}, \tag{3.9}$$

where N is the total number of elements in the assembly in all possible quantum states. It is important to note that, thermodynamically, η is the *chemical potential*, μ, divided by kT: $\eta \equiv \mu/kT$ (*cf.* Sect. 24 and Chap. 5 of Landau and Lifshitz [La58]; and Sect. 9.12 of this book).

The positive sign in (3.8) applies to elements which obey *Fermi-Dirac* statistics (*fermions*), *i.e.*, particles with half-integral spins; for example, electrons, positrons, protons, neutrons, neutrinos, μ mesons, etc. This is the case which will be of interest to us when we take up the study of the white dwarf stars (Chaps. 24 and 25). The negative sign applies to particles which obey *Bose-Einstein* statistics (*bosons*), *i.e.*, particles with integral (or zero) spin; for example, photons, π mesons, etc. For *photons* we must set $\eta = 0$ because the number of photons in a system is not conserved (see Sect. 9.12 and Landau and Lifshitz [La58, Sect. 60]).

The system is said to be "non-degenerate" when $(-\eta)\gg 1$; in this case (3.8) becomes

$$dN = dg\, e^{\eta} e^{-\varepsilon/kT}, \tag{3.10}$$

which is the distribution law for *Maxwell-Boltzmann* (or "classical") systems. If $\eta \gg 1$, the system is said to be "highly degenerate" (or simply "degenerate"); intermediates cases ($\eta \simeq 0$, say -5 to $+10$) are referred to as "partially degenerate." Cases of partially and highly degenerate systems are treated in Chap. 24.

3.3 Statistical Weight

3.3a For Discrete Energy Levels

In the case of a system with discrete energy levels, such as an atom, dg becomes g_i, the "degree of degeneracy" of the level, or the number of quantum states all of which correspond to the same energy ε_i of the i^{th}

energy level. For example, the statistical weight of an atom whose total angular momentum quantum number is J is

$$g(J) = 2J+1\,. \tag{3.11a}$$

The statistical weight of the n^{th} level of a hydrogen-like atom is

$$g(n) = 2n^2\,. \tag{3.11b}$$

The statistical weight of an individual *Zeeman state* (the ultimate unit into which a *level* may be divided) is

$$g(M_J) = 1\,. \tag{3.11c}$$

3.3b For Continuous Energy Levels

A free "particle" (we are here considering photons, for example, as "particles") is a system which has continuous energy levels. In order to be able to count the number of quantum states which a free particle may have within a given energy range, we must temporarily introduce an artifice which will cause the energy levels of the particle to become discrete. We can do this by localizing our free particle in some finite region of space. The particle is then "bound" in this region of space, and we know from general principles of quantum theory that bound systems have discrete energy levels. The qualitative reason for this discreteness is that the de Broglie probability waves which represent the particles must be confined within the limits of the system, *i.e.*, must have nodes at the "boundaries" of then system. This permits only certain discrete wavelengths, and hence only discrete momenta and energies, to exist for such a bound system. We can then make the region of space in which the particle is confined as large as we please, so that the particle may again be regarded as essentially "free."

We consider, then, a large box with rigid walls of volume L^3, within which is confined a single "free" particle. The de Broglie wavelength of the particle is

$$\lambda = h/p \quad (h = \text{Planck's constant}). \tag{3.12a}$$

Now the only values which λ can have are those corresponding to standing waves (representing "stationary states"), since the waves must have nodes on the boundaries. It follows, moreover, from quantum theory that a given standing wave can be made up only of an *integral* number of *whole* wave lengths (in the language of quantum mechanics, it is said that the system must obey "periodic boundary conditions", *cf.* Schiff [Sc55, p. 43]). For a given standing wave we may resolve its length along the three coordinate

axes, so that λ_x, λ_y, and λ_z are the projections of the length λ onto the three coordinate axes. The x-component of momentum p_x is related to λ_x by

$$\lambda_x = h/p_x, \tag{3.12b}$$

and similarly for the other two components.

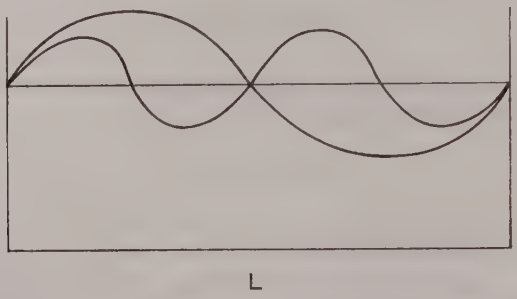


Fig. 3.1 Standing waves in a box with sides L.

In order for complete standing waves to exist in the box, we must have (see Fig. 3.1)

$$L = n_x\lambda_x, \quad n_x = 1, 2, \ldots$$

or

$$\lambda_x = L/n_x = h/p_x,$$

whence p_x can have only the values

$$p_x = n_x h/L, \quad n_x = \pm 1, \pm 2, \ldots, \tag{3.13}$$

where we have allotted negative values to n_x in order to include negative values of p_x. Similar expressions also hold for the y and z components of $\mathbf{p}$. Thus the momentum of a particle in a box of volume L^3 can have only the discrete values

$$p^2 = \frac{h^2}{L^2}(n_x^2 + n_y^2 + n_z^2), \quad n_{x,y,z} = \pm 1, \pm 2, \ldots . \tag{3.14}$$

The allowed energies, E, which the particle may have may be determined from (3.14), along with the relation $E = E(p)$ between energy and momentum for the particle. For example, if the particle is a non-relativistic material particle, we have $E = p^2/2m$.

In order now to determine the number of allowed quantum states per unit volume of momentum space, we lay off the allowed values of p_x, p_y, p_z along the three Cartesian axes in momentum space (see Fig. 3.2). Each lattice point then represents an allowed quantum state, *i.e.*, an allowed state

of motion of the particle, and the resulting allowed momentum of the particle may be pictured as a vector extending from the origin of coordinates to the point in question. Clearly, the direction of the vector **p** gives the direction of motion of the particle in space.

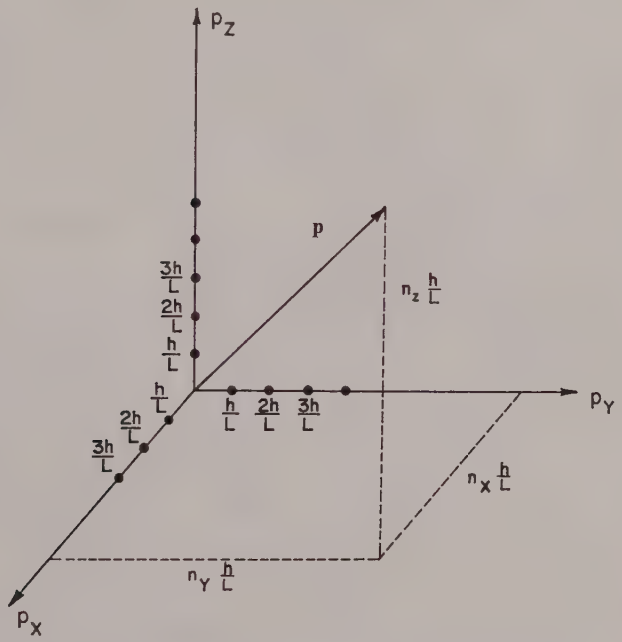


Fig. 3.2 Momentum p of a particle in momentum space.

Consider now a "box" in momentum space with one corner at the origin, having sides of length $n_x h/L$, $n_y h/L$, and $n_z h/L$. The number of lattice points, or quantum states, contained therein is

$$(n_x+1)(n_y+1)(n_z+1),$$

and the "volume" of the box is

$$n_x n_y n_z \frac{h^3}{L^3}.$$

Hence the number of lattice points, or quantum states, per unit volume of momentum space is

$$\left(1+\frac{1}{n_x}\right)\left(1+\frac{1}{n_y}\right)\left(1+\frac{1}{n_z}\right)\frac{L^3}{h^3} \simeq \frac{L^3}{h^3} \tag{3.15}$$

if we take $n_{x,y,z}$ very large. Taking $n_{x,y,z}$ very large clearly means making the sides of our box very large, so that the "unit," h/L, in momentum space will be very small compared with p_x, p_y, and p_z. Since $p = h/\lambda$, λ being the de Broglie wavelength corresponding to the momentum p, then the condition that $n_{x,y,z}$ be very large means that $L \gg \lambda$, or that the dimensions of the box must be large compared with the de Broglie wavelength of the particle. (This is physically clear since n_x, for example, is the number of de Broglie wavelengths that will fit into one side of the box.) Consider, as an example, an electron with $E = 1$ Kev (corresponding to $T \sim 10^7$°K) and $p = (2mE)^{(1/2)} \sim (2 \times 10^{-27} \times 10^{-9})^{(1/2)} \sim 10^{-18}$ gm-cm/sec. The corresponding de Broglie wavelength is then $\lambda \sim 6 \times 10^{-27}/10^{-18} \sim 10^{-8} - 10^{-9}$ cm. For stellar conditions, then, we may say that L must be large compared to atomic dimensions in order for this treatment to be valid.

The result (3.15) means that we have, on the average, just one lattice point per lattice cube of volume h^3/L^3 in momentum space. (Each lattice cube has eight corners, or eight lattice points; but each lattice point is common to eight lattice cubes; hence, on the average, there is one lattice point per lattice cube.) The box need not be cubical, or even rectangular. We may thus say that, on the average, there are $1/h^3$ lattice points, or quantum states for the "free" particle, per unit "volume" of *phase* space (which, for translational motion of a single particle, is the six-dimensional "space" whose coordinates are the three momentum components p_x, p_y, p_z and the three components x, y, z of the position vector $\mathbf{r}$). Hence the volume in phase space occupied by one quantum state is h^3; *this is then the "volume" of a "unit cell" in phase space.* Thus the "statistical weight" of a free particle in a large box of volume V, corresponding to momenta in the range $(p, p+\mathrm{d}p)$ and in directions confined to the solid angle $\mathrm{d}\Omega$, is (see Fig. 3.3)

$$\mathrm{d}g = V \cdot \frac{p^2 \mathrm{d}p \mathrm{d}\Omega}{h^3}. \tag{3.16}$$

In our phase-space picture, $\mathrm{d}g$ is the number of "unit cells" lying in the volume $V \cdot p^2 \mathrm{d}p \cdot \mathrm{d}\Omega$ of phase space, each unit cell containing just one quantum state for the particle.

If our particles may each have two spin directions or (as would be the case with photons) two directions of polarization, then the statistical weight would be just twice that given by (3.16). In general, if the spin angular momentum of each of the particles is $\sqrt{S(S+1)}\,\hbar$ ($\hbar \equiv h/2\pi$), then (3.16) must be multiplied by $(2S+1)$. However, if the particle spins are all aligned in the same direction, or (in the case of photons) if the particles are "polarized," then (3.16), as it stands, gives the correct statistical weight.

We note that nothing pertaining to the physical nature of the "particles" has entered into the derivation of (3.16). Hence (3.16) is valid for either non-relativistic or relativistic material particles or for photons or quasi-photons (see Sect. 2.10).

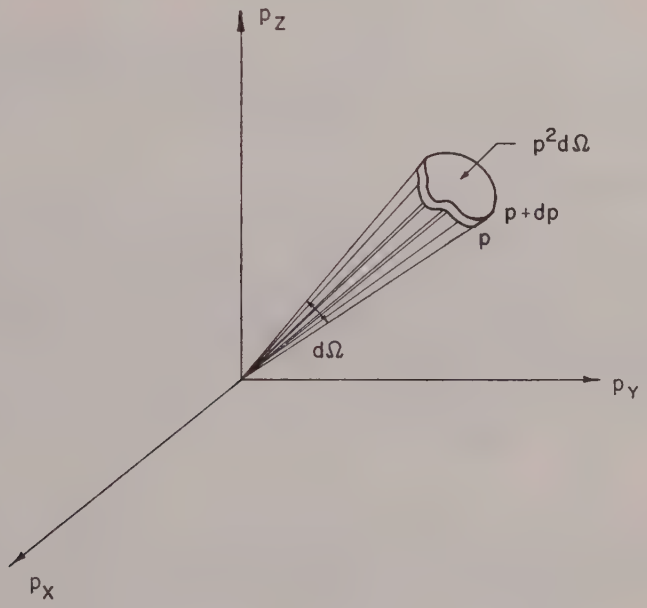


Fig. 3.3 An element of "volume" in momentum space.

In view of our "phase space" interpretation, we see that the factor $[e^{-\eta+\varepsilon/kT}\pm 1]^{-1}$ in the basic distribution law (3.8) may be interpreted as the number of "particles" occupying a "unit cell" of "volume" h^3 in phase space.

3.4 Distribution Laws for Matter

3.4a Boltzmann Equation

For non-degenerate particles of a given kind ($-\eta \gg 1$, atoms or ions, for example) we have from (3.8) for the numbers of particles in *discrete* states i and j, both types being in the same stage of ionization,

$$N_i = g_i e^{\eta} e^{-\chi_i/kT},$$
$$N_j = g_j e^{\eta} e^{-\chi_j/kT},$$

where the g's are the statistical weights of the two energy levels and the χ's

are the excitation potentials (energies above the ground level) of the two levels. Dividing one equation by the other, we obtain

$$\frac{N_i}{N_j} = \frac{g_i}{g_j} e^{-\chi_{ij}/kT}, \tag{3.17}$$

where $\chi_{ij} = \chi_i - \chi_j$, for the relative numbers of atoms or ions in the two states i and j in a given stage of ionization in statistical equilibrium. Equation (3.17) is often referred to as the *Boltzmann equation.*

3.4b Maxwell-Boltzmann Distribution Law

We consider non-degenerate systems ($-\eta \gg 1$) of non-relativistic mass points, each of mass m, and take $\varepsilon = p^2/2m$ as the total (kinetic) energy of a particle. We have for the number of particles in volume V having momenta in the range $(p,\ p+dp)$ (including all possible directions, $d\Omega \to 4\pi$, and assuming isotropy with respect to directions of momenta)

$$dN(p) = V \cdot g \frac{4\pi p^2 dp}{h^3} e^{\eta} e^{-p^2/2mkT},$$

where g stand for a "partition function" for the "internal" states of the particles (electronic or spin, for example). Letting n stand for the total number of particles per unit volume of the given kind, we have

$$n = \frac{1}{V} \int_0^\infty dN(p) = e^{\eta} g \int_0^\infty \frac{4\pi p^2 dp}{h^3} e^{-p^2/2mkT},$$

where the integral in the above equation has the value

$$\int_0^\infty \frac{4\pi p^2 dp}{h^3} e^{-p^2/2mkT} = \frac{(2\pi mkT)^{(3/2)}}{h^3}.$$

Hence we have

$$e^{\eta} = \frac{1}{g} \frac{nh^3}{(2\pi mkT)^{(3/2)}},$$

whence we obtain for the distribution law of momenta

$$\frac{dn(p)}{n} = \frac{4\pi}{(2\pi mkT)^{(3/2)}} e^{-p^2/2mkT} p^2 dp. \tag{3.18}$$

(A corresponding equation for $dn(p)/n$ in the extreme relativistic case is (10.65b).)

The distribution law for kinetic energies follows from (3.18) by using the (non-relativistic) relation $E = p^2/2m$; we obtain

$$\frac{dn(E)}{n} = \frac{2}{\pi^{(1/2)}} \frac{1}{(kT)^{(3/2)}} e^{-E/kT} E^{(1/2)} dE . \tag{3.19}$$

3.4c *Saha Ionization Equation*

We consider a system composed of three "kinds" of particle: (a) r-times ionized "atoms" (r = number of electrons lost from an atom) of a given kind; (b) $(r+1)$-times ionized "atoms" of the same kind; and (c) free electrons. For each type of particle (a) or (b) we consider only the total numbers of atoms or ions in all states of excitation for the given stage of ionization. We assume that the (a)- and (b)-type particles are non-degenerate ($-\eta_{r+1} \gg 1$, $-\eta_r \gg 1$) and non-relativistic. We regard an r-times ionized "atom" as a "compound" system, *i.e.*, as a "molecule" consisting of an $(r+1)$-times ionized atom and an electron bound together. We may then apply the thermodynamic theory of chemical equilibrium to the ionization process we are considering.

Consider the general "chemical" reaction

$$\sum_i \nu_i A_i = 0 , \tag{3.20}$$

where the A_i denote the various "reactants" and the ν_i (the *stoichiometric coefficients*) are integers; this reaction may be, for example, a true chemical reaction involving molecules, an ionization process involving ions and electrons, a nuclear reaction involving nuclei, etc. A specific example of each of these three kinds of reactions follows:

Chemical: $2H_2 + O_2 - 2H_2O = 0$: $\nu(H_2) = 2$, $\nu(O_2) = 1$, $\nu(H_2O) = -2$;
Ionization: $H^+ + e - H = 0$: $\nu(H^+) = 1$, $\nu(e) = 1$, $\nu(H) = -1$;
Nuclear: $2He^4 - Be^8 = 0$: $\nu(He^4) = 2$, $\nu(Be^8) = -1$.

The condition for chemical equilibrium can then be shown (see, for example, Chap. 10 of Landau and Lifshitz [La58] and Sect. 9.12 of this book) to be $\sum_i \nu_i \mu_i = 0$, where the μ_i denote the chemical potentials of the various reactants. Assuming all reactants to have the same temperature, this condition may also be written as

$$\sum_i \nu_i \eta_i = 0 , \tag{3.21}$$

since (see Sect. 3.2) $\eta_i \equiv \mu_i/kT$, where the η_i denote the degeneracy parameters of the various reactants. In the case of our ionization process the appropriate "reaction" is:

((r+1)-times ionized "atom") + (e) − (r-times ionized "atom") = 0,

whence the condition for equilibrium is from (3.21)

$$\eta_r = \eta_{r+1} + \eta_e \,. \tag{3.22}$$

For the energies of the three kinds of particle we have (assuming the atoms and ions to be non-relativistic)

$$\left.\begin{aligned} \varepsilon_{r,i} &= p^2/2m_r + \phi_{r,i}\,, \\ \varepsilon_{r+1,j} &= p^2/2m_{r+1} + \phi_{r+1,j} + I_r\,, \end{aligned}\right\} \tag{3.23}$$

ε = kinetic energy of a free electron,

where I_r is the ionization potential of an r-times ionized atom, or the energy required to remove an electron from the ground state of the r-times ionized atom; the ϕ's are the excitation potentials of the excited levels above the ground states of the corresponding "atoms"; and m_r and m_{r+1} denote the masses of the ions ($m_r = m_{r+1} + m_e$). If we were considering the ionization of a particular "atom", with the free electron having a particular kinetic energy, then the two p's in (3.23) would not be the same, but would be related to each other and to ε by momentum conservation. Since, however, we are interested only in the relative numbers of atoms and ions having all possible kinetic energies, we consider the three kinetic energies in (3.23) to be unrelated. For a given kinetic energy, the $(r+1)$-times ionized "atom" is considered to have an excess energy over that of the r-times ionized "atom" of $(\phi_{r+1,j} - \phi_{r,i}) + I_r$. We then have from (3.8) and (3.22) for the numbers per unit volume of the three "types" of particle:

$$n_{r+1} = B_{r+1}(T) e^{-I_r/kT} e^{\eta_{r+1}} \int_0^\infty \frac{4\pi p^2 \mathrm{d}p}{h^3} e^{-p^2/2m_{r+1}kT}, \tag{3.24a}$$

$$n_r = B_r(T) e^{\eta_{r+1}} e^{\eta_e} \int_0^\infty \frac{4\pi p^2 \mathrm{d}p}{h^3} e^{-p^2/2m_r kT}, \tag{3.24b}$$

$$n_e = 2 \int_0^\infty \frac{4\pi p^2 \mathrm{d}p}{h^3} \frac{1}{e^{-\eta_e + \varepsilon/kT} + 1}\,, \tag{3.24c}$$

where the factor 2 in the third equation takes into account the two possible spin states of an electron, and where $B_r(T)$ is the "partition function" for the r-times ionized "atom":

$$B_r(T) \equiv \sum_i g_{r,i} e^{-\phi_{r,i}/kT}, \tag{3.25}$$

the summation being carried out over all bound states of the "atom".* Integrating (3.24a,b) we obtain

$$n_{r+1} = e^{\eta_{r+1}} B_{r+1}(T) e^{-I_r/kT} \frac{(2\pi m_{r+1} kT)^{(3/2)}}{h^3}, \tag{3.26a}$$

$$n_r = e^{\eta_{r+1}} e^{\eta_e} B_r(T) \frac{(2\pi m_r kT)^{(3/2)}}{h^3}. \tag{3.26b}$$

Forming the quantity $n_e\, n_{r+1}/n_r$, we have

$$\frac{n_{r+1} n_e}{n_r} = (n_e e^{-\eta_e}) \frac{B_{r+1}(T)}{B_r(T)} \left(\frac{m_{r+1}}{m_r}\right)^{(3/2)} e^{-I_r/kT}. \tag{3.27}$$

In the form (3.27) of the Saha equation, the *electrons* may be either partially degenerate or relativistic or both. The quantity $e^{-\eta_e}$ is obtained as a function of n_e and T from (3.24c). We note that (3.27) could also have been derived by substituting into (3.22) directly the expressions giving the η's as functions of the n's and T (see Sect. 9.12).

If the electrons are both non-degenerate ($-\eta_e \gg 1$) and non-relativistic ($\varepsilon_e = p^2/2m_e$), then (3.24c) becomes

$$n_e = 2e^{\eta_e} \int_0^\infty \frac{4\pi p^2 \mathrm{d}p}{h^3} e^{-p^2/2m_e kT}$$

which gives

$$e^{-\eta_e} = \frac{2(2\pi m_e kT)^{(3/2)}}{n_e h^3}. \tag{3.28}$$

Thus (3.27) becomes

$$\frac{n_{r+1} n_e}{n_r} = \frac{2B_{r+1}(T)}{B_r(T)} \frac{(2\pi mkT)^{(3/2)}}{h^3} e^{-I_r/kT}, \tag{3.29}$$

where

$$m = \frac{m_e m_{r+1}}{m_e + m_{r+1}} \approx m_e,$$

which is the usual form of the Saha equation.

* The well-known divergence of the right side of (3.25) (due to the facts that $\phi_{r,i}/kT$ approaches a constant for sufficiently large values of i and that $g_{r,i}$ generally increases rapidly with increasing i) can be taken care of by summing over only a finite number of levels: Perturbations due to surrounding particles will "smear out" the highly excited levels, thus depressing the continuum and leaving only a finite number of discrete levels in the "atom". Such a depression of the continuum would also result in a lowering of the value of I_r, and hence in an increase in the value of the ratio n_{r+1}/n_r (*cf.* Sect. 15.5a).

3.5 Planck Radiation Law

Here we regard our "particles "as photons or quasi-photons (photons in a material (dispersive) medium, see Sect. 2.10) which may have two independent directions of polarization. If the radiation is unpolarized (as we assume it to be in dealing with black body radiation, which is known to be unpolarized), we have for the number of quantum states per unit volume per photon, corresponding to momenta in the range $(p,\ p+\mathrm{d}p)$ and having directions within the solid angle $\mathrm{d}\Omega$,

$$\mathrm{d}g = \frac{2p^2\mathrm{d}p\mathrm{d}\Omega}{h^3},$$

where again the factor 2 accounts for the two possible directions of polarization (see paragraph following (3.16)). Since (see Sect. 2.10a) the momentum of a quasi-photon of frequency ν is $p = \mu_\nu h\nu/c$, where μ_ν is the (real) refractive index, we have (remembering that μ_ν is in general a function of ν)

$$\mathrm{d}g = \frac{2\mu_\nu{}^2\nu^2}{c^3}\frac{\partial(\mu_\nu\nu)}{\partial\nu}\mathrm{d}\nu\mathrm{d}\Omega = \frac{2\mu_\nu{}^2\nu^2\mathrm{d}\nu\mathrm{d}\Omega}{c^2v_g},$$

where (see (2.131)) $v_g = c[\partial(\mu_\nu\nu)/\partial\nu]^{-1}$ is the group velocity of the electromagnetic wave, *i.e.*, the velocity of propagation of the quasi-photons. Since each quantum state (*i.e.*, each quasi-photon) is characterized by the energy $h\nu$, we then have that the "specific" energy density of radiation in the frequency range $(\nu,\ \nu+\mathrm{d}\nu)$ and having directions within $\mathrm{d}\Omega$ associated with these $\mathrm{d}g$ states is

$$h\nu\mathrm{d}g = \frac{2\mu_\nu{}^2h\nu^3\mathrm{d}\nu\mathrm{d}\Omega}{c^2v_g}.$$

Recalling that v_g is the velocity of propagation of the quasi-photons, it follows that the specific *intensity* (energy per unit area per unit time per unit solid angle per unit frequency interval) associated with these $\mathrm{d}g$ states is (*cf.* Sect. 2.3)

$$\frac{v_g}{\mathrm{d}\Omega\mathrm{d}\nu}\cdot\frac{2\mu_\nu{}^2h\nu^3\mathrm{d}\nu\mathrm{d}\Omega}{c^2v_g} = \mu_\nu{}^2\cdot\frac{2h\nu^3}{c^2}. \tag{3.30}$$

The factor in (3.8), $[e^{-\eta+\varepsilon/kT}\pm1]^{-1}$, giving the number of photons occupying a unit cell in phase space corresponding to the frequency, direction, and polarization of the radiation (sometimes called the "occupation" number), then becomes $[e^{h\nu/kT}-1]^{-1}$ (since $\eta = 0$ for photons in thermodynamic

equilibrium, see Sect. 9.12),* so that we finally obtain for the radiation field in an enclosure in *t.e.*

$$I_\nu = {\mu_\nu}^2 \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/kT}-1}. \tag{3.30'}$$

The factor multiplying ${\mu_\nu}^2$ in (3.30') is called $B_\nu(T)$, the *Planck function*:

$$B_\nu(T) = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/kT}-1}. \tag{3.31}$$

Note that, since $B_\nu(T)$ is isotropic, then $I_\nu = {\mu_\nu}^2 B_\nu(T)$ is also isotropic (assuming μ_ν to be isotropic, *cf.* Chap. 2). Hence, in an enclosure in *t.e.* the vector net flux $\mathbf{F}_\nu$ (see (2.16)) vanishes identically, as it must in accordance with the second law of thermodynamics (*cf.* Chap. 9).

3.6 Relations Among the Einstein Coefficients

In terms of atomic parameters, the (time-independent) equation of transfer for radiation of frequency ν ($\approx \nu_0 = (E_2 - E_1)/h$) and for coherent scattering was (see (2.106))

$$\frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{I_\nu}{{\mu_\nu}^2}\right) = \kappa_\nu^{(a)}\left[1-\frac{N_2B_{21}\psi(\nu)}{N_1B_{12}\phi(\nu)}\right]\left\{\frac{A_{21}\psi^*(\nu)}{B_{21}{\mu_\nu}^2\psi(\nu)}\left[\frac{N_1B_{12}\phi(\nu)}{N_2B_{21}\psi(\nu)}-1\right]^{-1}-\frac{I_\nu}{{\mu_\nu}^2}\right\}$$

$$+\kappa_\nu^{(s)}\left\{\int_{4\pi}\frac{I_\nu(\mathbf{n}')}{{\mu_\nu}^2}p_\nu(\mathbf{n}',\mathbf{n})\frac{\mathrm{d}\omega_{\mathbf{n}'}}{4\pi}-\frac{I_\nu}{{\mu_\nu}^2}\right\}, \tag{3.32}$$

where $\kappa_\nu^{(a)} \equiv h\nu_0 N_1 B_{12}\phi(\nu)/\rho$ and $\kappa_\nu^{(s)} \equiv N_e\sigma_\nu/\rho$; that is, $\kappa_\nu^{(a)}$ is the "true" absorption coefficient and $\kappa_\nu^{(s)}$ is the "direct scattering" absorption coefficient (which neglects induced scattering). We recall that only bound-bound transitions have been included explicitly in the "true" emission and absorption terms in (3.32). Other terms would be present on the right side of (3.32) if other physical processes, such as bound-free and free-free transitions, were to be included explicitly (*cf.* comments at end of Sect. 2.9b). In addition, we recall (*cf.* Sect. 2.8) that effects of dispersion, if present, are assumed included in $\kappa_\nu^{(a)}$, $\psi^*(\nu)$, $\psi(\nu)$, $\phi(\nu)$, and $\kappa_\nu^{(s)}$ (see later in this section).

* It has been shown by Britten and Chappell [Ch65a] and by Osborne [Os63a] that this factor $[e^{h\nu/kT}-1]^{-1}$ applies to quasi-photons as well as to "bare" photons (photons in vacuo) in thermodynamic equilibrium.

We now apply (3.32) to the case of *t.e.* In this case we have (see (3.4)) $I_\nu/\mu_\nu^2 = B_\nu(T)$, where $B_\nu(T)$ is both homogeneous and isotropic. Hence the left side of (3.32) is zero; also, the second curly bracket in (3.32) vanishes because of the isotropy of $B_\nu(T)$. Since $B_\nu(T)$ (which replaces each (I_ν/μ_ν^2) on the right side of (3.32)) is independent of the physical nature of the detailed atomic processes, it follows that each of the "absorption" terms, of which only the one representing bound-bound transitions is included explicitly in (3.32), must vanish separately.* Assuming the square bracket outside the curly bracket in this term to be non-zero, it follows that the first curly bracket in (3.32) must vanish. Hence we have

$$B_\nu(T) = \frac{A_{21}\psi^*(\nu)}{B_{21}\mu_\nu^2\psi(\nu)} \frac{1}{\dfrac{N_1 B_{12}\phi(\nu)}{N_2 B_{21}\psi(\nu)} - 1}. \tag{3.33}$$

But we know that Boltzmann's equation

$$\frac{N_2}{N_1} = \frac{g_2}{g_1} e^{-h\nu_0/kT} \tag{3.33'}$$

($\nu_0 = (E_2 - E_1)/h$) also is valid in *t.e.* Thus (3.33) becomes

$$B_\nu(T) = \frac{A_{21}\psi^*(\nu)}{B_{21}\mu_\nu^2\psi(\nu)} \frac{1}{\dfrac{g_1 B_{12}\phi(\nu)}{g_2 B_{21}\psi(\nu)} e^{h\nu_0/kT} - 1} \tag{3.34a}$$

$$= \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/kT} - 1} \tag{3.34b}$$

* The equation that results when the left side and the scattering term (second curly bracket in (3.32)) are zero, but when several different kinds of "absorption" processes are simultaneously present, can be written schematically in the form

$$\sum_i a_i \{ A_i - B_\nu(T) \} = 0,$$

where the summation is over all relevant processes. The solution of this equation is

$$B_\nu(T) = (\sum_i a_i A_i)/\sum_i a_i .$$

Now, if B_ν(T) is to be independent of the physical nature of the detailed atomic processes, then $B_\nu(T)$ cannot be a function of any of the a_i. Excluding the trivial case where $A_i \equiv 0$ (all i), the only way that $B_\nu(T)$ can be independent of all the a_i is for all the A_i to be equal to one another, and hence also to be equal to $B_\nu(T)$:

$$B_\nu(T) \equiv A_i \text{ (all } i),$$

Q.E.D.

from (3.31). Comparison of (3.34a) and (3.34b) then yields the relations

$$\frac{A_{21}\psi^*(\nu)}{B_{21}\psi(\nu)} = \mu_\nu{}^2 \frac{2h\nu^3}{c^2}, \tag{3.34c}$$

$$\frac{g_1 B_{12}\phi(\nu)}{g_2 B_{21}\psi(\nu)} = e^{h(\nu-\nu_0)/kT} \approx 1. \tag{3.34d}$$

However, A_{21} and B_{21} are by assumption constant over the narrow frequency range $\Delta\nu$ centered about $\nu_0 = (E_2 - E_1)/h$ within which the line shape functions differ significantly from zero. Hence the quantity $[\psi^*(\nu)/\psi(\nu)\mu_\nu{}^2]$ can differ from a constant only to the extent that ν^3 differs from $\nu_0{}^3$ within the "line". Since the "line" is of negligible width in most cases of interest ($\Delta\nu/\nu_0 \ll 1$), this difference between ν^3 and $\nu_0{}^3$ may be neglected. Consideration of the special case where effects of dispersion are absent ($\mu_\nu = 1$, $\psi^*(\nu) = \psi_a^*(\nu)$, $\psi(\nu) = \psi_a(\nu)$, see Sect. 2.8) shows (recalling the normalization of $\psi_a^*(\nu)$ and $\psi_a(\nu)$) that the value of this constant must be unity. Hence, we have

$$\psi^*(\nu) = \psi(\nu)\mu_\nu{}^2 \tag{3.34e}$$

in general, and

$$\psi_a^*(\nu) = \psi_a(\nu) \tag{3.34e'}$$

in the absence of effects of dispersion. Equation (3.34*e*′) shows that the line shape functions for spontaneous and induced emission are identical in the absence of dispersive effects. (This result was first explicitly stated by Oxenius [Ox64].) Moreover, it is clear from Sect. 3.5 that the right side of (3.34c) is just a statistical weight factor whose value is clearly independent of any assumptions regarding whether or not the system is in *t.e.* Equations (3.34e) and (3.34e′) are therefore *generally* valid (provided that $\Delta\nu/\nu_0 \ll 1$), and are not restricted to *t.e.*

Indeed, that (3.34e′) must be generally valid follows directly from the general quantum mechanical principle regarding induced emission stated in Sect. 2.8a2.

Similarly, since g_1, g_2, and B_{12} are independent of frequency "within the line," it follows from (3.34d) (assuming the right side to be equal to unity) that the quantity $[\phi(\nu)/\psi(\nu)]$ must be a constant, independent of ν. Again, the value of this constant must be unity because of the normalization of $\phi_a(\nu)$ and $\psi_a(\nu)$ (the line shape functions in the absence of dispersive effects). It follows, then, that in *t.e.*

$$\phi(\nu) = \psi(\nu) \tag{3.34f}$$

in general, and

$$\phi_a(\nu) = \psi_a(\nu) \tag{3.34f'}$$

in the absence of dispersive effects. Thus the line shape functions for absorption and induced emission are identical in *t.e.*, whether dispersive effects are present or not. It is important to note that these last results (*i.e.*, (3.34f) and (3.34f')), in contrast to (3.34e) and (3.34e') are *not* generally valid, as the validity of (3.34d) rests explicitly on the *t.e.* assumption.

In view of (3.34e) (generally valid) and (3.34f) (valid in *t.e.* but not generally), (3.34c) and (3.34d) yield the two "universal" relations between the three Einstein coefficients:

$$\frac{A_{21}}{B_{21}} = \frac{2h\nu^3}{c^2} \tag{3.35a}$$

and

$$g_1 B_{12} = g_2 B_{21}\,. \tag{3.35b}$$

Note that (3.35b), although obtained by assuming *t.e.*, involves only atomic parameters and therefore is *universally* valid.

Equation (3.35a), however, is not strictly universal, as the value of A_{21}/B_{21} depends on the state of polarization of the radiation (assuming that the basic definitions of A_{21} and B_{21} given in Sect. 2.8, which are without regard to the polarization of the radiation, are retained).

To consider effects of polarization, recall that, according to the quantum theory of radiation (see Sect. 2.8a2), the probability of emission of a photon (or quasi-photon, *cf.* Sect. 2.10a) into a certain unit cell in phase space of "volume" h^3 corresponding to given energy (frequency), direction, and polarization, is proportional to $(1+n)$, where n is the number of such photons (or quasi-photons) already occupying that cell. On the other hand, we saw from Sect. 3.5 that in *t.e.* the specific intensity was given as the product of n and a statistical weight factor, where clearly only the value of n is dependent on the assumption of *t.e.* We shall denote this statistical weight factor by W, where $(W/h\nu)$ is the number of states corresponding to photons of frequency ν per unit frequency range, flowing in unit time across unit area into unit solid angle about some direction. Hence, in a perfectly general case we can write for the "occupation" of a particular unit cell in phase space:

$$n = I_\nu / W, \tag{3.36}$$

where I_ν is $h\nu$ times the number of photons in unit frequency interval flowing in unit time across unit area into unit solid angle about some direction (*cf.* also (2.60)). Now n clearly cannot depend on the state of polarization of the radiation, since our definition of the "unit cell" in phase space already takes polarization into account (the unit cell is defined with respect to a given state of polarization). However, if unpolarized radiation characterized by a given number of photons of given frequency flowing across some area into some

solid angle in some time is "polarized" (for example, by passing it through an ideal polarizing filter), it is known that the intensity of the radiation will be very nearly halved (due to the removal of half of the photons), *i.e.*, that I_ν (unpolarized) $= 2I_\nu$ (polarized). Hence, W must be twice as great for unpolarized radiation as for polarized radiation of given frequency, direction, etc. Since (see Sect. 3.5) $W = \mu_\nu{}^2(2h\nu^3/c^2)$ for unpolarized radiation, we then have the following universal results:

$$n = \frac{I_\nu}{\mu_\nu{}^2}\frac{c^2}{2h\nu^3} \quad \text{(unpolarized radiation)}, \tag{3.36b}$$

$$n = \frac{I_\nu}{\mu_\nu{}^2}\frac{c^2}{h\nu^3} \quad \text{(polarized radiation)}. \tag{3.36c}$$

In the case of bound-bound transitions the developments leading up to (3.35), along with the relation (see (3.4)) $I_\nu = \mu_\nu{}^2 B_\nu(T)$, show that for unpolarized radiation $W = \mu_\nu{}^2 A_{21}/B_{21}$ in this case. Hence, for *polarized* radiation we must have, in contrast to (3.35a), $A_{21}/B_{21} = h\nu^3/c^2$. However, since in this book we shall practically always be dealing with unpolarized radiation, we shall, unless we specify otherwise, always use the value of (A_{21}/B_{21}) given in (3.35a).

We consider now briefly effects of a non-unity (real) refractive index on the line shape functions $\psi^*(\nu)$, $\psi(\nu)$, and $\phi(\nu)$ for, respectively, spontaneous emission, induced emission, and absorption (see also Sect. 2.9). We define the functions $f^*(\mu_\nu)$, $f(\mu_\nu)$, and $g(\mu_\nu)$ by the relations

$$\begin{aligned} \psi^*(\nu) &= f^*(\mu_\nu)\psi_a^*(\nu), \\ \psi(\nu) &= f(\mu_\nu)\psi_a(\nu), \\ \phi(\nu) &= g(\mu_\nu)\phi_a(\nu), \end{aligned} \tag{3.37a}$$

where (see Sect. 2.8) the subscript a implies the absence of dispersive effects ($\mu_\nu = 1$). It follows that $f^* = f = g = 1$ when $\mu_\nu = 1$. Using the first equation of (3.37a) in the general result (3.34e) and noting (3.34e′), we obtain the general relation

$$f^*(\mu_\nu) = \mu_\nu{}^2 f(\mu_\nu). \tag{3.37b}$$

In the special case of *t.e.* the relations (3.34f) and (3.34f′) are valid (they may in some cases also be valid outside of *t.e.*, see Sect. 3.1). These relations, along with the last two equations of (3.37a), lead to the result

$$f(\mu_\nu) = g(\mu_\nu), \tag{3.37c}$$

valid in *t.e.* but not generally. (We note, however, that (3.37c) would be valid outside of *t.e.* if (3.34f) and (3.34f′) were.)

If (3.37c) is assumed to be valid, then it and (3.37b) provide two relations between three functions of μ_ν. Specification of any one of these functions would then fix the other two.

For example, it was shown in Sect. 2.10b on the basis of dispersion theory that the relation

$$g(\mu_\nu) = 1/\mu_\nu \tag{3.37d}$$

is valid at least in a number of special cases. Granting (3.37d) and assuming (3.37c) to be valid, we have from (3.37b) and (3.37c) that $f(\mu_\nu) = 1/\mu_\nu$ and $f^*(\mu_\nu) = \mu_\nu$. In this case the effects of a non-unity refractive index on the line-shape functions are given explicitly by the relations

$$\begin{aligned} \psi^*(\nu) &= \mu_\nu \psi_a^*(\nu), \\ \psi(\nu) &= (1/\mu_\nu)\psi_a(\nu), \\ \phi(\nu) &= (1/\mu_\nu)\phi_a(\nu). \end{aligned} \tag{3.37e}$$

It should be emphasized that (3.37e), derived on the assumption of *t.e.*, are not generally valid (even granting validity of (3.37d)). On the other hand, they may be valid even outside of *t.e.* in some cases (*cf.* Sect. 3.1). Equation (3.37b), however, *is* generally valid and (3.37d) *may* be (see Sect. 2.10b, in particular the precautionary remarks following (2.141)).

Finally, we write down an expression for the ratio of the induced emission to the absorption in *t.e.* in the case of bound-bound transitions. Using (3.34f), (3.35b), and the Boltzmann equation (3.33′), we see that this ratio is

$$\frac{N_2 B_{21}\psi(\nu)}{N_1 B_{12}\phi(\nu)} = e^{-h\nu_0/kT} \approx e^{-h\nu/kT} (\nu \approx \nu_0), \tag{3.38}$$

and this result is valid in *t.e.* whether dispersive effects are present or not.

3.7 Properties of Black Body Radiation

3.7a Integrated Planck Function

This is defined by the relation

$$B(T) \equiv \int_0^\infty B_\nu(T)\mathrm{d}\nu. \tag{3.39}$$

Using (3.31), we have

$$B(T) = \frac{2h}{c^2}\int_0^\infty \frac{\nu^3 \mathrm{d}\nu}{e^{h\nu/kT} - 1}.$$

Letting $\alpha = h\nu/kT$ gives

$$B(T) = \frac{2h}{c^2}\left(\frac{kT}{h}\right)^4 \int_0^\infty \frac{\alpha^3 d\alpha}{e^\alpha(1-e^{-\alpha})}.$$

By expanding the term in parenthesis in the denominator and integrating term by term, we obtain

$$\int_0^\infty \frac{\alpha^3 d\alpha}{e^\alpha(1-e^{-\alpha})} = 6\sum_{n=0}^{\infty} \frac{1}{(n+1)^4} = \frac{\pi^4}{15}.$$

Thus the integrated Planck function becomes

$$B(T) = \frac{2\pi^4 k^4}{15c^2h^3} T^4 \equiv \frac{\sigma}{\pi} T^4, \tag{3.40}$$

where σ, the Stefan-Boltzmann constant, is given by

$$\sigma = \frac{2\pi^5 k^4}{15c^2h^3}. \tag{3.41}$$

3.7b Energy Density of Black Body Radiation

We had (see Sect. 2.4)

$$u_\nu = \frac{4\pi}{c} J_\nu,$$

where

$$J_\nu = \frac{1}{4\pi}\int_{4\pi} I_\nu d\omega$$

But $I_\nu = B_\nu(T)$ in *t.e.* (for $\mu_\nu = 1$) and is isotropic; hence we have

$$u_\nu = \frac{4\pi}{c} B_\nu(T)$$

or

$$u_\nu = \frac{8\pi h\nu^3}{c^3} \frac{1}{e^{h\nu/kT} - 1} \tag{3.42}$$

for the monochromatic energy density of black body radiation.

3.7c Integrated Energy Density

This is given by

$$u = \int_0^\infty u_\nu \mathrm{d}\nu = \frac{4\pi}{c} \int_0^\infty B_\nu(T)\mathrm{d}\nu = \frac{4\pi}{c} B(T)$$

or

$$u = \frac{8\pi^5 k^4}{15c^3h^3} T^4 \equiv aT^4. \tag{3.43}$$

Thus the radiation constant, a, is given by

$$a = \frac{8\pi^5 k^4}{15c^3h^3}. \tag{3.44}$$

Also we have

$$u = \frac{4\pi}{c} B(T) = \frac{4\pi}{c} \cdot \frac{\sigma}{\pi} T^4 = aT^4.$$

Thus the relation between a and σ is

$$a = \frac{4\sigma}{c}. \tag{3.45}$$

3.7d Integrated Radiation Pressure

In *t.e.* and for $\mu_\nu = 1$, we have $I_\nu = B_\nu(T)$, and $I = \int_0^\infty I_\nu \mathrm{d}\nu = B(T)$. Since $B(T)$ is isotropic, then $p_r = (1/3)u$ (see Sect. 2.5), so that the integrated radiation pressure for *t.e.* is given by

$$p_r = (1/3)aT^4. \tag{3.46}$$

We have now assembled the more important basic formulae that describe the properties of matter and radiation under the ideal conditions of *t.e.* It will be shown in subsequent chapters that most of these formulae are valid to high accuracy throughout most of the stellar interior, in spite of the fact that perfect *t.e.* does not obtain there.

CHAPTER 4

Local Thermodynamic Equilibrium (LTE)

We wish now to consider to what extent the conditions in the interior of a star can be represented, *locally*, by the ideal condition of thermodynamic equilibrium (*t.e.*). Since in this chapter we are applying the concept of *t.e.* to stars, we shall, in accordance with our convention stated in Sect. 3.1, interpret *t.e.* as implying *thermal* equilibrium and *ionization-excitation* equilibrium, but not necessarily *nuclear* equilibrium.

Let us first consider the question of how well the thermal equilibrium requirement is satisfied in a star. For this purpose we may estimate the average temperature gradient in the sun, for example, by taking $T_c \sim 10^7\,^\circ\mathrm{K}$ for the central temperature and $R \sim 10^{11}$ cm for the solar radius; we obtain

$$\left|\frac{\overline{dT}}{dr}\right| \sim \frac{T_c}{R} \sim 10^{-4}\,^\circ\mathrm{K/cm} \quad \textit{or} \quad 1^\circ\mathrm{K\ in\ } 10^4\ \mathrm{cm}.$$

This is an extremely small temperature gradient. Furthermore, the opacity of stellar material is typically so great throughout most of the interior that direct radiation arriving at a given point can have originated no further away from the point than, say, some fraction of a millimeter (which is the geometrical distance corresponding, in a typical case, to a few photon mean free paths). Thus, at least as far as direct exchanges of radiant energy are concerned, the material in a small neighborhood (having a radius of, say, a few photon mean free paths) about any given point is effectively totally "shielded" from its surroundings. Moreover, since under comparable conditions the mean free path for collisions of an atom or ion with other particles (particularly electrons, in the stellar interior) is generally orders of magnitude smaller than the photon mean free path, the material in such a neighborhood is even more effectively shielded from its surroundings, as far as direct particle-particle interactions are concerned, than in the case of particle-photon interactions. These considerations, together with the smallness of the temperature gradient (which prevents the existence of any

appreciable temperature differences between opposite sides of the neighborhood), suggest that the material in the neighborhood of the point can be considered to a high order of accuracy as being effectively enclosed adiabatically and at constant temperature, *i.e.*, effectively in thermal equilibrium at the local temperature in the vicinity of the point.

Considering next the other requirement of *t.e.* (ionization-excitation equilibrium), we note that the "reaction times" for ionization and excitation processes may be taken, very roughly, to be of the general order of the mean time between successive encounters of an atom or ion with photons or other particles; and these "encounter times" are, under typical stellar interior conditions, quite small (considerably less than one second, say, in most cases) and are at any rate nearly always small compared with the times of interest (>1 second, say). The ionization-excitation equilibrium requirement therefore also appears to be well satisfied at a typical point in the stellar interior. It is therefore quite reasonable to expect the stellar material to approach, locally, the conditions of *t.e.* to very high accuracy—probably to much greater accuracy, in fact, than is ever realized in any terrestrial laboratory.

Hence we would expect the radiation field, for example, at a typical interior point to be describable approximately by the Planck function $B_\nu(T)$, evaluated at the temperature at that point (assuming temporarily that the (real) refractive index μ_ν is unity):

$$I_\nu \simeq B_\nu(T), \tag{4.1}$$

so that, locally

$$j_\nu \simeq \kappa_\nu B_\nu(T). \tag{4.2}$$

It must be emphasized that these relations are only approximate; for, if they were exactly true, the radiation field would have to be perfectly isotropic, and the entire star would have to be at a uniform temperature, in which case there could be no net transfer of radiation from the interior outward, and the star could not lose energy by radiation. Indeed, the star would necessarily be in *t.e.* with "outer space" and would therefore have the temperature of space. We showed, however (Chap. 1), that the mean temperature of the stellar material must be several millions of degrees K; moreover, since the surface temperatures are observed to be of the order of only thousands of degrees, then there must be a temperature gradient within the star. Hence there must be a net outward flow of radiant energy, so that I_ν must have a maximum in the outward direction and a minimum in the inward direction rather than being perfectly isotropic.

While we shall obtain, later, a reasonably rigorous solution of the equation of transfer for the deep interior of a star, it is instructive to attempt to

determine here from the equation of transfer an order-of-magnitude estimate of the degree to which the radiation field in the deep interior of a star would be expected to deviate from Planckian radiation, still assuming that $\mu_\nu = 1$. For this purpose we shall let $ds = dr$, so that we are considering a *radial* ray; then the equation of transfer (see (2.91)) becomes

$$\frac{dI_\nu}{\rho dr} = j_\nu - \kappa_\nu I_\nu .$$

We suppose that $j_\nu \simeq \kappa_\nu B_\nu(T)$, so that we can write

$$I_\nu - B_\nu(T) \simeq \frac{dI_\nu}{d\tau_\nu},$$

where $d\tau_\nu = -\kappa_\nu \rho\, dr$ is the element of (normal) optical depth. Since we know on physical grounds that $I_\nu \simeq B_\nu(T)$, we can approximate the *difference* between I_ν and $B_\nu(T)$ by setting $I_\nu = B_\nu(T)$ in the right side of the above equation:

$$I_\nu - B_\nu(T) \simeq \frac{dB_\nu}{d\tau_\nu}.$$

or

$$\frac{I_\nu}{B_\nu(T)} - 1 \simeq \frac{1}{B_\nu}\frac{dB_\nu}{d\tau_\nu}$$

To obtain an order-of-magnitude estimate of the value of the quantity on the right side, we replace both B_ν(T) and dB_ν by the value of $B_\nu(T)$ at the center of the star (since the surface value is negligible in comparison) and $d\tau_\nu$ by the optical depth from the surface to the center of the star, say $d\tau_\nu \sim \kappa_\nu \rho R$, where R is the radius of the star and κ_ν and ρ are representative interior values of absorption coefficient and density. Thus we have

$$\frac{I_\nu}{B_\nu(T)} - 1 \sim \frac{1}{\kappa_\nu \rho R},$$

where we note that $\kappa_\nu \rho R$ is of the order of magnitude of the number of photon mean free paths in a radius of the star. Typical values for a stellar interior are $\kappa_\nu \sim 10^2$ cm^2/gm, $\rho \sim 10$ gm/cm^3, and $R \sim 10^{10}$ cm; these give

$$\frac{I_\nu}{B_\nu(T)} - 1 \sim 10^{-13},$$

so that the departures of I_ν from $B_\nu(T)$ in a stellar interior should indeed be small (for further discussion, see Sect. 6.4 and Chap. 7).

In Sect. 4.1 we shall discuss the mathematical expression of the condition of LTE and shall derive the form taken by the equation of transfer under

this condition. In Sect. 4.2 we discuss the effect and importance of scattering on the form of the equation of transfer in the stellar interior, still assuming LTE. For generality, the developments in Sects. 4.1 and 4.2 are carried through with a general value of the (real) refractive index μ_ν, even though μ_ν is essentially unity in most applications of interest.

Conditions for the validity of the LTE assumption in stellar interiors are derived and discussed in Chap. 7.

4.1 Equation of Transfer for LTE

In order to derive the equation of transfer in a form appropriate to the condition of LTE, we need a mathematical expression of this condition. We have just seen that the departures of the radiation field in a stellar interior from Planckian radiation are expected to be small (assuming that the (real) refractive index μ_ν is unity). Nevertheless, (4.1) and (4.2) (taken as exact equalities) clearly cannot be used as the mathematical expressions of the condition of LTE, for these would lead to the conclusion that $B_\nu(T)$, and hence T, are constant throughout the star; and we know that this is not the case in reality. We therefore need an expression of the condition of LTE which does not require the radiation field to be isotropic.

For this purpose it is customary to suppose that, loosely speaking, the *matter* at each point in a star is in a condition of *t.e.*, locally, corresponding to the *kinetic* temperature at that point (more precisely, the *electron kinetic* temperature). More specifically, the relevant distribution laws describing the state of the *matter* are supposed to have their *t.e.* forms—all being characterized by a common temperature at each point, the local electron kinetic temperature T_k.* Thus the Maxwell-Boltzmann velocity distribution law for *electrons* (if non-degenerate and non-relativistic), the Boltzmann law giving the distribution of *atoms* and *ions* over their various energy levels, and the Saha ionization equation are all assumed to be valid, the common temperature parameter in all these laws at each point being the local value of T_k. In other words, we are assuming that we have local *chemical equilibrium* among molecules, atoms, ions, electrons, and photons. For example, the Boltzmann equation,

$$\frac{N_2}{N_1} = \frac{g_2}{g_1} e^{-h\nu/kT_k}, \tag{4.3}$$

* Note that these statements do not apply to the *nuclear* matter, in accordance with our convention regarding the meaning of *t.e.* as applied to stars; see Sect. 3.1 and the introduction to this chapter.

where $h\nu = E_2 - E_1$, is assumed to be valid at each point, and this equation may, in fact, be taken in our work as the mathematical expression of the condition of LTE.

The conditions for the validity of the assumption of LTE in a star will be discussed in Chap. 7. Here we only note that the presence of large enough densities can be shown to be a *sufficient* condition that (4.3) be valid locally. The frequency of atomic collisions (predominantly with electrons) in a star increases with increasing density, and sufficiently frequent collisions will maintain the Boltzmann distribution despite any possible tendencies which the radiation field may have to destroy this distribution. Thus, even the presence of a non-Planckian radiation field, or a Planckian radiation field characterized by a temperature different from T_k, will have no appreciable effect on the distribution of atoms and ions over their various energy levels, provided that the frequency of collisions is great enough, and we will then have LTE.

A high frequency of collisions, however, is not a *necessary* condition for the validity of LTE; in fact, LTE can obtain in regions of very low density, where the rate of radiative transitions among atomic energy levels may far exceed the rate of collisional transitions. In this case, however, as will be shown in Chap. 7, the material must be, locally, optically very *opaque* (in the relevant frequencies); by this we mean that the optical thickness, say $\Delta\tau_\nu$ (*i.e.*, the number of photon mean free paths), separating regions of appreciably different physical characteristics (such as T_k) must be very large. An average photon (of the appropriate frequency) would then have to undergo a great many (of the order of $(\Delta\tau_\nu)^2$) successive absorptions and re-emissions (or, loosely speaking, "scatterings") in order to "diffuse" from one region to another of appreciably different physical characteristics. The ratio of collisional to radiative atomic transitions need then only be large enough to insure that such a diffusing photon be destroyed (for example, by being absorbed and then converted to kinetic energy by a collisional de-excitation of the atom) long before it has diffused an appreciable distance from its place of origin. All the photons present at a given point must then have originated at points having practically the same (kinetic) temperature as at the point in question.

This rapid "thermalization" of photons through occasional absorptions followed by collisional de-excitations (and hence transformations of radiant energy into kinetic energy) insures an intimate interaction between the radiation and the matter and forces the photons at a given point to have a spectral energy distribution close to that of a Planckian corresponding to a temperature practically equal to the local value of T_k. Since the radiation intensity is accordingly almost $B_\nu(T_k)$ at each point (assuming that $\mu_\nu = 1$),

the great preponderance of radiative transitions over collisional ones does not appreciably alter the distribution of atoms and ions over their *t.e.* distributions corresponding to T_k, and we have LTE. We note that in this second, low-density case it is still, ultimately, material particle collisions that determine the distribution of atoms and ions over their various energy levels.

Because both the "opaqueness" of the matter and the collision frequency increase rapidly in descending into a star, LTE is very likely to obtain throughout practically the whole star except perhaps in the very outermost regions in or above the photosphere.

We note that the LTE assumption, as exemplified mathematically by (4.3), imposes no conditions, immediately, on the *radiation* field; (4.3) is primarily a statement describing the local condition of the *matter*, even though the computed radiation field will be *affected* by this assumption.

We now proceed to write the (time-independent) equation of transfer in a form appropriate to the condition of LTE. We write this equation in the form (see (2.90))

$$\frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{I_\nu}{{\mu_\nu}^2}\right)=\left(\frac{j_\nu}{{\mu_\nu}^2}\right)-\kappa_\nu\left(\frac{I_\nu}{{\mu_\nu}^2}\right). \tag{4.4}$$

We wrote the equation of transfer for bound-bound transitions and coherent scattering, in terms of the basic atomic parameters, in the form (see (2.106) and the paragraphs following that equation)

$$\frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{I_\nu}{{\mu_\nu}^2}\right)=\kappa_\nu^{(a)}\left[1-\frac{N_2B_{21}\psi(\nu)}{N_1B_{12}\phi(\nu)}\right]\left\{\frac{A_{21}}{B_{21}}\left[\frac{N_1B_{12}\phi(\nu)}{N_2B_{21}\psi(\nu)}-1\right]^{-1}-\frac{I_\nu}{{\mu_\nu}^2}\right\}+$$

$$+\kappa_\nu^{(s)}\left\{\int_{4\pi}\frac{I_\nu(\mathbf{n}')}{{\mu_\nu}^2}\,p_\nu(\mathbf{n}',\mathbf{n})\frac{\mathrm{d}\omega_{\mathbf{n}'}}{4\pi}-\frac{I_\nu}{{\mu_\nu}^2}\right\}, \tag{4.5}$$

where $\kappa_\nu^{(a)}\equiv h\nu_0N_1B_{12}\phi(\nu)/\rho$, $\kappa_\nu^{(s)}\equiv N_e\sigma_\nu/\rho$, $\psi(\nu)$ and $\phi(\nu)$ are the line shape functions for induced emission and absorption, and we have used the general result (see (3.34e)) that $\psi^*(\nu)=\psi(\nu)\mu_\nu{}^2$. But we have the universal relations

$$\frac{A_{21}}{B_{21}}=\frac{2h\nu^3}{c^2},\quad g_1B_{12}=g_2B_{21},$$

so that (4.5) becomes

$$\frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{I_\nu}{{\mu_\nu}^2}\right)=\kappa_\nu^{(a)}\left[1-\frac{N_2g_1\psi(\nu)}{N_1g_2\phi(\nu)}\right]\left\{\frac{2h\nu^3}{c^2}\left[\frac{N_1g_2\phi(\nu)}{N_2g_1\psi(\nu)}-1\right]^{-1}-\frac{I_\nu}{{\mu_\nu}^2}\right\}+$$

$$+\kappa_\nu^{(s)}\left\{\int_{4\pi}\frac{I_\nu(\mathbf{n}')}{{\mu_\nu}^2}\,p_\nu(\mathbf{n}',\mathbf{n})\frac{\mathrm{d}\omega_{\mathbf{n}'}}{4\pi}-\frac{I_\nu}{{\mu_\nu}^2}\right\}. \tag{4.6}$$

Under general conditions $\psi(\nu) \neq \phi(\nu)$. In many cases of interest, however, these two line shape functions can be shown to be identical (see Oxenius [Ox64] and Hummer [Hu65]); as, for example, in the case of "complete redistribution" [Hu65]. Also, they are identical in thermodynamic equilibrium (see Sect. 3.6). We may therefore expect them to be *nearly* identical under conditions of LTE. In any case it is completely adequate for work in stellar interiors to assume that

$$\psi(\nu) = \phi(\nu), \tag{4.6'}$$

and we shall make this assumption, unless we indicate otherwise, throughout the remainder of this book.

We thus see that, aside from the scattering term in (4.6), the form of the equation of transfer depends (in view of (4.6')) only on the assumption we make regarding the quantity $N_2 g_1/N_1 g_2$, or the manner in which the atoms are distributed over their various energy levels. Applying the condition of LTE,

$$\frac{N_2}{N_1} = \frac{g_2}{g_1} e^{-h\nu_0/kT} \approx \frac{g_2}{g_1} e^{-h\nu/kT}$$

($h\nu_0 = E_2 - E_1$), where $T \equiv T_k$, the electron kinetic temperature, we obtain

$$\frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{I_\nu}{\mu_\nu^2}\right) = \kappa_\nu^{(a)}(1-e^{-h\nu/kT})\left\{\frac{2h\nu^3}{c^2}\frac{1}{e^{h\nu/kT}-1} - \frac{I_\nu}{\mu_\nu^2}\right\} +$$

$$+\kappa_\nu^{(s)}\left\{\int_{4\pi}\frac{I_\nu(\mathbf{n}')}{\mu_\nu^2}\, p_\nu(\mathbf{n}', \mathbf{n})\frac{\mathrm{d}\omega_{\mathbf{n}'}}{4\pi} - \frac{I_\nu}{\mu_\nu^2}\right\} =$$

$$= \kappa'_\nu B_\nu(T) - \kappa'_\nu\left(\frac{I_\nu}{\mu_\nu^2}\right) + \kappa_\nu^{(s)}\left\{\int_{4\pi}\frac{I_\nu(\mathbf{n}')}{\mu_\nu^2}\, p_\nu(\mathbf{n}', \mathbf{n})\frac{\mathrm{d}\omega_{\mathbf{n}'}}{4\pi} - \frac{I_\nu}{\mu_\nu^2}\right\}, \tag{4.7}$$

where $B_\nu(T)$ is the Planck function (see (3.5)) and κ'_ν denotes a *new* absorption coefficient,

$$\kappa'_\nu = \kappa_\nu^{(a)}(1-e^{-h\nu/kT}), \tag{4.8}$$

which corrects for the effects of induced emission. The term $\exp(-h\nu/kT)$ in (4.8) represents the ratio of the induced emission to the "true" absorption (*cf.* Sect. 3.2). Thus, we are actually treating the induced emission as a negative absorption; it is possible to do this only because the induced emission is precisely in the same direction as the inducing radiation. Note again that the correction factor $(1-e^{-h\nu/kT})$ is applied only to the "true" atomic absorption and *not* the scattering "absorption."

If we had considered explicitly other emission and absorption mechanisms, such as bound-free and free-free transitions, we would have obtained on the right side of (4.7) two additional pairs of terms having exactly the same form as the first pair, one pair for each of these mechanisms. Factoring out the common factors involving ν, T, and I_ν/μ_ν^2 we see that $\kappa_\nu^{(a)}$ and κ_ν' in (4.7) and (4.8) can now be interpreted as the sum of all the monochromatic mass absorption coefficients for all relevant physical absorption mechanisms that may be effective. Hence the form (4.7) of the equation of transfer is not restricted only to absorption and emission by bound-bound transitions, as all our explicit derivations leading up to this equation have assumed.

(We recall that we had, in general (with $\psi(\nu) = \phi(\nu)$),

$$\kappa_\nu' = \kappa_\nu^{(a)}\left(1 - \frac{N_2 g_1}{N_1 g_2}\right),$$

where $N_2 g_1/N_1 g_2$ is the ratio of the induced emission to the "true" absorption. In LTE, $N_2 g_1/N_1 g_2 = e^{-h\nu/kT}$, which is always less than one. Hence κ_ν' can never be negative in LTE, whence *maser* or *laser* action can never occur under equilibrium conditions. The principle of maser or laser amplification is based on the possibility of making $N_2 g_1 > N_1 g_2$, *i.e.*, in overpopulating a higher energy level by means of some non-equilibrium mechanism. This leads to a negative κ_ν', which means that a beam of radiation would become more intense (be amplified) as it progressed through the medium, instead of becoming attenuated.)

We write the equation of transfer (4.7) under conditions of LTE as

$$\frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{I_\nu}{\mu_\nu^2}\right) = \kappa_\nu' B_\nu(T) - \kappa_\nu'\left(\frac{I_\nu}{\mu_\nu^2}\right) + \kappa_\nu^{(s)}\left(\frac{\mathscr{J}_\nu}{\mu_\nu^2}\right) - \kappa_\nu^{(s)}\left(\frac{I_\nu}{\mu_\nu^2}\right), \tag{4.9}$$

where

$$\mathscr{J}_\nu \equiv \int_{4\pi} p_\nu(\mathbf{n}', \mathbf{n}) I_\nu(\mathbf{n}') \frac{\mathrm{d}\omega_{\mathbf{n}'}}{4\pi}. \tag{4.10}$$

Equation (4.9) is sometimes written in the form

$$\frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{I_\nu}{\mu_\nu^2}\right) = (\kappa_\nu' + \kappa_\nu^{(s)}) \cdot \left[\frac{\kappa_\nu' B_\nu(T) + \kappa_\nu^{(s)}(\mathscr{J}_\nu/\mu_\nu^2)}{\kappa_\nu' + \kappa_\nu^{(s)}}\right] - (\kappa' + \kappa_\nu^{(s)})\left(\frac{I_\nu}{\mu_\nu^2}\right),$$

where we shall henceforth write, for brevity,

$$\kappa_\nu \equiv \kappa_\nu' + \kappa_\nu^{(s)} \tag{4.11}$$

for the *total* absorption coefficient, properly corrected for induced emission and direct scattering. The product of μ_ν^2 and the quantity in square brackets,

which is the ratio of the sum of all "source" terms to the total absorption coefficient, is called the *source function*, and we write it in the following form:

$$S_\nu = (1-\sigma)\mu_\nu{}^2 B_\nu(T) + \sigma \mathscr{J}_\nu , \tag{4.12}$$

or

$$S_\nu = (1-\sigma)B_\nu(T) + \sigma \mathscr{J}_\nu \tag{4.12'}$$

if $\mu_\nu = 1$, where

$$\sigma \equiv \frac{\kappa_\nu^{(s)}}{\kappa_\nu} \quad \text{and} \quad 1-\sigma = \frac{\kappa_\nu'}{\kappa_\nu} \tag{4.13a}$$

denote, respectively, the ratio of the scattering to the total absorption coefficient and the ratio of the "true" absorption to the total absorption coefficient.

Note that under conditions of thermodynamic equilibrium (*not* LTE!) we have (*cf.* Chap. 3) $I_\nu = \mu_\nu{}^2 B_\nu(T)$, whence $\mathscr{J}_\nu = \mu_\nu{}^2 B_\nu(T)$, so that

$$S_\nu = \mu_\nu{}^2 B_\nu(T) \quad (t.e.), \tag{4.13'}$$

where $B_\nu(T)$ is constant in space. Hence S_ν becomes $\mu_\nu{}^2 B_\nu(T)$ in *t.e.*, even when scattering is present.

In astrophysics it is customary to make the simplifying assumption that $p_\nu(\mathbf{n}', \mathbf{n}) = 1$ (isotropic scattering), so that

$$\mathscr{J}_\nu \equiv \int_{4\pi} p_\nu(\mathbf{n}', \mathbf{n}) I_\nu(\mathbf{n}') \frac{d\omega_{\mathbf{n}'}}{4\pi} = \int_{4\pi} I_\nu(\mathbf{n}') \frac{d\omega_{\mathbf{n}'}}{4\pi} \equiv J_\nu ;$$

i.e., for isotropic scattering $\mathscr{J}_\nu$ becomes the average intensity J_ν. Thus the source function, under conditions of LTE and for isotropic, coherent scattering, becomes

$$S_\nu = (1-\sigma)\mu_\nu{}^2 B_\nu(T) + \sigma J_\nu \tag{4.13b}$$

or

$$S_\nu = (1-\sigma)B_\nu(T) + \sigma J_\nu \tag{4.13b'}$$

if $\mu_\nu = 1$. We may therefore finally write the equation of transfer in the form

$$\frac{1}{\rho}\frac{d}{ds}\left(\frac{I_\nu}{\mu_\nu{}^2}\right) = \kappa_\nu \left(\frac{S_\nu}{\mu_\nu{}^2}\right) - \kappa_\nu \left(\frac{I_\nu}{\mu_\nu{}^2}\right), \tag{4.14}$$

or in the form

$$\frac{dI_\nu}{\rho ds} = \kappa_\nu S_\nu - \kappa_\nu I_\nu , \tag{4.14'}$$

if $\mu_\nu = 1$ or a constant, where S_ν is given either by (4.12), (4.12'), (4.13b), or (4.13b').

We note that, because of the presence of the scattering term in (4.12) or (4.13b), $S_\nu/\mu_\nu{}^2$ does not equal $B_\nu(T)$ in general, even when LTE obtains. As

we shall see in the next section, however, S_ν/μ_ν^2 does in fact become equal to $B_\nu(T)$ at large optical depths in a star, even when electron scattering predominates over absorption ($\sigma \simeq 1$). Nevertheless, for the present we shall use the equation of transfer in the general form (4.14) (or (4.14′)).

To conclude, we see that we can write the general (time-independent) equation of transfer,

$$\frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{I_\nu}{\mu_\nu^2}\right)=\left(\frac{j_\nu}{\mu_\nu^2}\right)-\kappa_\nu\left(\frac{I_\nu}{\mu_\nu^2}\right), \tag{4.15}$$

or

$$\frac{\mathrm{d}I_\nu}{\rho\mathrm{d}s}=j_\nu-\kappa_\nu I_\nu \tag{4.15′}$$

if $\mu_\nu = 1$ or a constant, in a form appropriate to LTE, with coherent scattering included, by formally regarding κ_ν as given by

$$\kappa_\nu=\kappa_\nu^{(a)}\,(1-e^{-h\nu/kT})+\kappa_\nu^{(s)} \tag{4.16}$$

and replacing j_ν by $\kappa_\nu S_\nu$, where S_ν is given by (4.12), (4.12′), (4.13b), or (4.13b′).

4.2 Departure of S_ν/μ_ν^2 from $B_\nu(T)$, Assuming LTE

Here we shall show that, even if coherent scattering is present, the source function S_ν approaches $\mu_\nu^2 B_\nu(T)$ (or $B_\nu(T)$ if the refractive index $\mu_\nu = 1$), assuming LTE to obtain, at sufficiently great optical depths in a star. For LTE and with coherent, isotropic scattering present, we have (*cf.* (4.13b))

$$S_\nu=(1-\sigma)\mu_\nu^2 B_\nu(T)+\sigma J_\nu. \tag{4.17}$$

In Chap. 6 we shall obtain from a formal solution of the equation of transfer, some expansions for some of the radiative quantities in terms of the quantity (S_ν/μ_ν^2) and its derivatives with respect to normal optical depth τ_ν. One of these expansions may be written in the form

$$J_\nu = S_\nu(1+\delta_\nu), \tag{4.18}$$

where δ_ν involves (S_ν/μ_ν^2) and its derivatives; for $\tau_\nu \gg 1$ (in the deep interior), we have (*cf.* (6.30a))

$$\delta_\nu=\frac{1}{3}\frac{1}{(S_\nu/\mu_\nu^2)}\frac{\mathrm{d}^2(S_\nu/\mu_\nu^2)}{\mathrm{d}\tau_\nu^2}+\frac{1}{5}\frac{1}{(S_\nu/\mu_\nu^2)}\frac{\mathrm{d}^4(S_\nu/\mu_\nu^2)}{\mathrm{d}\tau_\nu^4}+\cdots* \tag{4.19}$$

* See footnote page 127.

Substituting (4.18) into (4.17), we obtain

$$S_\nu = (1-\sigma)\mu_\nu{}^2 B_\nu(T) + \sigma S_\nu(1+\delta_\nu),$$

which gives for $S_\nu/\mu_\nu{}^2$:

$$\frac{S_\nu}{\mu_\nu{}^2} = B_\nu(T) \cdot \frac{1}{1-\dfrac{\sigma}{1-\sigma}\delta_\nu}. \tag{4.20}$$

We see from (4.20) that the condition that $S_\nu/\mu_\nu{}^2 \to B_\nu(T)$ is then that

$$\frac{\sigma}{1-\sigma}\delta_\nu \ll 1. \tag{4.21}$$

To order of magnitude, $|\delta_\nu| \sim 1/|\Delta\tau_\nu|^2$ (at great optical depths), where $\Delta\tau_\nu \sim \kappa_\nu \rho \ell$ is the increment in optical depth over which $S_\nu/\mu_\nu{}^2$ (or S_ν if $\mu_\nu = 1$) changes by an appreciable amount (for example, by an appreciable fraction of itself), and ℓ is the corresponding geometrical (radial) distance; in a very crude approximation, ℓ may be comparable, in the deep interior, to the stellar radius R or some fraction thereof. Note that $\Delta\tau_\nu$ is also equal to the number of photon mean free paths in the geometrical distance ℓ. Hence, according to the theory of random walk, $|\Delta\tau_\nu|^2$ is of the order of the total number of absorptions and scatterings N that an "average" photon will undergo in diffusing through the distance ℓ. Accordingly, $\delta_\nu \sim 1/N$. Note that $|\Delta\tau_\nu|^2 (\sim N)$ generally increases rapidly with increasing depth in a star. Inspection of (4.18) and (4.19) then illustrates the important result that in regions of large optical depth, $J_\nu \to S_\nu$, *independently of any other effects.* However, the question of how $S_\nu/\mu_\nu{}^2 \to B_\nu(T)$ is seen form (4.21) to involve scattering effects also. Note that $\sigma/(1-\sigma) = \kappa_\nu^{(s)}/\kappa_\nu'$, *i.e.*, the ratio of the scattering absorption coefficient to the "true" absorption coefficient (corrected for induced emission). Hence (4.21) can also be written as

$$|\Delta\tau_\nu|^2 \cdot \frac{1-\sigma}{\sigma} = |\Delta\tau_\nu|^2 \cdot \frac{\kappa_\nu'}{\kappa_\nu^{(s)}} \sim N \cdot \frac{\kappa_\nu'}{\kappa_\nu^{(s)}} \gg 1. \tag{4.22}$$

Since N is very large in the deep interior of a star ($\sim 10^{24}$ if ℓ is set equal to R and if $\mu_\nu \sim 1$), then (4.22) states that the ratio $\kappa_\nu^{(s)}/\kappa_\nu'$ cannot exceed unity by more than, say 15 or so orders of magnitude; but it can still be very large

* The use of *normal* optical depths implies that the curvature of the ray paths is being neglected, which as we shall see (Chap. 6) is amply justified for most problems in stellar interiors. The first term of (4.18) is not affected at all at great optical depths by curvature of the ray paths, and the order of magnitude of δ_ν in (4.18) should not be seriously affected by curvature of the ray paths (*cf.* Chap. 6).

compared to unity and still permit (4.22) to be satisfied (provided that N is sufficiently large, *i.e.*, provided that the material is sufficiently "opaque"). Recall that N is equal to the total number of absorptions (followed by re-emissions) and scatterings that an "average" photon undergoes in diffusing through the distance ℓ over which $S_\nu/\mu_\nu{}^2$ may change appreciably, and that $\kappa_\nu'/\kappa_\nu^{(s)}$ is essentially the ratio of the probability of absorption (followed by re-emission) to that of scattering for a photon. Hence, (4.22) may also be regarded as stating that the photon must undergo many "true" absorptions and successive re-emissions in diffusing through the distance ℓ (even though the photon may be scattered many more times than it is absorbed and re-emitted), in order for $S_\nu/\mu_\nu{}^2$ to be approximately equal to $B_\nu(T)$. The many "true" absorptions insure that the radiation is strongly "coupled" to the matter, and thus that the radiation temperature will tend to be the same as the matter temperature, *i.e.*, that $S_\nu/\mu_\nu{}^2 \simeq B_\nu(T)$.

In the outer layers of all except the very hottest stars, $\kappa_\nu'/\kappa_\nu^{(s)}$ exceeds unity, usually by a large factor, so that (4.22) can be satisfied even at moderately small optical depths. In the interiors of very hot stars, usually $\kappa_\nu^{(s)}/\kappa_\nu'$ exceeds unity, but never by very many orders of magnitude. Since in these deeper regions, however, $|\Delta\tau_\nu|$ may already exceed 10^{10} or so, then (4.22) is satisfied in the interior also. We may therefore set $S_\nu/\mu_\nu{}^2 = B_\nu(T)$ or $S_\nu = B_\nu(T)$ if $\mu_\nu = 1$ (*provided* that LTE, *i.e.*, (4.3), obtains), to high accuracy throughout most of the interiors of essentially all stars, including even very hot stars whose opacity arises primarily from electron scattering. Consideration of the validity of the LTE assumption for stellar interiors is found in Chap. 7.

CHAPTER 5

Thermal and Radiative Equilibrium

As used in the theory of the stellar interior, the term *thermal equilibrium*, to be defined in Sect. 5.1, has a different meaning from the term as used in thermodynamics (*cf.* Chaps. 3, 4, 7, and 9). We shall use this term in the "stellar interior" sense in this chapter and, generally, throughout the book except in cases where the thermodynamic meaning is obviously appropriate. The reader should have no difficulty in deciding which of the two meanings is intended in any given context.

The term *radiative equilibrium*, to be defined in Sect. 5.2, is widely used in discussions of stellar interiors and stellar atmospheres. Since, as we shall see, radiative equilibrium is a special case of thermal equilibrium, we consider the more general case of thermal equilibrium first (Sect. 5.1), and then specialize the discussion (Sect. 5.2) to the case of radiative equilibrium. We assign a special section to radiative equilibrium because of its general importance in astrophysics and because it has been the subject of much discussion in the literature pertaining to stellar interiors and stellar atmospheres.

5.1 Thermal Equilibrium

By *thermal equilibrium* we shall mean the maintenance of a *steady state* condition throughout a star over time intervals of the order of the "nuclear" time t_{nuc}, or at least over time intervals long compared with the "Kelvin" time t_{K} (*cf.* Chap. 0). The only way that such a steady state condition can be realized over such long time intervals is for the star to be, at least on the average, in almost perfect *energy* balance; *i.e.*, the total rate at which energy is irreversibly released by sources (such as nuclear sources) in any small region of the star must be equal, at least on the average, to the rate at which energy is carried away from the region by all the various transport mechanisms (radiation, convection, etc.) that may be operative. Because t_{K} is essentially

the "relaxation time" for departures from thermal equilibrium (energy balance), it is clear that considerations of thermal equilibrium for a star as a whole are meaningful only in connection with time intervals longer than t_K. In the case of a pulsating star, for example, thermal equilibrium certainly does *not* obtain over times comparable to the pulsation period (*cf.* Chap. 27), although it might obtain, on the average, over *many* pulsation periods. A star with *no* nuclear energy sources clearly cannot be in thermal equilibrium (unless it emits no energy, as in the case of a "black dwarf," *cf.* Chap. 25), because its radius will decrease through gravitational contraction by an appreciable amount in a time of the order of t_K (*cf.* Sect. 17.4).

Since in connection with questions of thermal equilibrium we are always concerned with "long" time intervals, we shall in this chapter and, in fact, throughout the book always assume, unless we specifically state otherwise, that all rates of energy generation and all energy fluxes shall have been suitably averaged over time, if necessary, so as to remove any short-term fluctuations. (In the case of convective transfer, for example, the convective flux at a given point may fluctuate with time, and the time scale for the fluctuations may be of the order of or less than the mean "eddy lifetime" (*cf.* Chap. 14)).

Let $\mathbf{F}$ denote the total vector net flux of energy (energy per unit area per unit time) in some direction at some arbitrary fixed point, including contributions from whatever transport mechanisms are effective, and let ε denote the total amount of energy generated in the vicinity of the point, per unit mass and time, by the sources. Considering some arbitrary fixed volume V enclosed by the fixed closed surface S in the star, the condition of thermal equilibrium clearly demands that

$$\oint_S \mathbf{F} \cdot d\mathbf{S} = \int_V \rho \varepsilon dV,$$

where ρ is the local mass density, and where the fluid is assumed to be, at least as far as macroscopic motions are concerned, at rest. Applying the divergence theorem to the left side of this equation and making use of the arbitrariness of the volume V, we have

$$\nabla \cdot \mathbf{F} = \rho \varepsilon \tag{5.1}$$

as an expression of the condition of thermal equilibruim.

More explicitly, we might write, for example, with considerable generality,

$$\mathbf{F} = \mathbf{F}_{\text{rad}} + \mathbf{F}_{\text{cond}} + \mathbf{F}_{\text{conv}} + \mathbf{F}_{\text{neut}} + \mathbf{F}_{\text{mass}}, \tag{5.2}$$

where the first three terms on the right denote, respectively, the radiative,

conductive, and convective fluxes; $\mathbf{F}_{\text{neut}}$ denotes the flux carried by neutrinos; and $\mathbf{F}_{\text{mass}}$ represents a flux of (kinetic and potential) energy resulting from the net transfer of matter (for example, $\mathbf{F}_{\text{mass}}$ might represent the energy flux carried by particles escaping from the surface regions of a star undergoing mass loss—in this case $\mathbf{F}_{\text{mass}}$ would be essentially zero in the interior regions and would rise to a finite value in the surface regions). Of course, all five kinds of energy transfer indicated in (5.2) would not, in general, simultaneously occur in any one region of a star.

We might also write, for example,

$$\varepsilon = \varepsilon_{\text{nuc}} + \varepsilon_{\text{vis}}\,, \tag{5.3}$$

where the two terms on the right denote the rates of energy production per unit mass from, respectively, nuclear sources and viscous heating arising from fluid motions. Thus we would have, in thermal equilibrium,

$$\nabla \cdot (\mathbf{F}_{\text{rad}} + \mathbf{F}_{\text{cond}} + \mathbf{F}_{\text{conv}} + \mathbf{F}_{\text{neut}} + \mathbf{F}_{\text{mass}}) = \rho(\varepsilon_{\text{nuc}} + \varepsilon_{\text{vis}})\,, \tag{5.4}$$

where ε_{vis} is customarily neglected (since it is indeed negligible in most astrophysically interesting cases, *cf.* Sect. 27.6d), as it will be throughout this book except where stated otherwise; $\mathbf{F}_{\text{neut}}$ and $\mathbf{F}_{\text{mass}}$ are probably important only for certain kinds of stars and in certain stages of stellar evolution (*cf.* Chap. 26); and $\mathbf{F}_{\text{cond}}$ is appreciable only in highly degenerate stars (*cf.* Sect. 16.7).

It is important to note that it is customary to exclude from ε_{nuc} any nuclear energy which appears *directly* in the form of neutrinos (*i.e.*, energy carried by neutrinos which are *by-products of the nuclear reactions*), since neutrinos, once formed, generally escape readily from the star with essentially no further interaction (see the end of Sect. 17.16); the energy contained in these neutrinos is therefore essentially never available to the star at all and so may be omitted forthwith from the energy balance (this direct neutrino energy "loss" accompanying certain nuclear reactions is taken into account in the calculation of the Q's in the theory of nuclear reactions (*cf.* Chap. 17)). With this convention, then, the neutrino flux $\mathbf{F}_{\text{neut}}$ in (5.4) must be interpreted as *that flux of neutrinos which does not originate directly in nuclear reactions;* as we shall see in Chap. 17, there is reason to believe that a number of processes for the direct production of neutrinos, quite independently of nuclear reactions, may exist. We shall henceforth, unless we state explicitly otherwise, always interpret ε_{nuc} and $\mathbf{F}_{\text{neut}}$ in this manner.

This characteristic of neutrinos that, once formed, they essentially never interact with stellar material permits a special interpretation to be given to the (non-nuclear) neutrino flux $\mathbf{F}_{\text{neut}}$ in (5.4). If we assume that there are no

sinks of neutrinos, then $\nabla \cdot \mathbf{F}_{\text{neut}}$ can never be negative: $\nabla \cdot \mathbf{F}_{\text{neut}} \geq 0$. We may then define a "neutrino energy production rate" $\varepsilon'_{\text{neut}}$ by the equation

$$\nabla \cdot \mathbf{F}_{\text{neut}} \equiv \rho \varepsilon'_{\text{neut}}, \tag{5.5}$$

where $\varepsilon'_{\text{neut}} \geq 0$ and gives the rate per unit mass at which (non-nuclear) neutrino energy is being generated at the expense of other forms of energy (such as photonic). (Equation (5.5) may also formally be derived by setting up an equation of transfer for the "specific intensity" of neutrino energy and then setting the "absorption coefficient" equal to zero; see (5.16) in the next section.) We may then transfer $\nabla \cdot \mathbf{F}_{\text{neut}}$ to the right side of (5.4), which may then be written in the form

$$\nabla \cdot \mathbf{F}_{\text{non-neut}} = \rho(\varepsilon_{\text{nuc}} + \varepsilon_{\text{vis}} - \varepsilon'_{\text{neut}}), \tag{5.6}$$

where $\mathbf{F}_{\text{non-neut}}$ denotes the total net *non-neutrinic* (*i.e.*, "sensible") flux. Since $\varepsilon'_{\text{neut}} \geq 0$ (if we assume that there are no sinks for neutrinos), then, *from the standpoint of the non-neutrinic forms of energy*, this term acts as an energy *sink*: once non-neutrinic energy has been converted into neutrino energy, it escapes directly from the star and is thus no longer available for conversion into other forms of energy. It is therefore customary to write (5.6) in the form

$$\nabla \cdot \mathbf{F}_{\text{non-neut}} = \rho(\varepsilon_{\text{nuc}} + \varepsilon_{\text{vis}} + \varepsilon_{\text{neut}}), \tag{5.7}$$

where $\varepsilon_{\text{neut}}(\leq 0) \equiv -\varepsilon'_{\text{neut}}$ is regarded as another "source" term, but is always negative (this convention is followed in Chap. 17). It is thus seen that $\nabla \cdot \mathbf{F}_{\text{non-neut}}$ could even become negative in certain regions of a star (conversion of other forms of energy into neutrino energy) if $|\varepsilon_{\text{neut}}| > |\varepsilon_{\text{nuc}}|$ (as can probably happen in some phases of stellar evolution, *cf.* Sects. 26.4 and 26.5).

We note that there is no advantage to treating the other fluxes in (5.4) as we have treated $\mathbf{F}_{\text{neut}}$ because of the possibility that they may have a negative divergence. A negative divergence amounts to an "absorption" of the "carriers" (photons, particles, convecting elements) of a particular mode of energy transfer, *i.e.*, a transformation of one form of energy into another through interaction of the "carriers" with the surrounding matter.

If we now apply (5.1) to a star and assume spherical symmetry, then $\mathbf{F}$ will be directed radially and will depend only on the distance r from the center of the star: $|\mathbf{F}| \equiv F(r)$. In spherical polar coordinates and for spherical symmetry (5.1) becomes

$$\nabla \cdot \mathbf{F} = \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}(r^2 F) = \frac{1}{4\pi r^2}\frac{\mathrm{d}}{\mathrm{d}r}(4\pi r^2 F) = \rho\varepsilon.$$

Defining

$$L(r) \equiv 4\pi r^2 F(r) \tag{5.8}$$

as the "interior luminosity" (net rate of flow of energy outward through a sphere of radius r), we have

$$\frac{dL(r)}{dr} = 4\pi r^2 \rho\varepsilon \tag{5.9}$$

as an alternative expression of the condition of thermal equilibrium in a spherically symmetric star. Equation (5.9) is one of the basic equations of stellar structure. The total luminosity of the star is then given, in a steady state (over time intervals long compared with t_K), by

$$L = \int_0^R 4\pi r^2 \rho\varepsilon dr. \tag{5.10}$$

If $\varepsilon = \varepsilon_{nuc} + \varepsilon_{vis}$, then L in (5.10) must be associated, in general, with the *total* rate of loss of energy from the star, in the form of photons, (non-nuclear) neutrinos, and particle emission (mass loss). If, on the other hand, $\varepsilon = \varepsilon_{nuc} + \varepsilon_{vis} + \varepsilon_{neut}$, then L in (5.10) may include (in general) only photons and particles, but not neutrinos.

We note a simpler derivation of (5.9). Since ε denotes the net local rate of energy production per unit mass and $dM(r) = 4\pi r^2 \rho dr$ denotes the mass of an elementary spherical shell of thickness dr, we have $dL(r) = \varepsilon dM(r) = 4\pi r^2 \rho\varepsilon dr$, which is (5.9).

In the case of a star in hydrostatic equilibrium with no energy sources ($\varepsilon = 0$ everywhere) but which is nevertheless radiating energy, it is clear that $\nabla \cdot \mathbf{F}$ cannot be zero throughout the entire star, and hence that (5.1) cannot be satisfied, *i.e.*, the star cannot be in thermal equilibrium. The star is, in fact, slowly contracting in this case, with the energy lost through the stellar surface being supplied by the gravitational energy released through the contraction (*cf.* Sect. 17.4). In this case it is shown in Sect. 17.6 that $\nabla \cdot \mathbf{F}$ is given by the time rate of change per unit volume of certain local thermodynamic quantities (*cf.* (17.74), (17.75), (17.75′) below); this time rate of change per unit volume is often written as $\rho\varepsilon_{grav}$. With ε_{grav} included on the right sides of (5.4) or (5.6), these equations (and all subsequent equations based on them) are then formally valid for stars which, according to our definition, are not in thermal equilibrium (such as a gravitationally contracting star). In this book, however, we do not generally include the term ε_{grav} in this manner, so that (5.4) or (5.6) will usually be taken as the conditions for thermal equilibrium (exceptions to this practice will be pointed out when they are made).

5.2 Radiative Equilibrium

The important special case of thermal equilibrium in which all the (non-neutrinic) energy is carried by radiation,

$$\mathbf{F}_{\text{non-neut}} = \mathbf{F}_{\text{rad}} = \int_0^\infty \mathbf{F}_\nu d\nu$$

($\mathbf{F}_\nu$ = monochromatic vector net flux), possibly in only a portion of the star, is known as *radiative equilibrium*. In this case the condition of thermal equilibrium, (5.1), becomes the condition of radiative equilibrium:

$$\nabla \cdot \int_0^\infty \mathbf{F}_\nu d\nu = \rho\varepsilon, \tag{5.11}$$

where ε, the net local rate of energy production per unit mass, may in general include $\varepsilon_{\text{neut}}$ (see the preceding Sect. 5.1).*

Equation (5.11) can be written in an alternative form by making use of the (time-independent) equation of transfer in the general form

$$\frac{1}{\rho}\frac{d}{ds}\left(\frac{I_\nu}{\mu_\nu^2}\right) = \frac{j_\nu - \kappa_\nu I_\nu}{\mu_\nu^2}, \tag{5.12}$$

where μ_ν is the (real) refractive index and where, as was pointed out in Sect. 2.9, the differentiation d/ds must be taken along a (possibly curved) ray trajectory. With d/ds written out explicitly, (5.12) assumes the form

$$\frac{1}{\rho}\mathbf{n}\cdot\nabla\left(\frac{I_\nu}{\mu_\nu^2}\right) + \frac{1}{\rho}\frac{d\mathbf{n}}{ds}\cdot\nabla_n\left(\frac{I_\nu}{\mu_\nu^2}\right) = \frac{j_\nu - \kappa_\nu I_\nu}{\mu_\nu^2}, \tag{5.13}$$

where $\mathbf{n}$ is a unit vector in some arbitrarily selected direction of observation at the point $\mathbf{r}$ under consideration ($\mathbf{n}$ is also the unit tangent vector of the ray path followed by the photons passing through $\mathbf{r}$ in direction $\mathbf{n}$); d$\mathbf{n}$/ds is the spatial rate of change of direction of the photons brought about by spatial variation of μ_ν following along the ray path specified by $\mathbf{r}$ and $\mathbf{n}$; ∇ is the ordinary spatial gradient vector operator; and ∇_n is the "directional"

* Note that the energy which is capable of heating up the stellar material produced by the sources is generally *not* immediately in the form of photons. For example, in the case of nuclear sources the energy generally appears either in the form of high energy gamma rays or as particle kinetic energy, or both. However, in a time of the order of a few particle or photon "collision" times, this energy will have been thermalized, or distributed to other particles and photons, and will result in a local heating of the matter in the vicinity of the point under consideration ; this excess local heat can then be transported as radiation if we have radiative equilibrium. In (5.11) it is assumed that this "thermalization time" is small compared with other times of interest (as it ordinarily is) and is accordingly neglected.

gradient vector operator (*cf.* Sect. 2.11). The interpretation of the two terms on the left side of (5.13) was explained in Sect. 2.9. We note that, since ∇ implies only *spatial* differentation, with direction (say $\mathbf{n}$) held fixed, we may write $\mathbf{n} \cdot \nabla(I_\nu/\mu_\nu^2) = \nabla \cdot (I_\nu \mathbf{n}/\mu_\nu^2)$, so that (5.13) may also be written in the form

$$\frac{1}{\rho} \nabla \cdot \left(\frac{I_\nu \mathbf{n}}{\mu_\nu^2} \right) + \frac{1}{\rho} \frac{d\mathbf{n}}{ds} \cdot \nabla_n \left(\frac{I_\nu}{\mu_\nu^2} \right) = \frac{j_\nu - \kappa_\nu I_\nu}{\mu_\nu^2}. \tag{5.13'}$$

We now multiply (5.13′) through by $d\omega_{\mathbf{n}}$, the element of solid angle about $\mathbf{n}$, and integrate over all solid angles. Noting the definition of the vector net flux (*cf.* Sect. 2.2),

$$\mathbf{F}_\nu = \int_{4\pi} I_\nu(\mathbf{n}) \mathbf{n} \, d\omega_{\mathbf{n}}, \tag{5.14}$$

and recalling (*cf.* Chap. 2) that in this book we consider only isotropic media, so that μ_ν is not a function of direction $\mathbf{n}$, we obtain from (5.13′) the relation

$$\frac{1}{\rho} \nabla \cdot \left(\frac{\mathbf{F}_\nu}{\mu_\nu^2} \right) + \frac{1}{\rho} \int_{4\pi} \frac{d\mathbf{n}}{ds} \cdot \nabla_n \left(\frac{I_\nu}{\mu_\nu^2} \right) d\omega_{\mathbf{n}} = \frac{1}{\mu_\nu^2} \int_{4\pi} (j_\nu - \kappa_\nu I_\nu) d\omega_{\mathbf{n}}. \tag{5.15}$$

Consider first the simple case where μ_ν is constant in space (however, see next paragraph). Then (*cf.* Sect. 2.9) $d\mathbf{n}/ds = 0$, so that the second term on the left side vanishes, and we have simply

$$\nabla \cdot \mathbf{F}_\nu = \rho \int_{4\pi} (j_\nu - \kappa_\nu I_\nu) d\omega_{\mathbf{n}}. \tag{5.16}$$

Using this result in (5.11) we have, finally

$$\int_0^\infty \int_{4\pi} (j_\nu - \kappa_\nu I_\nu) d\omega_{\mathbf{n}} d\nu = \varepsilon \tag{5.17}$$

as an alternative expression of the condition of radiative equilibrium.

The results (5.16) and (5.17), however, are valid also when the (real) refractive index μ_ν is *not* constant in space, as is expected on physical grounds. To see that this statement is true, it will be necessary to evaluate the integral appearing on the left side of (5.15). Calling this integral I for the moment, we first integrate by parts, to obtain

$$I \equiv \int_{4\pi} \frac{d\mathbf{n}}{ds} \cdot \nabla_n \left(\frac{I_\nu}{\mu_\nu^2} \right) d\omega_{\mathbf{n}} \tag{5.18}$$

$$I = \int_{4\pi} \nabla_n \cdot \left(\frac{I_\nu}{\mu_\nu^2} \frac{d\mathbf{n}}{ds} \right) d\omega_{\mathbf{n}} - \int_{4\pi} \frac{I_\nu}{\mu_\nu^2} \nabla_n \cdot \left(\frac{d\mathbf{n}}{ds} \right) d\omega_{\mathbf{n}}. \tag{5.19}$$

The first integral in (5.19), however, vanishes, according to Sect. 2.11, because the vector $(I_\nu/\mu_\nu^2)\mathrm{d}\mathbf{n}/\mathrm{d}s$ is always perpendicular to $\mathbf{n}$.

Consider now the second integral in (5.19). To evaluate the indicated "directional divergence," we make use of the geometrical optics equation (2.95) which gives $\mathrm{d}\mathbf{n}/\mathrm{d}s$ as a function of $\mathbf{n}$ and $\nabla\mu_\nu$. Noting that ∇_n does not operate on μ_ν because μ_ν is assumed not to be a function of direction (*cf.* Chap. 2), we have

$$\nabla_n \cdot \left(\frac{\mathrm{d}\mathbf{n}}{\mathrm{d}s}\right) = \nabla_n \cdot \left\{\frac{1}{\mu_\nu}\left[\nabla\mu_\nu - \mathbf{n}\,\mathbf{n}\cdot\nabla\mu_\nu\right]\right\} =$$

$$= -\frac{1}{\mu_\nu}\left[\nabla_n \cdot (\mathbf{nn})\right]\cdot\nabla\mu_\nu = -\frac{2}{\mu_\nu}\mathbf{n}\cdot\nabla\mu_\nu, \tag{5.20}$$

where the last equality follows from an identity established in Sect. 2.11. The integral I then has the value (*cf.* 5.14)

$$I = \frac{2}{\mu_\nu^3}\,\mathbf{F}_\nu \cdot \nabla\mu_\nu. \tag{5.21}$$

On the other hand, expanding out the first term in (5.15), we obtain $\nabla\cdot(\mathbf{F}_\nu/\mu_\nu^2) = (\nabla\cdot\mathbf{F}_\nu)/\mu_\nu^2 - (2/\mu_\nu^3)\mathbf{F}_\nu\cdot\nabla\mu_\nu$. It is obvious that the last term of this expression cancels the integral I, and we recover (5.16), and hence also (5.17), which are therefore also valid when μ_ν varies with position, Q.E.D.*

It is clear from the definitions of the mass emission and mass absorption coefficients j_ν and κ_ν that the two terms of the left side of (5.17) represent, respectively, the rate of *emission* of radiant energy in all frequencies into all directions per unit mass and the rate of *absorption* of radiant energy in all frequencies from all directions per unit mass; and the net outward rate of flow of radiant energy in all frequencies into all directions per unit mass must clearly equal the rate of energy production ε per unit mass in a condition of radiative equilibrium.

We note that the condition of radiative equilibrium (5.17) is formulated only with respect to the *integrated* net flux $\mathbf{F}$. Such a condition clearly cannot exist for the *monochromatic* net flux $\mathbf{F}_\nu$ in a star because of the presence of a temperature gradient. A temperature gradient will result in a redistribution of frequencies toward shorter wavelengths as one descends into a star (assuming the temperature to increase inward). Hence $\nabla\cdot\mathbf{F}_\nu \neq 0$ in general, even when no energy sources are present.

* The deduction of the equations (5.16) and (5.17) of radiative equilibrium from the more general form (5.13) of the equation of transfer (which includes effects of a spatially variable refractive index) has not, to the best of our knowledge, previously appeared in the published literature.

To examine further the implications of this fact, consider (for the sake of dealing with simpler formulas) (5.16) for the case of LTE. We recall that, to obtain the correct formulas for LTE, we simply replace j_ν by $\kappa_\nu S_\nu$, where

$$S_\nu = (1-\sigma)\mu_\nu{}^2 B_\nu(T) + \sigma J_\nu^1$$

if isotropic, coherent scattering is present, and we understand that κ_ν is given explicitly by

$$\kappa_\nu = \kappa_\nu^{(a)}(1-e^{-h\nu/kT}) + \kappa_\nu^{(s)}.$$

We note that S_ν is isotropic in this case, whence (5.16) becomes

$$\frac{1}{\rho}\nabla\cdot\mathbf{F}_\nu = 4\pi\kappa_\nu(S_\nu - J_\nu), \tag{5.22}$$

where we assume that $\sigma<1$.

Consider a region in the star (such as the outer envelope) where $\varepsilon = 0$, so that $F = \int_0^\infty F_\nu d\nu$ can vary only slowly with r (as $1/r^2$) for a star in a steady state. We also assume that κ_ν is a smooth and slowly varying function of ν (essentially a grey case), so that F_ν is also a smooth function, with a well-defined maximum. Hence, at two closely separated depths (Depth 2> Depth 1, say), the spectral distributions will be as shown schematically in

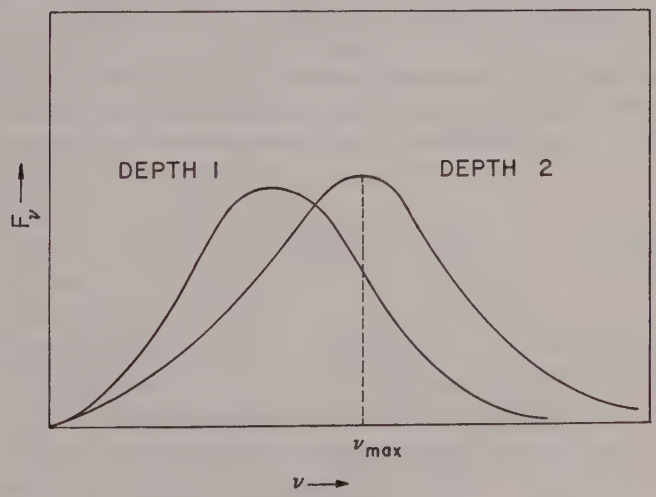


Fig. 5.1 Schematic spectral energy curves for two depths (Depth 2 > Depth 1).

Fig. 5.1, with the maximum of F_ν occurring at higher frequencies for Depth 2 than for Depth 1; note also that the areas under the two curves are practically the same (Area 2 slightly greater than Area 1).

Letting ν_{max} be the frequency at which F_ν is maximum at a given depth, we may draw the following conclusions from Fig. 5.1:

1. ν_{max} must be an increasing function of depth.
2. For $\nu > \nu_{max}$, $\nabla \cdot \mathbf{F}_\nu < 0$, whence $S_\nu < J_\nu$, so that absorption of radiation of frequency ν exceeds emission at that frequency.
3. For $\nu < \nu_{max}$, $\nabla \cdot \mathbf{F}_\nu > 0$, whence $S_\nu > J_\nu$, so that emission of radiation of frequency ν in this case exceeds absorption of radiation at that frequency.

We thus see that the temperature gradient provides, at each depth, a *sink* for high-frequency radiation and a *source* for low-frequency radiation. In other words, high-energy photons are transformed into low-energy photons as the radiation diffuses out toward the surface of the star (looking inward, toward the stellar center, one would see higher temperatures, and therefore bluer radiation, than when looking in the opposite direction). It is clear, therefore, that, because $S_\nu \neq J_\nu$ in general, *detailed balancing* does not apply even in a star in which LTE obtains; the essential reason is the presence of the temperature gradient, which clearly precludes perfect *t.e.*, even locally.*

* This conclusion that detailed balancing does not obtain even under conditions of LTE has been confirmed and discussed in a recent paper by Wildt [Wi66]. Wildt has also noted that the divergence of the total (integrated) photon flux is everywhere positive in a grey atmosphere in LTE. This is simply a consequence of the fact that high-energy (blue) photons have more energy than low-energy (red) photons. Since at each depth $\nabla \cdot \int_0^{\nu_{max}} \mathbf{F}_\nu d\nu + \nabla \cdot \int_{\nu_{max}}^{\infty} \mathbf{F}_\nu d\nu = 0$ in radiative equilibrium, and since the first term in this equation is positive (see above), it follows that the divergence of the total photon flux $\nabla \cdot \int_0^{\nu_{max}} (\mathbf{F}_\nu/h\nu)\, d\nu + \nabla \cdot \int_{\nu_{max}}^{\infty} (\mathbf{F}_\nu/h\nu)\, d\nu > 0$, since there are more red photons (photons with $\nu > \nu_{max}$) than there are blue photons (photons with $\nu \geq \nu_{max}$).

CHAPTER 6

Solution of the Equation of Transfer

In this chapter we shall obtain solutions of the equation of transfer that are particularly useful for work in stellar interiors. These solutions, although formally applicable in the stellar atmosphere also, are actually often not too useful in these outermost regions because our mathematical assumptions regarding convergence are often not justified.

We have, for generality, carried through in this chapter the case of a non-unity and spatially variable (real) refractive index μ_ν, although in most cases of interest μ_ν is very close to unity (see the introduction to Chap. 2 and Sect. 6.1). It will be easy for the reader who is not interested in this more general case simply to set $\mu_\nu = 1$ in the formulae of this chapter.

In this chapter we shall be concerned, specifically, with only a special kind of symmetry, namely *spherical symmetry* about the stellar center and *axial symmetry* about every radial direction. The work in this chapter will be almost exclusively formal; some physical applications will be made in Sect. 6.4.

In Sect. 6.1 we obtain *formal* solutions of the equation of transfer, based on certain boundary conditions at the stellar surface and in the deep stellar interior. In Sect. 6.2 we obtain, on the basis of these formal solutions of the equation of transfer, formal expressions for J_ν, F_ν, and $p_{r,\nu}$ in terms of an isotropic source function S_ν for the case where the ray paths can be considered "straight" (see Sect. 6.1). In Sect. 6.3 we apply the formal solutions and the above formal expressions to the case where the quantity (S_ν/μ_ν^2) can be written as a convergent power series about the optical depth of interest. Finally, in Sect. 6.4 we examine in a semi-quantitative way the convergence of the power series solutions and of the expansions based on them.

6.1 Formal Solutions

Under conditions of LTE (*cf.* Chap. 4), the (time-independent) equation of transfer is of the form

$$\frac{dI'_\nu}{\kappa_\nu \rho ds} = S'_\nu - I'_\nu, \tag{6.1}$$

where ds is an element of distance in some arbitrary direction along a (possibly curved) ray path, measured in the direction of motion of the photons following along the ray path, and primes denote *throughout this chapter* division by μ_ν^2 (except where stated otherwise), where μ_ν is the (real) refractive index. Hence

$$f' \equiv f/\mu_\nu^2, \tag{6.1'}$$

where f stands for any radiative quantity such as I_ν, S_ν, J_ν, etc. Clearly, $f'=f$ if $\mu_\nu = 1$. Hence, readers who are not interested in effects of dispersion can simply disregard the primes and set $\mu_\nu = 1$ in all the formulae in this chapter.

We now define the element of *optical depth* $d\sigma_\nu$, where σ_ν increases in the *backward* direction from the direction of observation (opposite to the direction of motion of the photons) along a (possibly curved) ray path:

$$d\sigma_\nu \equiv -\kappa_\nu \rho ds. \tag{6.2a}$$

For some purposes it is convenient to define the element of *normal optical depth* $d\tau_\nu$ as follows:

$$d\tau_\nu = -\kappa_\nu \rho dr, \tag{6.2b}$$

where dr is an element of radial distance r (say from the stellar center) corresponding to two points separated by the distance ds along a ray path. The convention usually adopted for *normal* optical depth τ_ν is that it always *increases inward* (*i.e.*, τ_ν increases as r decreases), irrespective of the direction of motion of the photons. The relation between $d\sigma_\nu$ and $d\tau_\nu$ is clearly

$$\frac{d\sigma_\nu}{d\tau_\nu} = \frac{ds}{dr} = \frac{1}{\mathbf{n}\cdot\mathbf{e}_r} = \frac{1}{\cos\theta} \equiv \frac{1}{\mu}, \tag{6.2c}$$

where θ is the angle between the unit vectors $\mathbf{n}$ (the instantaneous tangent vector of a ray path, $d\mathbf{s} = \mathbf{n}ds$) and $\mathbf{e}_r$ (which defines the outward radial direction from the origin of coordinates (usually the stellar center), $\mathbf{r} = \mathbf{e}_r r$) (see Fig. 6.1). The last equality in (6.2c) defines μ. The equation of transfer (6.1) can then be written in the two alternative forms:

$$\frac{dI'_\nu}{d\sigma_\nu} = I'_\nu - S'_\nu, \tag{6.3a}$$

$$\mu \frac{dI'_\nu}{d\tau_\nu} = I'_\nu - S'_\nu. \tag{6.3b}$$

We note that, in general, μ depends on the value of r (or of τ_ν) at different points along a particular ray path. This statement will be true if the ray path is curved, as will be the case if the (real) refractive index μ_ν is not spatially

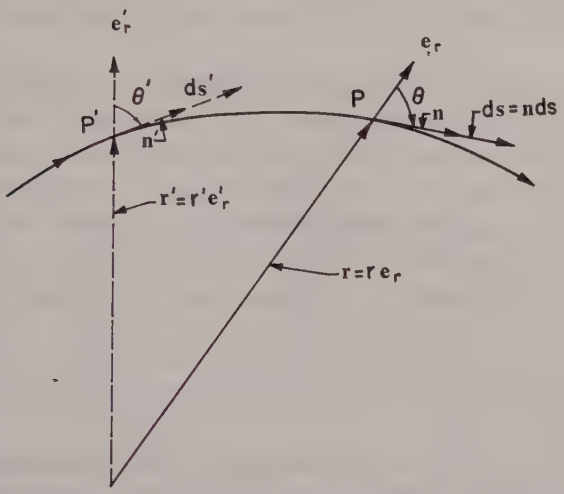


Fig. 6.1 Relations between unit vectors on a (curved) ray path.

constant (*cf.* Sect. 2.9); and also even if the ray path is straight but if the direction of $\mathbf{e}_r$ at point P', say, differs from that at point P (see Fig. 6.1). In any case, if μ in general depends on r along a ray path, then the form (6.3a) of the transfer equation is the more convenient; otherwise, the form (6.3b) is usually more convenient. For brevity, we shall refer to these two cases, respectively, as cases of ray paths which are "curved" (μ in general a function of r along a ray path) and "straight" (μ independent of r along every ray path), even though both cases can be realized with truly straight ray paths. In most of the conventional astrophysical literature (except in radio astronomy and possibly in studies of the solar corona) only the case of "straight" ray paths is considered. However, if the curvature of the stellar layers is sufficiently great (as, for example, in the case of red supergiant atmospheres), or if the ray paths are appreciably curved (as, for example, in the case of radio waves propagating through the solar corona), then it will be found convenient to use the form (6.3a) of the transfer equation.

It is clear intuitively that "straight" ray paths are an adequate approximation if the following order-of-magnitude conditions are satisfied:

$$\lambda_p \ll r, \tag{6.3c}$$

$$\lambda_p \ll \lambda_\mu, \tag{6.3d}$$

where λ_p is the average photon mean free path and λ_μ is the local radius of curvature of a typical ray path passing through the point under consideration. Conditions (6.3c) and (6.3d) are based on the fact that most of the contributions to the radiation intensity at a given point originate at points removed from the given point by distances no greater than a few average photon mean free paths.

It is clear (*cf.* Chap. 4) that (6.3c) is well satisfied throughout most of a stellar interior where typical values of λ_p and r are $\lambda_p \sim 10^{-1}$ cm and $r \sim 10^{10}$ cm. The condition (6.3d) is also likely to be well satisfied in those regions of the stellar interior where $\mu_\nu \sim 1$ and is not abruptly changing (spatially). If we assume complete ionization and use the approximate formula (2.163) (see also (2.157)) for μ_ν (this formula is of the form $\mu_\nu = \sqrt{1 - \text{const. } N_e}$, where N_e = electron density), it is easily seen that

$$\lambda_\mu = \frac{2\mu_\nu^2}{1-\mu_\nu^2}\lambda_{N_e}, \tag{6.3e}$$

where λ_{N_e} is the electron density scale height (equal to the mass density scale height in the case of complete ionization, if all electrons present are ionization electrons (*cf.* Chap. 15)) and where we have taken (*cf.* Sect. 2.9) $\lambda_\mu^{-1} = |\nabla\mu_\nu|/\mu_\nu$. Equation (6.3e) shows that $\lambda_\mu \gg \lambda_{N_e}$ (which is typically of the order of a stellar radius or some fraction thereof) if $\mu_\nu \approx 1$. If $\mu_\nu \ll 1$, however, then λ_μ could become very small, and curvature of the ray paths could be important. We shall often make use of the "straight" ray path approximation (μ independent of r along every ray path) in this book, even in cases where μ_ν is not strictly constant with position.

In the remainder of this section we shall drop the subscript ν ($I \equiv I_\nu$, etc.) from all quantities except refractive index μ_ν (to avoid confusion with $\mu = \cos\theta$), although we shall still be referring, in general, to monochromatic quantities.

We shall find it convenient to consider the equation of transfer in the form (6.3a) in obtaining the formal solution; this formal solution will then apply to both "curved" and "straight" ray paths.

We note, letting ζ denote the integration variable instead of σ (see (6.2a)), that (6.3a) may be written in the form

$$\frac{\mathrm{d}}{\mathrm{d}\zeta}(I'e^{-\zeta}) = -S'e^{-\zeta}. \tag{6.4}$$

We consider first the "outward" intensity and denote it by a superscript "+". By this we mean the intensity of radiation flowing at the point under

consideration in the general direction of the stellar surface ($0<\mu\leq 1$). We consider here only those rays extending to infinite optical depths in the "inward" direction.* We also consider that the outward intensity at optical depth σ along a ray path is made up of contributions from all optical depths ζ along the ray path greater than σ. Integrating from ∞ to σ and assuming that

$$\lim_{\sigma\to\infty}(I'e^{-\sigma})=0,$$

we obtain

$$I'^{+}(\sigma)=\int_{\zeta=\sigma}^{\infty}S'e^{-(\zeta-\sigma)}\mathrm{d}\zeta=\int_{\zeta-\sigma=0}^{\zeta-\sigma=\infty}S'e^{-(\zeta-\sigma)}\mathrm{d}(\zeta-\sigma)\quad(0<\mu\leq 1). \tag{6.5a}$$

We note that, in general, $\sigma=\sigma(\mu)$ at a given spatial point, because every direction of observation generally corresponds to a different ray path having a different optical depth down to the given spatial point. The interpretation of (6.5a) is that the contribution $S'\mathrm{d}\zeta$ to $I'^{+}(\sigma)$, arising from the element of optical depth $\mathrm{d}\zeta$, is attenuated by the factor $e^{-(\zeta-\sigma)}$, where $\zeta-\sigma$ is the optical distance along the ray path between the contribution at optical depth ζ and the "point of observation" at depth σ.

In the "straight" ray path approximation (μ independent of ζ along every ray path) we have (*cf.* (6.2c)

$$\mathrm{d}\zeta=\mathrm{d}t/\mu,\qquad\sigma=\int_{0}^{\tau}\frac{\mathrm{d}t}{\mu(t)}=\frac{\tau}{\mu}, \tag{6.5a'}$$

whence (6.5a) becomes

$$I'^{+}(\tau,\mu)=\int_{t=\tau}^{t=\infty}S'e^{-(t-\tau)/\mu}\frac{\mathrm{d}t}{\mu}=\int_{t-\tau=0}^{t-\tau=\infty}S'e^{-(t-\tau)/\mu}\frac{\mathrm{d}(t-\tau)}{\mu}\quad(0<\mu\leq 1), \tag{6.5b}$$

where τ is now *normal* optical depth (independent of μ at a given spatial point).

For the *inward* intensity ($-1\leq\mu<0$), we assume that no radiation is incident on the surface of the star. Recalling that, according to our definition, optical depth always increases *backward* along the ray path, and associating

* In the case of strongly curved ray paths the requirement of infinite optical depth backward along the ray path may exclude some of the rays having values of μ in the range ($0<\mu\leq 1$) at the point under consideration.

zero optical depth with the stellar surface, it is easy to see that in this case the formal solution of (6.4) is

$$I'^{-}(\sigma) = -\int_{0}^{\sigma} S' e^{\sigma-\zeta} d\zeta \quad (-1 \leq \mu < 0).^{*} \tag{6.6a}$$

In accordance with our convention regarding the sign of optical depth measured along the ray path, σ must be a negative number. In the "straight" ray path approximation we may use (6.5a′) in (6.6a), to obtain (remembering that τ is taken as a positive number, see sentence following (6.2b))

$$I'^{-}(\tau, \mu) = -\int_{t=0}^{t=\tau} S' e^{(\tau-t)/\mu} \frac{dt}{\mu} =$$

$$= \int_{\tau-t=0}^{\tau-t=\tau} S' e^{-(\tau-t)/(-\mu)} \frac{d(\tau-t)}{(-\mu)} \quad (-1 \leq \mu < 0). \tag{6.6b}$$

From the second equality in (6.6b) we note that, for great optical depths ($\tau \to \infty$), $I'^{-}(\tau,\mu) = I'^{+}(\tau,-\mu)$, or that I'^{-} is the same function of μ as I'^{+} is of $-\mu$. Hence both I'^{+} and I'^{-} have the same functional dependence on μ and both can be represented by the same equation, *viz.*, $I'(\tau,\mu) \equiv I'^{+}(\tau,\mu)$, where $(-1 \leq \mu \leq 1)$ in $I'(\tau,\mu)$.

We also note from (6.5a) that, if S' remains sensibly constant over several photon mean free paths, say over an interval in optical depth $\Delta\zeta \gtrsim 10$, we have $I'^{+}(\sigma) \approx S'(\sigma)$; if, moreover, $|\sigma|$ is large, we also have $I'^{-}(\sigma) \approx S'(\sigma) \approx I'^{+}(\sigma)$ and, consequently, $I^{-}(\sigma) \approx I^{+}(\sigma) = S(\sigma)$. Hence the local intensity $I(\sigma)$ is almost equal to the source function $S(\sigma)$ at great optical depths ($|\sigma| \gg 1$), provided that S' changes only slowly with optical depth. Also, $I^{+}(\sigma) = S(\sigma) =$ constant (independent of σ) if S' is strictly constant with optical depth. It is for these reasons that S is called the "source function."

6.2 Expressions for J'_ν, F'_ν, and $p'_{r,\nu}$ in Terms of S'_ν for an Isotropic Source Function

We again drop for convenience the subscript ν from all quantities except μ_ν (refractive index), and define the n^{th} *moment* of $I(\tau,\mu)$ (recall that we have

* We consider here only those rays whose "backward" extensions intersect the stellar surface. In the case of strongly curved ray paths this restriction could exclude some rays having values of μ in the range $(-1 \leq \mu < 0)$ at the point under consideration.

assumed axial symmetry about every radial direction, so that I is not a function of the azimuthal angle ϕ):

$$I_n(\tau) \equiv \int_{-1}^{1} I(\tau, \mu)\mu^n \mathrm{d}\mu, \tag{6.7}$$

where $\mu = \cos\theta$ and τ denotes *normal* optical depth (*cf.* Sect. 6.1). We then have (*cf.* Sects. 2.2, 2.4, and 2.5)

$$\left.\begin{aligned} J' &= \frac{1}{2} I'_0 \\ F' &= 2\pi I'_1 \\ \frac{p'_r}{\mu_v} &= \frac{2\pi}{c} I'_2 \end{aligned}\right\} \tag{6.8}$$

where a prime denotes division by $\mu_v{}^2$.

The "modified" moments $I'_n(\tau) \equiv I_n(\tau)/\mu_v{}^2(\tau)$ can be expressed in terms of S' and of known functions only in the case of the "straight" ray path approximation (*cf.* Sect. 6.1), which we adopt throughout this section except where we specify otherwise. Using, accordingly, the formal solutions (6.5b) and (6.6b) for $I'^{+}(\tau,\mu)$ and $I'^{-}(\tau,\mu)$, we obtain

$$I'_n(\tau) = (-1)^n \int_0^{\tau} S'(t)E_{n+1}(\tau - t)\mathrm{d}t + \int_{\tau}^{\infty} S'(t)E_{n+1}(t-\tau)\mathrm{d}t, \tag{6.9}$$

where $S'(t) \equiv S(t)/\mu_v{}^2(t)$, S has been assumed isotropic, and $E_n(x)$ is the *integral exponential function*, defined by

$$E_n(x) = \int_0^1 e^{-x/v} v^{n-1}\mathrm{d}v/v = x^{n-1}\int_x^{\infty} (e^{-u}/u^n)\mathrm{d}u = \int_1^{\infty} \frac{e^{-xw}}{w^n}\mathrm{d}w, \quad x \geq 0. \tag{6.10}$$

The following recursion relations are satisfied by the $E_n(x)$:

$$\left.\begin{aligned} (n-1)E_n(x) &= e^{-x} - xE_{n-1}(x) \\ E'_{n+1}(x) &= -E_n(x) \end{aligned}\right\} \quad (n > 1). \tag{6.11}$$

The $E_n(x)$ have the following properties:

$$E_0(x) = e^{-x}/x,$$

$$E_n(0) = 1/(n-1) \quad (n > 1).$$

$$x \to 0\colon \quad E_1(x) = -\gamma - \ln x + \sum_{\nu=1}^{\infty} (-1)^{\nu-1} x^{\nu}/(\nu \cdot \nu!).$$

(γ = Euler Mascheroni constant = 0.57722...)

$$x \to \infty\colon E_n(x) = e^{-x}/x \to 0. \tag{6.12}$$

We then have, for the desired quantities:

$$\left.\begin{aligned} J'(\tau) &= \frac{1}{2}\int_0^\infty S'(t)E_1(|\tau-t|)\mathrm{d}t \\ F'(\tau) &= 2\pi\int_\tau^\infty S'(t)E_2(t-\tau)\mathrm{d}t - 2\pi\int_0^\tau S'(t)E_2(\tau-t)\mathrm{d}t \\ \frac{p_r'(\tau)}{\mu_\nu(\tau)} &= \frac{2\pi}{c}\int_0^\infty S'(t)E_3(|\tau-t|)\mathrm{d}t\,. \end{aligned}\right\} \qquad (6.13)$$

These equations, along with (6.5b) and (6.6b), permit $I(\tau,\mu)$, $J(\tau)$, $F(\tau)$, and $p_r(\tau)$ to be computed at any normal optical depth τ, given the source function $S(\tau)$ and the refractive index $\mu_\nu(\tau)$ as known functions of τ, in the "straight" ray path approximation. If $\mu_\nu = 1$ everywhere, then the primes appearing in (6.13) are simply dropped.

6.3 Power Series Solution of the Equation of Transfer

We assume that we can expand $S'(\zeta) \equiv S(\zeta)/\mu_\nu^2(\zeta)$ in a convergent Taylor series about σ, the optical depth measured along a (possibly curved) ray path (we are still omitting the subscript ν from all quantities except refractive index μ_ν, although we are still considering monochromatic quantities in general):

$$\begin{aligned} S'(\zeta) &= S'(\sigma) + \left(\frac{\mathrm{d}S'}{\mathrm{d}\zeta}\right)_\sigma(\zeta-\sigma) + \frac{1}{2}\left(\frac{\mathrm{d}^2S'}{\mathrm{d}\zeta^2}\right)_\sigma(\zeta-\sigma)^2 + \cdots \\ &= \sum_{r=0}^\infty \frac{1}{r!}\left(\frac{\mathrm{d}^rS'}{\mathrm{d}\zeta^r}\right)_\sigma(\zeta-\sigma)^r = \sum_{r=0}^\infty (-1)^r\frac{1}{r!}\left(\frac{\mathrm{d}^rS'}{\mathrm{d}\zeta^r}\right)_\sigma(\sigma-\zeta)^r\,. \end{aligned} \qquad (6.14a)$$

In the "straight" ray path approximation we have $\zeta = t/\mu$ and $\sigma = \tau/\mu$, where μ in this approximation is independent of normal optical depth t or τ along every ray path (*cf.* Sect. 6.1). We obtain from (6.14a)

$$S'(t) = \sum_{r=0}^\infty \frac{1}{r!}\left(\frac{\mathrm{d}^rS'}{\mathrm{d}t^r}\right)_\tau(t-\tau)^r = \sum_{r=0}^\infty (-1)^r\frac{1}{r!}\left(\frac{\mathrm{d}^rS'}{\mathrm{d}t^r}\right)_\tau(\tau-t)^r\,. \qquad (6.14b)$$

Substituting (6.14a) into the formal solutions (6.5a) and (6.6a), we obtain

$$I'^{+}(\sigma) = \sum_{r=0}^\infty \frac{1}{r!}\left(\frac{\mathrm{d}^rS'}{\mathrm{d}\zeta^r}\right)_\sigma\int_0^\infty (\zeta-\sigma)^r e^{-(\zeta-\sigma)}\mathrm{d}(\zeta-\sigma) \qquad (0<\mu\leq 1)\,,$$

where the integral in the above equation is equal to the gamma function, $\Gamma(r+1) = r!$ (since r is an integer). Hence we have

$$I'^{+}(\sigma) = \sum_{r=0}^{\infty}\left(\frac{d^r S'}{d\zeta^r}\right)_\sigma = S'(\sigma)+\left(\frac{dS'}{d\zeta}\right)_\sigma+\left(\frac{d^2S'}{d\zeta^2}\right)_\sigma+\cdots \quad (0<\mu\leq 1), \tag{6.15a}$$

where the derivatives must be evaluated *along* the ray path under consideration. In terms of *normal* optical depth $\tau = \tau(\mu,\sigma)$ where $\mu(t)$ is in general a function of t along a ray path (*cf.* Sect. 6.1), we have

$$I'^{+}(\tau,\mu) = \sum_{r=0}^{\infty}\left[\left(\mu\frac{d}{dt}\right)^r S'\right]_\tau = S'(\tau)+\left(\mu\frac{dS'}{dt}\right)_\tau+\left[\mu\frac{d}{dt}\left(\mu\frac{dS'}{dt}\right)\right]_\tau+\cdots$$

$$(0<\mu(\tau)\leq 1). \tag{6.15b}$$

In the "straight" ray path approximation (μ independent of t along every ray path, *cf.* Sect. 6.1), (6.15b) reduces to the usual expansion:

$$I'^{+}(\tau,\mu) = \sum_{r=0}^{\infty}\mu^r\left(\frac{d^r S'}{dt^r}\right)_\tau = S'(\tau)+\mu\left(\frac{dS'}{dt}\right)_\tau+$$

$$+\mu^2\left(\frac{d^2S'}{dt^2}\right)_\tau+\cdots \quad (0<\mu\leq 1). \tag{6.15c}$$

Note that the first two terms in the expansion (6.15b) for I'^{+} are identical to the first two terms in the expansion (6.15c), so that effects of "curvature" of the ray paths do not appear explicitly in these two terms. Moreover, carrying out the differentiation in the third term on the right side of (6.15b), we have

$$\left[\mu\frac{d}{dt}\left(\mu\frac{dS'}{dt}\right)\right]_\tau = \mu^2(\tau)\left(\frac{d^2S'}{dt^2}\right)_\tau+\left(\frac{dS'}{dt}\right)_\tau\left[\frac{d}{dt}\left(\frac{1}{2}\mu^2\right)\right]_\tau,$$

where it is seen that the ratio of the last term to the next-to-last is of the general order of magnitude of either of the ratios λ_p/r or λ_p/λ_μ (λ_p = average photon mean free path, λ_μ = radius of curvature of ray paths, r = radial distance, *cf.* Sect. 6.1). Hence, again we may conclude that the "straight" ray path approximation is adequate as long as both of these ratios are small compared to unity.

Consider now the "inward" intensity $I^{-}(\sigma)$ $(-1\leq\mu<0)$. Substituting the expansion (6.14a) for S' into the formal solution (6.6a), we obtain

$$I'^{-}(\sigma) = \sum_{r=0}^{\infty}(-1)^r\frac{1}{r!}\left(\frac{d^r S'}{d\zeta^r}\right)_\sigma\int_{\zeta=0}^{\zeta=\sigma}(\sigma-\zeta)^r e^{\sigma-\zeta}\,d(-\zeta) \quad (-1\leq\mu<0),$$

where the integral can be evaluated by successive integration by parts. We finally obtain

$$I'^{-}(\sigma) = \sum_{r=0}^{\infty} \left\{ 1 - \frac{(-1)^r}{r!} e^{\sigma} [\sigma^r - r\sigma^{r-1} - r(r-1)\sigma^{r-2} - \cdots + \right.$$

$$\left. - \cdots + (-1)^{r-1} r!\sigma + (-1)^r r!] \right\} \left(\frac{d^r S'}{d\zeta^r} \right)_{\sigma} \quad (-1 \leq \mu < 0), \quad (6.16a)$$

which in the "straight" ray path approximation reduces to

$$I'^{-}(\tau, \mu) = \sum_{r=0}^{\infty} \left\{ 1 - \frac{(-1)^r}{r!} e^{\tau/\mu} \left[\left(\frac{\tau}{\mu} \right)^r - r \left(\frac{\tau}{\mu} \right)^{r-1} + r(r-1) \left(\frac{\tau}{\mu} \right)^{r-2} - \cdots + \right. \right.$$

$$\left. \left. - \cdots + (-1)^{r-1} r! \left(\frac{\tau}{\mu} \right) + (-1)^r r! \right] \right\} \left(\frac{d^r S'}{dt^r} \right)_{\tau} \mu^r \quad (-1 \leq \mu < 0). \quad (6.16b)$$

We note from (6.16b) that $I'^{-}(\tau,\mu)$ has the same functional dependence on μ as $I'^{+}(\tau,\mu)$ in the limit as $\tau \to \infty$, as was pointed out in Sect. 6.1.

In order to evaluate the corresponding expansions for the average intensity J, the net flux F, and the radiation pressure p_r, we shall consistently adopt the "straight" ray path approximation. It is most economical to evaluate the expansion for the moment integral $I_n(\tau)$ (*cf.* (6.7)), since J, F, and p_r can all be expressed in terms of this integral (see (6.8)). Substituting the formal solutions (6.5b) and (6.6b) into (6.7), we obtained (6.9). Substituting the expansion (6.14b) for $S'(t)$ into (6.9), we obtain

$$I'_n(\tau) = \sum_{r=0}^{\infty} \frac{1}{r!} \left(\frac{d^r S'}{dt^r} \right)_{\tau} \int_{\tau}^{\infty} (t-\tau)^r E_{n+1}(t-\tau) dt +$$

$$+ (-1)^n \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \left(\frac{d^r S'}{dt^r} \right)_{\tau} \int_{0}^{\tau} (\tau - t)^r E_{n+1}(\tau - t) dt =$$

$$= \sum_{r=0}^{\infty} \frac{1}{r!} \left(\frac{d^r S'}{dt^r} \right)_{\tau} \left\{ \int_{0}^{\infty} x^r E_{n+1}(x) dx + (-1)^{n+r} \int_{0}^{\tau} x^r E_{n+1}(x) dx \right\}. \quad (6.19)$$

Consider now the general integral

$$I \equiv \int_{a}^{b} x^m E_n(x) dx \qquad (m, n \geq 0,\ m+n > 0),$$

where m and n are positive integers or zero and a and b are arbitrary non-negative real numbers (we must have $m+n>0$ if a may take the value zero). Using the expression

$$E_n(x) = \int_0^1 e^{-x/\mu}\mu^{n-2}\mathrm{d}\mu\,, \tag{6.21}$$

we have

$$I = \int_a^b x^m\left[\int_0^1 e^{-x/\mu}\mu^{n-2}\mathrm{d}\mu\right]\mathrm{d}x = \int_0^1 \mu^{n-2}\left[\int_a^b x^m e^{-x/\mu}\mathrm{d}x\right]\mathrm{d}\mu\,,$$

interchanging the order of integration. The "middle" integral in the last term in the above expression is

$$\mathscr{J}_{m\,(\mathrm{a},\,\mathrm{b},\,\mu)} \equiv \int_a^b x^m e^{-x/\mu}\mathrm{d}x = \mu^{m+1}\int_{a/\mu}^{b/\mu} y^m e^{-y}\mathrm{d}y\,. \tag{6.22}$$

By successive integration by parts, we obtain

$$\mathscr{J}_{m(\mathrm{a},\,\mathrm{b},\,\mu)} = \mu^{m+1}\left\{-(x^m+mx^{m-1}+m(m-1)x^{m-2}+\cdots+m!)e^{-x}\right\}\Big|_{a/\mu}^{b/\mu} =$$

$$= \mu^{m+1}\sum_{\nu=0}^{m}\frac{m!}{(m-\nu)!}\left[\left(\frac{a}{\mu}\right)^{m-\nu}e^{-a/\mu}-\left(\frac{b}{\mu}\right)^{m-\nu}e^{-b/\mu}\right]. \tag{6.23}$$

Then

$$I = \int_a^b x^m E_n(x)\mathrm{d}x = \int_0^1 \mathscr{J}_{m(\mathrm{a},\,\mathrm{b},\,\mu)}\,\mu^{n-2}\mathrm{d}\mu =$$

$$= \sum_{\nu=0}^{m}\frac{m!}{(m-\nu)!}\left[a^{m-\nu}\int_0^1 \mu^{n+\nu-1}e^{-a/\mu}\mathrm{d}\mu - b^{m-\nu}\int_0^1 \mu^{n+\nu-1}e^{-b/\mu}\mathrm{d}\mu\right]$$

or, from (6.21),

$$I = \int_a^b x^m E_n(x)\mathrm{d}x = \sum_{\nu=0}^{m}\frac{m!}{(m-\nu)!}\left[a^{m-\nu}E_{n+\nu+1}(\mathrm{a}) - b^{m-\nu}E_{n+\nu+1}(\mathrm{b})\right]$$

$$(m,\, n\geq 0,\, m+n>0)\,. \tag{6.24}$$

In particular, we have

$$\left.\begin{aligned}\int_0^\tau x^m E_n(x)\mathrm{d}x &= \frac{m!}{m+n} - \sum_{\nu=0}^{m}\frac{m!}{(m-\nu)!}\,\tau^{m-\nu}E_{n+\nu+1}(\tau)\\ \int_0^\infty x^m E_n(x)\mathrm{d}x &= \frac{m!}{m+n}\,.\end{aligned}\right\}\quad (m,\, n\geq 0,\, m+n>0) \tag{6.25}$$

The expansion for $I'_n(\tau)$, (6.19), then becomes

$$I'_n(\tau) = \sum_{r=0}^{\infty} \left\{ \frac{1+(-1)^{n+r}}{n+r+1} + \right.$$

$$\left. + \frac{(-1)^{n+r+1}}{r!} \sum_{v=0}^{r} \frac{r!}{(r-v)!} \tau^{r-v} E_{n+v+2}(\tau) \right\} \left(\frac{\mathrm{d}^r S'}{\mathrm{d}t^r} \right)_\tau , \qquad (6.26)$$

where $I'_n(\tau) \equiv I_n(\tau)/\mu_v{}^2(\tau)$ and $S'(t) \equiv S(t)/\mu_v{}^2(t)$.

We then have for $J'(\tau)$, $F'(\tau)$, and $p'_r(\tau)/\mu_v(\tau)$ in the "straight" ray path approximation, using (6.8),

$$J'(\tau) = \frac{1}{2} I'_0(\tau) =$$

$$= \frac{1}{2} \sum_{r=0}^{\infty} \left\{ \frac{2}{r+1} \delta_{r,e} + \frac{(-1)^{r+1}}{r!} \sum_{v=0}^{r} \frac{r!}{(r-v)!} \tau^{r-v} E_{v+2}(\tau) \right\} \left(\frac{\mathrm{d}^r S'}{\mathrm{d}t^r} \right)_\tau , \qquad (6.27a)$$

$$F'(\tau) = 2\pi I'_1(\tau) =$$

$$= 2\pi \sum_{r=0}^{\infty} \left\{ \frac{2}{r+2} \delta_{r,0} + \frac{(-1)^r}{r!} \sum_{v=0}^{r} \frac{r!}{(r-v)!} \tau^{r-v} E_{v+3}(\tau) \right\} \left(\frac{\mathrm{d}^r S'}{\mathrm{d}t^r} \right)_\tau , \qquad (6.27b)$$

$$p'_r(\tau)/\mu_v(\tau) = \frac{2\pi}{c} I'_2(\tau) =$$

$$= \frac{2\pi}{c} \sum_{r=0}^{\infty} \left\{ \frac{2}{r+3} \delta_{r,e} + \frac{(-1)^{r+1}}{r!} \sum_{v=0}^{r} \frac{r!}{(r-v)!} \tau^{r-v} E_{v+4}(\tau) \right\} \left(\frac{\mathrm{d}^r S'}{\mathrm{d}t^r} \right)_\tau , \qquad (6.27c)$$

where

$$\delta_{r,e} \equiv \frac{1}{2}[1+(-1)^r] = \begin{cases} 1 \text{ if } r \text{ is even} \\ 0 \text{ if } r \text{ is odd}, \end{cases}$$

$$\delta_{r,0} \equiv \frac{1}{2}[1+(-1)^{r+1}] = \begin{cases} 1 \text{ if } r \text{ is odd} \\ 0 \text{ if } r \text{ is even}. \end{cases}$$

The above equations could also, of course, have been obtained by substituting the expansions (6.15c) and (6.16b) for I' directly into (6.7).

Writing out the leading few terms explicitly, we have

$$J'(\tau) = \left[1 - \frac{1}{2} E_2(\tau) \right] S'(\tau) + \left[\frac{1}{2} \{ \tau E_2(\tau) + E_3(\tau) \} \right] \frac{\mathrm{d}S'}{\mathrm{d}\tau} +$$

$$+\left[\frac{1}{3}-\frac{1}{4}\{\tau^2E_2(\tau)+2\tau E_3(\tau)+2E_4(\tau)\}\right]\frac{d^2S'}{d\tau^2}+$$

$$+\left[\frac{1}{2\cdot 3!}\{\tau^3E_2(\tau)+3\tau^2E_3(\tau)+6\tau E_4(\tau)+6E_5(\tau)\}\right]\frac{d^3S'}{d\tau^3}+$$

$$+\left[\frac{1}{5}-\frac{1}{2\cdot 4!}\{\tau^4E_2(\tau)+4\tau^3E_3(\tau)+12\tau^2E_4(\tau)+24\tau E_5(\tau)+24E_6(\tau)\}\right]\frac{d^4S'}{d\tau^4}$$

$$+\cdots, \tag{6.28a}$$

$$F'(\tau)=2\pi E_3(\tau)S'(\tau)+\pi\left[\frac{4}{3}-2\{\tau E_3(\tau)+E_4(\tau)\}\right]\frac{dS'}{d\tau}+$$

$$+\pi\left[\{\tau^2E_3(\tau)+2\tau E_4(\tau)+2E_5(\tau)\}\right]\frac{d^2S'}{d\tau^2}+$$

$$+\pi\left[\frac{4}{5}-\frac{1}{3}\{\tau^3E_3(\tau)+3\tau^2E_4(\tau)+6\tau E_5(\tau)+6E_6(\tau)\}\right]\frac{d^3S'}{d\tau^3}+$$

$$+\pi\left[\frac{1}{12}\{\tau^4E_3(\tau)+4\tau^3E_4(\tau)+12\tau^2E_5(\tau)+24\tau E_6(\tau)+24E_7(\tau)\}\right]\frac{d^4S'}{d\tau^4}$$

$$+\cdots, \tag{6.28b}$$

$$3\cdot\frac{c}{2\pi}p'_r(\tau)/\mu_v(\tau)=\left[2-3E_4(\tau)\right]S'(\tau)+\left[3\{\tau E_4(\tau)+E_5(\tau)\}\right]\frac{dS'}{d\tau}+$$

$$+\left[\frac{6}{5}-\frac{3}{2}\{\tau^2E_4(\tau)+2\tau E_5(\tau)+2E_6(\tau)\}\right]\frac{d^2S'}{d\tau^2}+$$

$$+\left[\frac{1}{2}\{\tau^3E_4(\tau)+3\tau^2E_5(\tau)+6\tau E_6(\tau)+6E_7(\tau)\}\right]\frac{d^3S'}{d\tau^3}+$$

$$+\left[\frac{6}{7}-\frac{1}{8}\{\tau^4E_4(\tau)+4\tau^3E_5(\tau)+12\tau^2E_6(\tau)+24\tau E_7(\tau)+24E_8(\tau)\}\right]\frac{d^4S'}{d\tau^4}$$

$$+\cdots. \tag{6.28c}$$

For $\tau = 0$, $E_n(0) = 1/(n-1)$ $(n>1)$, and we have

$$J'(0) = \frac{1}{2}S'(0)+\frac{1}{4}\left(\frac{dS'}{d\tau}\right)_0+\frac{1}{6}\left(\frac{d^2S'}{d\tau^2}\right)_0+\frac{1}{8}\left(\frac{d^3S'}{d\tau^3}\right)_0+\frac{1}{10}\left(\frac{d^4S'}{d\tau^4}\right)_0 + \cdots, \tag{6.29a}$$

$$F'(0) = \pi S'(0)+\frac{2\pi}{3}\left(\frac{dS'}{d\tau}\right)_0+\frac{\pi}{2}\left(\frac{d^2S'}{d\tau^2}\right)_0+\frac{2\pi}{5}\left(\frac{d^3S'}{d\tau^3}\right)_0+\frac{\pi}{3}\left(\frac{d^4S'}{d\tau^4}\right)_0 + \cdots, \tag{6.29b}$$

$$3\cdot\frac{c}{2\pi}p_r'(0)/\mu_v(0) = S'(0)+\frac{3}{4}\left(\frac{dS'}{d\tau}\right)_0+\frac{3}{5}\left(\frac{d^2S'}{d\tau^2}\right)_0+\frac{3}{6}\left(\frac{d^3S'}{d\tau^3}\right)_0+\frac{3}{7}\left(\frac{d^4S'}{d\tau^4}\right)_0 + \cdots. \tag{6.29c}$$

For $\tau\to\infty$, $E_n(\tau)\to e^{-\tau}/\tau \to 0$, and we have

$$J'(\tau) = \sum_{r=0}^{\infty}\frac{1}{r+1}\delta_{r,e}\left(\frac{d^rS'}{dt^r}\right)_\tau = S'(\tau)+\frac{1}{3}\frac{d^2S'}{d\tau^2}+\frac{1}{5}\frac{d^4S'}{d\tau^4}+\cdots, \tag{6.30a}$$

$$F'(\tau) = 2\pi\sum_{r=0}^{\infty}\frac{2}{r+2}\delta_{r,0}\left(\frac{d^rS'}{dt^r}\right)_\tau = \frac{4\pi}{3}\frac{dS'}{d\tau}+\frac{4\pi}{5}\frac{d^3S'}{d\tau^3}+\frac{4\pi}{7}\frac{d^5S'}{d\tau^5}+\cdots, \tag{6.30b}$$

$$\frac{c}{2\pi}p_r'(\tau)/\mu_v(\tau) = \frac{2}{3}\sum_{r=0}^{\infty}\frac{3}{r+3}\delta_{r,e}\left(\frac{d^rS'}{dt^r}\right)_\tau = \frac{2}{3}\left\{S'(\tau)+\frac{3}{5}\frac{d^2S'}{d\tau^2}+\frac{3}{7}\frac{d^4S'}{d\tau^4}+\cdots\right\}. \tag{6.30c}$$

In terms of the energy density $u(\tau)$ (*cf.* (2.24′)), we have, for $\tau\to\infty$,

$$\frac{p_r}{\frac{1}{3}u} = \frac{v_g}{v_p}\frac{S'(\tau)+\frac{3}{5}\frac{d^2S'}{d\tau^2}+\frac{3}{7}\frac{d^4S'}{d\tau^4}+\cdots}{S'(\tau)+\frac{1}{3}\frac{d^2S'}{d\tau^2}+\frac{1}{5}\frac{d^4S'}{d\tau^4}+\cdots} = \frac{v_g}{v_p}\left(1+\frac{4}{15}\frac{1}{S'}\frac{d^2S'}{d\tau^2}+\cdots\right), \tag{6.31}$$

where v_g is the group velocity and v_p $(= c/\mu_v)$ is the phase velocity of the quasi-photons (see Sect. 2.10 and *cf.* (2.29′)). The ratio $v_g/v_p = 1$ if $\mu_v = 1$, and all primes in (6.31) are then dropped.

6.4 Convergence

For $\tau_\nu \to \infty$ (which is the case we are most interested in for the study of the stellar interior), we note that the successive derivatives of $S'_\nu \equiv S_\nu/\mu_\nu^2$ in the expansions for J'_ν, u'_ν, F'_ν, and $p'_{r,\nu}/\mu_\nu$ in the "straight" ray path approximation differ in their order by just two in successive terms. Thus, an order-of-magnitude estimate of the rapidity of convergence of these expansions at great optical depths can be obtained by forming the quantity

$$\frac{|(n+1)^{\text{st}}\text{ term}|}{|n^{\text{th}}\text{ term}|} \sim \frac{\left|\left(\frac{\mathrm{d}}{\mathrm{d}\tau_\nu}\right)^{n+2} S'_\nu\right|}{\left|\left(\frac{\mathrm{d}}{\mathrm{d}\tau_\nu}\right)^{n} S'_\nu\right|} \sim \frac{1}{(\Delta\tau_\nu)^2} \sim \frac{1}{(\kappa_\nu \rho \ell)^2},$$

where $\Delta\tau_\nu \sim \kappa_\nu \rho \ell$ is the optical thickness (or the number of photon mean free paths) over which S'_ν changes appreciably and ℓ is the corresponding radial distance.

In the deep interior ℓ may be comparable to R, the total stellar radius, in which case the above ratio may be of the order of 10^{-24} for typical conditions in a stellar interior. We see, then, that these expansions converge extremely rapidly in the deep interior and the first terms, in fact, will give an accuracy, in the deep interior, of about 1 part in some 10^{24} (except in the case of the expansion for I'_ν). (We note that a large value of $\Delta\tau_\nu$ is equivalent to S' being a slowly varying function of τ_ν, *cf.* Sect. 6.1.) Moreover, it was shown in Sect. 4.2 that $S'_\nu \simeq B_\nu(T)$ (assuming LTE) in the deep interior, even if electron scattering significantly (but not completely) predominates over absorption. With $S'_\nu = B_\nu(T)$, it is seen that these first terms (with the exception of that for F'_ν) are just the expressions that are valid in strict *t.e.* Moreover, these first terms (including that for F'_ν) involve only the first *two* terms in the expansion (6.15c) for I'^{+}_ν ($\approx I'^{-}_\nu$ at large optical depths), and these first two terms in (6.15c) are identical to the first two terms in (6.15b), which applies to "curved" ray paths. These first terms are therefore not affected at great optical depths by our "straight" ray path approximation, and are valid even for "curved" ray paths (as may be seen by substituting (6.15b) and (6.16a) for great optical depths directly into (6.7)). Hence, to very high accuracy in the deep interior (about one part in 10^{24}), we have

$$\left.\begin{aligned} J'_\nu &= B_\nu, & F'_\nu &= \frac{4\pi}{3}\frac{\mathrm{d}B_\nu}{\mathrm{d}\tau_\nu}, \\ u'_\nu &= \frac{4\pi}{v_g} B_\nu, & p_{r,\nu} &= \frac{1}{3} u_\nu \frac{v_g}{v_p}, \end{aligned}\right\} \tag{6.32}$$

where, if $\mu_\nu = 1$, $v_g = v_p = c$ (see Sect. 2.10) and all primes are dropped.

Consider now the question of the rapidity of convergence of the expansion for I_ν' in the "straight" ray path approximation at great optical depths (again setting $S_\nu' = B_\nu(T)$):

$$I_\nu'(\mathbf{r},\mathbf{n}) \equiv I_\nu(\mathbf{r},\mathbf{n})/\mu_\nu{}^2(\mathbf{r}) = B_\nu(T) + \mu\frac{dB_\nu}{d\tau_\nu} + \mu^2\frac{d^2B_\nu}{d\tau_\nu{}^2} + \cdots, \tag{6.33}$$

We could treat this expansion just as we have the others, in which case we would conclude that the ratios of the successive terms are of the order of magnitude of $(\kappa_\nu\rho\ell)^{-1} \sim 10^{-13}$ if $\ell \simeq R$. However, an alternative treatment emphasizes more clearly the physical reason for the great rapidity of convergence.

We retain only the first two terms on the right side of (6.33) and use the first term in the expansion for F_ν', *i.e.*, $(4\pi/3)\,dB_\nu/d\tau_\nu$, to eliminated $B_\nu/d\tau_\nu$. We obtain

$$I_\nu'(\mathbf{r},\mathbf{n}) = B_\nu(T) + \frac{3}{4\pi}F_\nu'\cos\theta = B_\nu(T) + \frac{3}{4\pi}\mathbf{n}\cdot\mathbf{F}_\nu', \tag{6.34}$$

where $\mathbf{F}_\nu$ is the vector net flux (*cf.* Sect. 2.2); these equations show that I_ν has a maximum in the outward direction and a minimum in the inward direction, as we expect (recall that μ_ν is assumed to be independent of direction). These equations also show that I_ν' differs from $B_\nu(T)$ (assumed equal to S_ν') only because of the "leakage" of radiation represented by the non-equilibrium term $\mathbf{F}_\nu'$.

To examine in a crude way the relative orders of magnitude of the two terms on the right side of (6.34), it is sufficient to use integrated quantities. Also, we set $\mu_\nu = 1$ for simplictiy. Thus we have

$$\frac{|2^{\text{nd}}\text{ term}|}{|1^{\text{st}}\text{ term}|} \sim \frac{\frac{3}{4\pi}F}{B(T)} = \frac{3}{4\pi}\frac{L(r)}{4\pi r^2\left(\frac{\sigma}{\pi}T^4\right)} = \frac{3}{4\pi}\frac{L(r)}{4r^2\sigma T^4}, \tag{6.35}$$

where $L(r) = 4\pi r^2 F$ is the net rate at which energy flows outward through a sphere of radius r, and $B(T)$ is the integrated Planck function. For the sun, we take $L(r) \sim 4\times10^{33}$ erg/sec, $r \sim 10^{10}$ cm, $\sigma \simeq 6\times10^{-5}$ erg/sec/cm^2/deg^4, and $T \simeq 10^7$°K; then (6.35) gives

$$\frac{|2^{\text{nd}}\text{ term}|}{|1^{\text{st}}\text{ term}|} \sim 4\times10^{-12}.$$

Physically, the smallness of the second term on the right side of (6.34) in comparison to the first in the deep interior can be understood as follows: The first term ($B_\nu(T)$ or $B(T)$) increases rapidly with increasing temperature, whereas the second term ($\propto F_\nu$ or F) depends only weakly on temperature (in the case of F, we have $F \propto 1/r^2$ if radiative equilibrium obtains outside the energy-producing region; F is therefore independent of temperature). As one descends into a star, the increasing temperature causes the first term to become rapidly much larger than the second, so that I_ν rapidly approaches $B_\nu(T)$ (assumed equal to S_ν here) in the deep interior. We may also say that the "leakage", represented by the term involving F, remains almost constant with depth and thus becomes relatively less and less important with increasing depth.

This interpretation makes it clear that the relative size of these two terms in (6.34) depends on the extent of the temperature increase from the surface to the interior regions (roughly speaking, the center) of the star. In fact, since outside the energy producing regions $F = (R/r)^2 \sigma T_e^4 \approx \sigma T_e^4$ for $r \approx R$, T_e being the effective temperature of the star, and, since $B(T) = \sigma T^4/\pi$, where T means either central or mean temperature, it is seen that the ratio of the second term to the first is of the order of T_e^4/T^4 (this is also evident from (6.35)). On the other hand, we see from the original expansion for I_ν that this ratio is also of the order of $1/(\kappa_\nu \rho R) \sim 1/n$, where n is the number of photon mean free paths in a stellar radius. Thus we have

$$\frac{T^4}{T_e^4} \sim n \sim \kappa_\nu \rho R \sim 10^{12}, \tag{6.36}$$

where the numerical value follows from taking $\kappa_\nu \sim 10\,\mathrm{cm}^2/\mathrm{gm}$, $\rho \sim 10\,\mathrm{gm/cm}^3$, and $R \sim 10^{10}$ cm. Equation (6.36) shows that the ratio of central to surface temperature ought to be of the general order of magnitude of $10^3 - 10^4$ for a star predominantly in radiative equilibrium.*

The equation obtained by taking only the first term in the expansion for F_ν',

$$F_\nu' = \frac{4\pi}{3} \frac{dB_\nu}{d\tau_\nu} \tag{6.37}$$

or, in vector notation,

$$\mathbf{F}_\nu' = -\frac{4\pi}{3} \frac{1}{\kappa_\nu \rho} (\nabla B_\nu), \tag{6.38}$$

* We are indebted to Prof. E.E. Salpeter for first pointing out this last interpretation of (6.34) to us.

is sometimes called the "diffusion" approximation because it is of the same form as other "diffusion" equations, such as the equation of heat conduction. This may be seen more explicitly by writing (6.38) in the form

$$\mathbf{F}_\nu' = -\left[\frac{4\pi}{3}\frac{1}{\kappa_\nu \rho}\left(\frac{\partial B_\nu}{\partial T}\right)\right]\nabla T\,, \tag{6.39}$$

where the quantity in square brackets plays the role of a "thermal conductivity." In the case of (6.39), however, the "thermal conductivity" is a strongly varying function of ρ and T (and ν, in the monochromatic case), in contrast to the usual case of ordinary thermal conductivity. The quantity in square brackets is often called the "radiative conductivity." Note that $\mathbf{F}_\nu' \equiv \mathbf{F}_\nu/\mu_\nu^2$ reduces to $\mathbf{F}_\nu$ if $\mu_\nu = 1$.

The fact that the two ways of estimating the value of the ratio of the second to the first term in (6.34) (*i.e.*, by using (6.35) and by using the quantity $(\kappa_\nu \rho R)^{-1}$) agree to order of magnitude is just a consequence of (6.37). As a stellar application of this equation, we may use it to estimate the luminosity of a star which is in radiative equilibrium throughout an appreciable fraction of its volume. We use only integrated quantities, set $\mu_\nu = 1$, replace F by $L/4\pi R^2$, $\mathrm{d}/\mathrm{d}\tau$ by $1/(\kappa\rho R)$, and $B(T)$ by $\sigma T^4/\pi$. Here L and R are luminosity and radius, respectively, of the star, and κ, ρ, and T are mean or central values (*i.e.*, the values representative of some typical point in a stellar interior). We then have from (6.37)

$$\frac{L}{4\pi R^2} \sim \frac{4\pi}{3}\frac{\frac{\sigma}{\pi}T^4}{\kappa\rho R} \tag{6.40a}$$

or

$$L \sim \frac{4\pi ac}{3}\, R \cdot \frac{T^4}{\kappa\rho} \simeq 10^{-3}\, R \cdot \frac{T^4}{\kappa\rho}\ \mathrm{erg/sec}\,, \tag{6.40b}$$

where all quantities in (6.40b) are in c.g.s. units. For the sun, we take $T = 10^7{}^\circ\mathrm{K}$, $R = 10^{11}$ cm, $\kappa = 100\ \mathrm{cm}^2/\mathrm{gm}$, and $\rho = 10\ \mathrm{gm/cm}^3$. We then obtain

$$L_\odot \sim 10^{33}\ \mathrm{erg/sec}$$

for the luminosity of the sun, which is not unexpectedly in good order-of-magnitude agreement with the observed value of $\sim 4 \times 10^{33}$erg/sec. We note that the assumption that the transfer of energy throughout most of the star was by radiation was essential for this result (the problem of calculating the luminosity of completely convective stars is discussed in Sect. 23.5).

From the foregoing developments it is clear that the condition for applicability of the diffusion approximation is that all terms in the expansion for F_ν' be small compared to the first. Since the ratio of all succeeding terms to the first is roughly $1/(\Delta\tau_\nu)^2 \sim 1/(\kappa_\nu \rho \ell)^2$, where $\Delta\tau_\nu$ and ℓ are, respectively, the optical thickness and corresponding geometrical radial distance over which S_ν' changes appreciably, we see that the condition for validity of the diffusion approximation is, roughly, that $(\Delta\tau_\nu)^2 \gg 1$, or that the material be *optically opaque* (in the relevant frequencies). If we take $\Delta\tau_\nu = \tau_\nu$ then $(\Delta\tau_\nu)^2 \gg 1$ implies that $\tau_\nu^2 \gg 1$; hence, we may also say, roughly speaking, that the diffusion approximation becomes valid at sufficiently great optical depths. (The other approximation we have made, *viz.*, replacing S_ν' by $B_\nu(T)$, is also normally valid for sufficiently great optical depths (*cf.* Sect. 4.2 and Chap. 7), so that, roughly, the same condition may be used for both approximations.)

CHAPTER 7

Conditions for LTE

In Chap. 4 it was pointed out that the central question in the whole problem of the validity of the assumption of LTE in a star is the question of the distribution of atoms and ions over their various energy levels, or, more briefly, of the level population of the atoms and ions. If this level population is described locally by the Boltzmann formula at the local electron kinetic temperature, we have LTE; otherwise, we do not. We must, therefore, in considering conditions for LTE, examine the factors which in a general, non-*t.e.* situation determine the atomic level population. The most important of these factors are *radiative* transitions among atomic energy levels (produced by absorption and emission of photons) and *collisional* transitions (produced by inelastic or superlastic collisions with other particles, predominantly electrons in the stellar case). If the system of interest is in a *statistically steady state*, the level populations can be found by setting up and solving the steady state equations for each level, taking all relevant processes into account (*cf.* Thomas and Athay [Th61a, Chap. 4] and Thomas [Th65, Lect. 3]).* Since radiative transitions are involved, in general, then it is clear that the radiation field must be known before the level populations can be determined.

The radiation field, however, is determined in general from the solution of the equation of transfer, which is seen from the general form (4.5) to depend, in turn, on the atomic level populations; *i.e.*, the *source function* (*cf.* Sect. 4.1) depends on these level populations. The problem of the level populations is therefore coupled to the radiation transfer problem, and these two problems must, in fact, be solved simultaneously in a general case. Only by such a simultaneous solution of these two problems can one, in principle, rigorously answer the question of the extent to which LTE obtains in stars.

* The term *statistically steady state* refers to a situation in which the atomic level populations are, statistically, constant in time; or, more generally, sufficiently slowly varying that they may be considered constant over the times of interest for the system. It is important to realize that the statistically steady state is *not* in general the same as statistical equilibrium (*i.e.*, thermodynamic equilibrium); for example, detailed balancing (*cf.* Chap. 3) is expected to obtain in the latter condition but not necessarily in the former.

Fortunately, the assumption of LTE is probably valid throughout essentially the entire stellar interior, as was argued from a mostly qualitative standpoint in Chap. 4 and as will be commented on further in Sect. 7.3. In the regions in and above the photosphere, however, the condition of LTE may break down and the only recourse, if a rigorously correct answer is desired, is to simultaneously solve the statistically steady state equations and the equation of transfer, as has been emphasized, for example, by Thomas [Th65] (further references to the literature may be found here). Moreover, in the absence of the LTE assumption, it is necessary to have values for all the relevant transition rates before the coupled problem can be solved; these are of course not needed if LTE obtains, since the level populations then depend on only a single parameter, the electron kinetic temperature.*

One reason the coupled problem is so difficult is that the problem is, in a true sense of the word, a *many-body* problem. The reason is that the atomic level populations at a given point in a stellar envelope or atmosphere, for example, are determined *not* by purely local physical conditions, but rather by conditions in the *entire* envelope or atmosphere. That this last statement is true may be easily seen by noting that the solution of the equation of transfer (which determines, in part, the level populations) involves an integration over the entire emitting region (see, for example, the formal solutions of the equation of transfer in Sect. 6.1).

In the present chapter we shall merely outline the general problem (Sect. 7.1) and a procedure for schematically solving the coupled statistically steady state-radiation transfer problem in the very simple case of a two-level atom (Sect. 7.2). We shall then use this solution to obtain a schematic condition for the validity of the assumption of LTE in a star (Sect. 7.3). This schematic condition can easily be very roughly evaluated numerically in the stellar interior, and we shall use this evaluation to discuss semi-quantitatively the applicability of LTE to stellar interiors (Sect. 7.3).

As in previous chapters, we consider in this chapter, for generality, the case of a non-unity and spatially variable refractive index μ_ν.

7.1 Equation of Transfer and the Excitation and Kinetic Temperatures

The equation of transfer, for "true" absorption by bound-bound transitions together with coherent, isotropic scattering is, in a general non-*t.e.*

* We are here and throughout this chapter neglecting electrostatic and other interactions among particles. Such interactions can cause the level populations to depend on particle density as well as on temperature (see Sect. 15.5a).

case (see (2.106) and the pragraphs following that equation),

$$\mu_\nu^2 \frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{I_\nu}{\mu_\nu^2}\right) = \kappa_\nu'\left\{\mu_\nu^2 \frac{A_{21}}{B_{21}}\left[\frac{N_1B_{12}}{N_2B_{21}}-1\right]^{-1} - I_\nu\right\} + \kappa_\nu^{(s)}\{J_\nu - I_\nu\} \tag{7.1a}$$

$$= (\kappa_\nu' + \kappa_\nu^{(s)})\left\{\frac{\kappa_\nu'}{\kappa_\nu' + \kappa_\nu^{(s)}} \cdot \mu_\nu^2 \frac{A_{21}}{B_{21}}\left[\frac{N_1B_{12}}{N_2B_{21}}-1\right]^{-1} + \right.$$

$$\left. + \frac{\kappa_\nu^{(s)}}{\kappa_\nu' + \kappa_\nu^{(s)}} J_\nu\right\} - (\kappa_\nu' + \kappa_\nu^{(s)})I_\nu, \tag{7.1b}$$

where κ_ν' and $\kappa_\nu^{(s)}$ are, respectively, the mass absorption coefficients for absorption (corrected for induced emission) and (coherent, isotropic) scattering, $J_\nu = (4\pi)^{-1}\int_{4\pi} I_\nu \mathrm{d}\omega$ is the average intensity, and the frequency ν is that appropriate to the separation in energy of the levels 1 and 2 ($h\nu = E_2 - E_1$). Recall (see Sect. 4.1) that we have assumed that the "line shape functions" $\phi(\nu)$ and $\psi(\nu)$ for, respectively, absorption and induced emission, are identical. We have also made use of the general relation (see (3.34e)) $\psi^*(\nu) = \psi(\nu)\mu_\nu^2$, where μ_ν is the (real) refractive index, assumed isotropic (*cf.* Chap. 2). The quantity in curly brackets in (7.1b), representing the ratio of all "source" terms to the total effective absorption coefficient $\kappa_\nu \equiv \kappa_\nu' + \kappa_\nu^{(s)}$, is the *source function* S_ν, in terms of which (7.1) may be written as

$$\frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{I_\nu}{\mu_\nu^2}\right) = \kappa_\nu\left(\frac{S_\nu}{\mu_\nu^2}\right) - \kappa_\nu\left(\frac{I_\nu}{\mu_\nu^2}\right). \tag{7.2}$$

We let $\sigma \equiv \kappa_\nu^{(s)}/\kappa_\nu$ and $1-\sigma = \kappa_\nu'/\kappa_\nu$ (as was done in Chap. 4), and write S_ν in the form

$$S_\nu = (1-\sigma)\mu_\nu^2 B_\nu(T_{\mathrm{ex}}) + \sigma J_\nu \tag{7.3}$$

(*cf.* (4.13b)), which we see is no longer the LTE form and where we have followed the notation of Thomas [Th65] in letting

$$B_\nu(T_{\mathrm{ex}}) \equiv \frac{A_{21}}{B_{21}}\left[\frac{N_1B_{12}}{N_2B_{21}}-1\right]^{-1} \tag{7.4a}$$

$$= \frac{A_{21}}{B_{21}}\left[\frac{N_1g_2}{N_2g_1}-1\right]^{-1}. \tag{7.4b}$$

In (7.4b) we have made use of the universal relation $g_1B_{12} = g_2B_{21}$ (*cf.* Sect. 3.6). The "excitation" temperature T_{ex} is that value of the temperature

parameter in the Boltzmann equation which will give the actual relative populations of levels 1 and 2:

$$\frac{N_2}{N_1} = \frac{g_2}{g_1} e^{-h\nu/kT_{ex}} . \tag{7.5}$$

Using the universal relation $A_{21}/B_{21} = 2h\nu^3/c^2$ (*cf.* Sect. 3.6) and (7.5) in (7.4b), we see that the right side is indeed the Planck function for the temperature T_{ex}. Specification of T_{ex} (or $B_\nu(T_{ex})$) for all pairs of levels of interest (in general, *different for each pair*) is thus seen to be equivalent to the specification of the actual level populations.

We consider now the electron kinetic temperature T_k, which is the temperature parameter in the velocity distribution formula (assumed Maxwellian) which will give the actual velocity distribution of the free electrons. We consider for the moment the case of *t.e.* where all temperatures have the same value, T, and where we assume that we have detailed balancing (*cf.* Chap. 3). Considering only *collisional* transitions between any two levels, say 1 and 2, we may then say that the rate of collisional transitions $1 \rightarrow 2$ is equal to rate of collisional transitions $2 \rightarrow 1$. We define the collisional transition probability per unit time per atom or ion, b_{ij}, by the following statement: $N_1 b_{12} \mathrm{d}t$ denotes the number of collisional transitions $1 \rightarrow 2$ per unit volume in time $\mathrm{d}t$, and $N_2 b_{21} \mathrm{d}t$ denotes the number of collisional transitions $2 \rightarrow 1$ per unit volume in time $\mathrm{d}t$. We then have from the principle of detailed balancing

$$N_1 b_{12} = N_2 b_{21}$$

or

$$\frac{N_1}{N_2} = \frac{b_{21}}{b_{12}} . \tag{7.6}$$

We know, however, that the Boltzmann formula

$$\frac{N_1}{N_2} = \frac{g_1}{g_2} e^{h\nu/kT} , \tag{7.7}$$

is also valid in *t.e.*,* whence we have, in *t.e.*,

$$\frac{b_{21}}{b_{12}} = \frac{g_1}{g_2} e^{h\nu/kT} . \tag{7.8}$$

But the value of the left side of (7.8) is independent of the *t.e.* assumption and depends only on the velocity distribution of the particles which effect the

* See the footnote page 159.

collisional transitions and on the nature of the atoms under consideration. Assuming these particles to be electrons, we have

$$\left.\begin{aligned} b_{12} &= \int_{v_0}^{\infty} \mathrm{d}N_e(v)\sigma_{12}(v)v, \\ b_{21} &= \int_{0}^{\infty} \mathrm{d}N_e(v)\sigma_{21}(v)v, \end{aligned}\right\} \tag{7.9}$$

where $\mathrm{d}N_e(v)$ is the number of electrons per unit volume having speeds in the range $(v, v + \mathrm{d}v)$, $(1/2)m_e{v_0}^2 = h\nu$, and σ_{12} and σ_{21} are the corresponding collision cross sections. In a general, non-*t.e.* situation, then, we may *define* the kinetic temperature T_k to be that value of T in (7.8) which will yield the *actual* value of the ratio b_{21}/b_{12}:

$$\frac{g_2 b_{21}}{g_1 b_{12}} = e^{h\nu/kT_k}. \tag{7.10}$$

In a general situation for which there exists no simple law describing the electron velocity distribution, T_k will generally have a different value for each pair of levels. However, if the electron velocity distribution is the same as that appropriate to *t.e.* (Maxwellian, for example), then only *one* value of T_k, namely that value which appears in the velocity distribution law, will correctly give the values of the left side of (7.10) for *all* pairs of levels (see (7.8) and the sentence following this equation). Using (7.10) and the relation $A_{21}/B_{21} = 2h\nu^3/c^2$ in the Planck formula for $B_\nu(T_k)$, we see that we may also write

$$B_\nu(T_k) = \frac{A_{21}}{B_{21}}\left[\frac{g_2 b_{21}}{g_1 b_{12}} - 1\right]^{-1}. \tag{7.11}$$

Thus, specification of $B_\nu(T_k)$ or of T_k is equivalent to specification of the ratio b_{21}/b_{12}.

If $B_\nu(T_{ex}) = B_\nu(T_k)$, *i.e.*, if $T_{ex} = T_k$, we see by comparison of (7.4b) and (7.11) and from (7.10) that we would have

$$\frac{N_1}{N_2} = \frac{b_{21}}{b_{12}} = \frac{g_1}{g_2} e^{h\nu/kT_k}, \tag{7.12}$$

which states that the atomic level populations are just those which would obtain if collision processes alone were operative (*i.e.*, no radiative transitions) at the local electron kinetic temperature T_k and which, according to our definition (*cf.* Sect. 4.1 and the introductory paragraphs of this chapter), is precisely the condition for LTE. We may then take the equality

$$B_\nu(T_{ex}) = B_\nu(T_k) \quad (\text{or } T_{ex} = T_k) \tag{7.13}$$

as an equivalent expression of the necessary and sufficient condition that LTE obtains.

7.2 Statistically Steady State for a Two-Level Atom

We assume, merely for simplicity and for illustrative purposes, the simplest atom model, one possessing only the two levels 1 and 2 ($E_2 > E_1$). The numbers of atoms per unit volume in the two levels are N_1 and N_2 and the statistical weights are g_1 and g_2. We assume the system to be in a statistically steady state, for which the condition is

$$\frac{dN_1}{dt} = \frac{dN_2}{dt} = 0, \tag{7.14}$$

where both radiative and collisional transitions are taken into account. Recalling the definitions of b_{12} and b_{21} given in Sect. 7.1, we have for level 1

$$\frac{dN_1}{dt} = 4\pi N_2 A_{21} \int_{\Delta\nu} \psi^*(\nu')d\nu' + N_2 B_{21} \int_{\Delta\nu} \int_{4\pi} I_\nu d\omega \psi(\nu')d\nu' + N_2 b_{21}$$

$$- N_1 B_{12} \int_{\Delta\nu} \int_{4\pi} I_\nu d\omega \phi(\nu')d\nu' - N_1 b_{12} = 0, \tag{7.15}$$

where the integrations over frequency ν' ($\approx \nu = (E_2 - E_1)/h$, where ν is the frequency at the "line center") are carried out over the narrow range of frequencies $\Delta\nu$ "in the line" within which the line shape functions $\psi^*(\nu')$ (for spontaneous emission), $\psi(\nu')$ (for induced emission), and $\phi(\nu')$ (for absorption) are appreciably different from zero. The A's and B's are the Einstein coefficients (*cf.* Sect. 2.8).

If we are to take effects of dispersion into account, we must regard them (*cf.* Sect. 2.8) as contained in the line shape functions, and we must make some assumptions about how the line shape functions are affected by dispersive effects. We shall assume (as we have already assumed in writing down the transfer equation in the form (7.1)) that the line shape functions for induced emission and absorption are identical, both when dispersive effects are present and when they are absent; *i.e.*, we assume that

$$\psi(\nu') \equiv \phi(\nu')$$

and

$$\psi_a(\nu') \equiv \phi_a(\nu'), \tag{7.15a}$$

where the subscript a means in the absence of dispersive effects. We also assume the validity of the relation

$$\phi(\nu') = \phi_a(\nu')/\mu_{\nu'} \tag{7.15b}$$

(see (3.37d) and the accompanying discussion; see also the precautionary remarks following (2.141)). The assumptions (7.15a) and (7.15b), along with the general relation (see (3.34e)) $\psi^*(\nu') \equiv \psi(\nu')\mu_{\nu'}{}^2$, then lead, as was shown near the end of Sect. 3.6, to the relations (see (3.37e))

$$\begin{aligned} \psi^*(\nu') &= \mu_{\nu'}\psi_a^*(\nu'), \\ \psi(\nu') &= (1/\mu_{\nu'})\psi_a(\nu'), \\ \phi(\nu') &= (1/\mu_{\nu'})\phi_a(\nu'). \end{aligned} \tag{7.15c}$$

Using these relations in the steady state condition (7.15) for level 1, we have

$$\frac{dN_1}{dt} = 4\pi N_2 A_{21} \int_{\Delta\nu} \mu_{\nu'}\psi_a^*(\nu')d\nu' + N_2 B_{21} \int_{\Delta\nu}\int_{4\pi} I_{\nu'}d\omega[\psi_a(\nu')/\mu_{\nu'}]d\nu' +$$

$$+ N_2 b_{21} - N_1 B_{12} \int_{\Delta\nu}\int_{4\pi} I_{\nu'}d\omega[\phi_a(\nu')/\mu_{\nu'}]d\nu' - N_1 b_{12} = 0. \tag{7.15d}$$

As we are here not interested in the details of line shapes, we shall for simplicity assume that $\psi_a(\nu')$ and $\phi_a(\nu')$ are both delta functions, $\delta(\nu-\nu')$ (note that $\psi_a(\nu')$ and $\phi_a(\nu')$ are properly normalized, *cf.* Sect. 2.8). Hence $\int_{\Delta\nu} \mu_{\nu'}\psi_a^*(\nu')d\nu' = \mu_\nu$, and

$$\int_{\Delta\nu} I_{\nu'}[\psi_a(\nu')/\mu_{\nu'}]d\nu' = \int_{\Delta\nu} I_{\nu'}[\phi_a(\nu')/\mu_{\nu'}]d\nu' = I_\nu/\mu_\nu.$$

Writing $J_\nu = (4\pi)^{-1}\int_{4\pi} I_\nu d\omega$ (average intensity), we have

$$\frac{N_2}{N_1}[4\pi A_{21}\mu_\nu + 4\pi B_{21}(J_\nu/\mu_\nu) + b_{21}] = 4\pi B_{12}(J_\nu/\mu_\nu) + b_{12}$$

or

$$\frac{N_2}{N_1} = \frac{4\pi B_{12}(J_\nu/\mu_\nu) + b_{12}}{4\pi A_{21}\mu_\nu + 4\pi B_{21}(J_\nu/\mu_\nu) + b_{21}}. \tag{7.16}$$

Substituting this expression for N_2/N_1 into (7.4b) for $B_\nu(T_{ex})$, we have

$$B_\nu(T_{ex}) = \frac{A_{21}}{B_{21}}\left[\frac{g_2}{g_1}\cdot\frac{4\pi A_{21}\mu_\nu + 4\pi B_{21}(J_\nu/\mu_\nu) + b_{21}}{4\pi B_{12}(J_\nu/\mu_\nu) + b_{12}} - 1\right]^{-1}.$$

Making use of the relation $g_1 B_{12} = g_2 B_{21}$, we may also write

$$B_\nu(T_{ex}) = \frac{4\pi A_{21}(g_2/g_1)(J_\nu/\mu_\nu) + b_{12}(A_{21}/B_{21})}{4\pi A_{21}\mu_\nu(g_2/g_1) + b_{12}\left[\dfrac{g_2 b_{21}}{g_1 b_{12}} - 1\right]} =$$

$$= \frac{(J_\nu/\mu_\nu^2) + b_{12}/4\pi B_{12}\mu_\nu}{1 + \dfrac{b_{21}}{4\pi A_{21}\mu_\nu}\left[1 - \dfrac{g_1 b_{12}}{g_2 b_{21}}\right]}. \tag{7.17}$$

We now define the quantity

$$\eta_\nu \equiv \frac{b_{21}}{4\pi A_{21}\mu_\nu}\left[1 - \frac{g_1 b_{12}}{g_2 b_{21}}\right] \tag{7.18}$$

which is, aside from the quantity in square brackets (which is practically always less than one), the ratio of the probability of a collisional $2 \to 1$ transition to the probability of a spontaneous radiative $2 \to 1$ transition and is therefore clearly a measure of the importance of collisional, as opposed to radiative, de-excitation;* η_ν is approximately proportional to the electron density (and hence to the mass density in the case of almost complete ionization, if all electrons present are ionization electrons, *cf.* Chap. 15). Using (7.11) and (7.18), we may finally write (7.17) in the form

$$B_\nu(T_{ex}) = \frac{(J_\nu/\mu_\nu^2) + \eta_\nu B_\nu(T_k)}{1 + \eta_\nu}. \tag{7.19}$$

A lucid interpretation of (7.19) has been given by Thomas [Th65, Lect. 3]: The terms in the numerator represent processes that populate the upper level: (J_ν/μ_ν^2) represents radiative, and $\eta_\nu B_\nu(T_k)$ collisional upward transitions (see (7.17)). On the other hand, the terms in the denominator represent processes that depopulate the upper level, normalized to spontaneous radiative de-excitation: the "1" represents this spontaneous radiative de-excitation and η_ν represents collisional de-excitation. Moreover, the terms $\eta_\nu B_\nu(T_k)$ and η_ν are, respectively, "source" and "sink" terms, since they represent, respectively, creation of photons through collisional excitations followed by radiative de-excitations and destruction of photons through

* Note (see the first equation of (7.15c)) that the factor μ_ν multiplying A_{21} (which is defined in the absence of dispersive effects) takes into account the effects of dispersion on "spontaneous" emission.

radiative excitations followed by collisional de-excitations. The terms (J_ν/μ_ν^2) and "1" are "storage and migration" terms, since they represent absorption followed by re-emission, or, loosely speaking, "scattering": through their action a photon is never destroyed (at least in our two-level atom with infinitely sharp levels, where this "scattering" may be regarded as essentially coherent), so it can disappear only by diffusing out through a boundary.

If we had considered an atom model with more than two levels, further terms would have appeared in both numerator and denominator of (7.19). Their interpretation as "source" and "sink" terms, however, as explained in the previous paragraph, would have remained essentially unaltered (*cf.* Thomas [Th65], Lect. 3). Consequently, our two-level atom contains the essential physics relevant to our present arguments.

At first sight (7.19) would appear to indicate that the presence or absence of LTE depends only on the value of η_ν: For $\eta_\nu \to \infty$ (collisional de-excitations much more frequent than radiative), we would have $B_\nu(T_{ex}) = B_\nu(T_k)$ and hence LTE. For $\eta_\nu \to 0$ (collisional de-excitation unimportant), we would have $B_\nu(T_{ex}) = (J_\nu/\mu_\nu^2)$, which would mean that the atomic level populations are determined entirely by the nature of the radiation field and not at all by collisions; moreover, in case η_ν were strictly zero, we would have a case of "monochromatic radiative equilibrium," or pure "scattering," and a photon of frequency ν could escape from the system only by diffusing out through a boundary, since in this case there would be no mechanism for the destruction of the photons. This analysis is incorrect in general, however (as has been pointed out by Thomas), since it implicitly assumes either that J_ν is specified *a priori* or that it is independent of the second term $\eta_\nu B_\nu(T_k)$ in the numerator of (7.19). The analysis would accordingly be correct only if J_ν *were* so specified (which could be done in the case of an optically thin gas being irradiated from some external source, for example, such as is the case in the solar corona) or *were* independent of $\eta_\nu B_\nu(T_k)$. In general, however, J_ν is not *a priori* specified and does depend on $\eta_\nu B_\nu(T_k)$; J_ν is, in fact, determined as a solution of the equation of transfer, and hence depends on the source function S_ν, which itself depends on η_ν and $B_\nu(T_k)$. The problem therefore reduces, in a general case, to the solution of the coupled problems of the atomic level populations and the radiation transfer problem, as was pointed out in the introductory paragraphs of this chapter. Equivalently, the solution of the coupled problem amounts to the solution of an integral equation, as will be shown in the next section.

7.3 Simultaneous Solution of the Equation of Transfer and the Statistically Steady State Equations for a Two-Level Atom

In the previous section it was pointed out that the simultaneous solution of the radiation transfer equation and the statistically steady state equations was equivalent to the solution of an integral equation. To obtain this integral equation, we substitute (7.19) for $B_\nu(T_{ex})$ into the expression for the source function (7.3). Denoting (as in Chap. 6) division by μ_ν^2 by a prime, we have

$$S'_\nu = (1-\sigma)B_\nu(T_{ex}) + \sigma J'_\nu = (1-\sigma)\cdot\frac{J'_\nu + \eta_\nu B_\nu(T_k)}{1+\eta_\nu} + \sigma J'_\nu =$$

$$= \frac{1+\sigma\eta_\nu}{1+\eta_\nu} J'_\nu + \frac{\eta_\nu(1-\sigma)}{1+\eta_\nu} B_\nu(T_k). \tag{7.20}$$

For the solution of the equation of transfer, we may use the formal solution (6.13), which gives J_ν as an integral over S_ν. This solution is valid in the "straight" ray path approximation (*cf.* Sect. 6.1), which is adequate for the stellar interior (see later in this section). For the case of no incident radiation on the surface of the star, we have

$$J'_\nu(\tau_\nu) = \frac{1}{2}\int_0^\infty S'_\nu(t_\nu) E_1(|\tau_\nu - t_\nu|)\,dt_\nu, \tag{7.21}$$

where $J'_\nu(\tau_\nu) \equiv J_\nu(\tau_\nu)/\mu_\nu^2(\tau_\nu)$, $S'_\nu(t_\nu) \equiv S_\nu(t_\nu)/\mu_\nu^2(t_\nu)$, and τ_ν is the normal optical depth to the point under consideration. The appropriate integral equation in the present case is then

$$S'_\nu(\tau_\nu) = \frac{1+\sigma\eta_\nu}{1+\eta_\nu}\cdot\frac{1}{2}\int_0^\infty S'_\nu(t_\nu) E_1(|\tau_\nu - t_\nu|)\,dt_\nu + \frac{\eta_\nu(1-\sigma)}{1+\eta_\nu} B_\nu(T_k), \tag{7.22}$$

which could in principle be solved to give $S'_\nu(\tau_\nu)$ as a function of τ_ν, provided that the electron density N_e and kinetic temperature T_k were specified functions of τ_ν and that the functional dependence of σ, η_ν, and μ_ν on N_e and T_k were known. Once $S'_\nu(\tau_\nu)$ is known, $J'_\nu(\tau_\nu)$ could be found from (7.21), and $B_\nu(T_{ex})$, that is, the atomic level populations, as a function of τ_ν would then be given by (7.19). Comparison of $B_\nu(T_{ex})$ with $B_\nu(T_k)$ would then yield precisely the departure from LTE at each point in the star.

A schematic solution of the integral equation (7.22) can be found by making use of the expansion for $J'_\nu(\tau_\nu)$ developed in Sect. 6.3, in which S'_ν is

expanded into a Taylor series about τ_ν. We may write this expansion in the form

$$J'_\nu(\tau_\nu) = S'_\nu(\tau_\nu)(1+\delta_\nu(\tau_\nu)), \tag{7.23}$$

where δ_ν depends on S'_ν and its derivatives at optical depth τ_ν. For $\tau_\nu \gg 1$, we have in the "straight" ray path approximation

$$\delta_\nu = \frac{1}{3}\frac{1}{S'_\nu}\frac{d^2 S'_\nu}{d\tau_\nu{}^2} + \frac{1}{5}\frac{1}{S'_\nu}\frac{d^4 S'_\nu}{d\tau_\nu{}^4} + \cdots. \tag{7.24}$$

Substituting J'_ν as given by (7.23) into (7.20), we have

$$S'_\nu = \frac{1+\sigma\eta_\nu}{1+\eta_\nu} S'_\nu(1+\delta_\nu) + \frac{\eta_\nu(1-\sigma)}{1+\eta_\nu} B_\nu(T_k),$$

which can be solved for S'_ν; solving and rearranging, we obtain

$$S'_\nu = B_\nu(T_k) \cdot \frac{1-\sigma}{1-(\delta_\nu/\eta_\nu)-\sigma(1+\delta_\nu)} \tag{7.25}$$

and

$$J'_\nu = B_\nu(T_k) \cdot \frac{(1-\sigma)(1+\delta_\nu)}{1-(\delta_\nu/\eta_\nu)-\sigma(1+\delta_\nu)}. \tag{7.26}$$

Using this value of J'_ν in the equation

$$B_\nu(T_{ex}) = \frac{J'_\nu + \eta_\nu B_\nu(T_k)}{1+\eta_\nu} \tag{7.27}$$

and rearranging, we obtain, finally,

$$B_\nu(T_{ex}) = B_\nu(T_k) \cdot \frac{1-\sigma(1+\delta_\nu)}{1-(\delta_\nu/\eta_\nu)-\sigma(1+\delta_\nu)}. \tag{7.28}$$

Note that the refractive index μ_ν does not appear explicitly in (7.28), although it does appear implicitly in the "small" quantity δ_ν and in η_ν (see (7.18)).

It should be emphasized that the solution (7.28) is actually only schematic, or formal, since the value of δ_ν, which is a function of S'_ν and its derivatives, can only be found accurately by actually solving the integral equation (7.22). Moreover, this solution (7.28) may not be useful in the extreme surface layers (particularly for strong absorption lines), since the power series solution (7.23) may not converge here. Nevertheless, (7.28) can be used to provide a schematic condition for the validity of LTE, which is useful at least in the stellar interior ($\tau_\nu \gg 1$), where the value of δ_ν can be estimated, at least

very roughly. It is clear from (7.28) that the condition that $B_\nu(T_{ex}) = B_\nu(T_k)$ (and hence that LTE obtain) is

$$|\delta_\nu/\eta_\nu| \ll |1-\sigma(1+\delta_\nu)| . \tag{7.29}$$

If $|\delta_\nu| \ll 1$ (as will likely be the case in the stellar interior, see next paragraph), then (7.29) is

$$|\delta_\nu/\eta_\nu| \ll 1-\sigma = \kappa_\nu'/\kappa_\nu = \kappa_\nu'/(\kappa_\nu'+\kappa_\nu^{(s)}) \tag{7.30}$$

(note that the primes in this equation do *not* represent division by $\mu_\nu{}^2$; see Sect. 7.1).

It is clear from (7.24) that, for $\tau_\nu \gg 1$, the order of magnitude of δ_ν is

$$|\delta_\nu| \sim 1/(\Delta\tau_\nu)^2 \sim 1/N , \tag{7.31}$$

where $\Delta\tau_\nu \simeq \kappa_\nu \rho \ell$ is the increment of normal optical depth over which S_ν' (or T_k) changes appreciably, ℓ being the corresponding geometrical (radial) distance, and N is the total number of scatterings and absorptions followed by re-emissions required by an "average" photon to diffuse through the distance ℓ.* In the deep interior, where ℓ may be comparable to R, the stellar radius, $(\Delta\tau_\nu)^2 \sim N$ is extremely large, say of the order of 10^{20} of so, whence $|\delta_\nu| \ll 1$; even at only moderate optical depths $|\delta_\nu|$ is likely to be <1, so that the approximate form (7.30) of the condition for LTE may be used in most cases. Using (7.31) in (7.30), we may write the approximate condition for LTE (valid at least for $\tau_\nu \gg 1$) in the form

$$\frac{\kappa_\nu'}{\kappa_\nu} N\eta_\nu \gg 1 . \tag{7.32}$$

The physical interpretation of (7.32) is clear and is the basis of the qualitative discussion presented in Sect. 4.2. The quantity $(\kappa_\nu'/\kappa_\nu)N$ is the total number of absorptions followed by re-emissions (with (true) scattering excluded) required for an average photon to diffuse through the distance ℓ over which S_ν' (or T_k) changes appreciably. Since η_ν is approximately the ratio of collisional to radiative de-excitations, then the quantity $(\kappa_\nu'/\kappa_\nu)N\eta_\nu$ is the number of times that an average photon will be destroyed by absorption followed by a collisional de-excitation (assuming the photon to be replaced by another one after each destruction) in diffusing through the distance ℓ, and (7.32) states that this number must be very large in order for LTE to obtain. The condition (7.32) may also be interpreted as saying that the

* We are tacitly assuming that the distance over which S_ν' changes significantly is comparable to the distance over which $B_\nu(T_k)$ changes significantly; this assumption is *a posteriori* justified in the stellar interior, as the immediately ensuing arguments will show.

distance from its place of origin through which an average photon may diffuse before being destroyed by absorption followed by a collisional de-excitation must be very small compared with the distance ℓ over which T_k changes appreciably. In other words, for LTE to obtain, the photons present at a given point must have originated at points differing only negligibly in (kinetic) temperature from that at the point under consideration. (See also the discussion in Sect. 4.2.)

The condition (7.32) makes it clear that (in contradiction to the expectation based on the naive interpretation of (7.19) presented in the previous section) it is not at all necessary for η_ν to be large compared to unity in order for LTE to obtain. If $\eta_\nu < 1$, however, the material must be *locally opaque*, according to (7.32), for LTE to obtain. In the deep interior η_ν could even be very small compared to unity (although it is not) and LTE would still obtain, because of the largeness of the quantity $(\kappa'_\nu/\kappa_\nu)N$ in (7.32).

We note from (7.25) that, in general, satisfaction of (7.29) (the condition for LTE) does *not* lead to the equality $S'_\nu = B_\nu(T_{ex}$ or $T_k)$, because of the possible presence of scattering (*cf.* Sect. 4.2).

Numerical estimates of the values of η_ν in stars (with $\mu_\nu = 1$) have been made by numerous workers, for example by Lecar [Le62] in the case of A0 stars and by Thomas [Th65, p.62] in the case of the sun. According to Thomas, for effective temperatures in the range $0.3 \times 10^4 \leq T_e(^\circ\mathrm{K}) \leq 10 \times 10^4$ and for transitions in the optical region of the spectrum, $\eta_\nu > 1$ for $N_e > 10^{15}$ cm^{-3}; $N_e = 10^{15}$ cm^{-3} (or $\rho \sim 10^{-8}$ gm/cm^3) at an optical depth in the continuum of about 10 in the sun. Also according to Thomas, $N_e < 10^{15}$ cm^{-3}, and hence $\eta_\nu < 1$, in the region of spectral line formation ($\tau_{\text{continuum}} \simeq 0.2$) for all spectral types; $\eta_\nu \sim 10^{-2}$ in the region of spectral line formation for the sun (Thomas [Th65, p. 62]). Lecar [Le62] also found $\eta_\nu < 1$ to be generally the case in A0 stellar atmospheres. Nevertheless, it is possible for LTE to obtain (except possibly in certain lines) in such atmospheres because of the possibility that the material may be very opaque in the relevent frequencies.

Since $N_e > 10^{15}$ (or $\rho > 10^{-8}$ gm/cm^3) throughout most of a typical stellar interior (whence $\eta_\nu > 1$) and since $(\kappa'_\nu/\kappa_\nu)N \gg 1$ also, the quantity $(\kappa'_\nu/\kappa_\nu)N\eta_\nu$ is likely to exceed unity by at least 20–30 powers of ten throughout most of a stellar interior. Hence the LTE assumption is probably realized to an extremely high order of accuracy throughout practically all of the stellar interior in essentially all cases.

CHAPTER 8

Radiative Temperature Gradient

The value of the temperature gradient at some point in a stellar interior will depend on the mode of energy transfer at that point. If the stellar material is stable against convection (*cf.* Chap. 13) and is non-degenerate (*cf.* Chap. 24), the mode of energy transfer will be radiative. (We do not consider energy transfer by neutrinos here, as they essentially do not interact with the stellar material and hence have no direct connection with the local temperature gradient, see Sects. 5.1 and 17.20.) If the material is convectively unstable, then the mode of energy transfer may be convective, radiative, or some combination of the two, depending on the "convective efficiency." If the material is significantly degenerate, then electron thermal conduction may contribute significantly (or overwhelmingly) to the total flux. Electron thermal conduction is discussed in Sect. 16.7, and convection is discussed in Chap. 14. The present chapter is concerned with the temperature gradient which exists when the local mode of energy transport is predominantly radiative.

In Sect. 8.1 we derive a general relation between the monochromatic radiation pressure tensor and the vector net flux of radiation, valid for an isotropic source function. In Sect. 8.2 we define and discuss the *Rosseland mean* mass absorption coefficient. We consider in this chapter not only the usual case of unity refractive index μ_ν, but also the case of a non-unity and spatially variable μ_ν.

8.1 General Relation Between Radiation Pressure Tensor and Vector Net Flux

The desired relation between the monochromatic radiation pressure tensor $\mathsf{p}_{r,\nu}$ (*cf.* Sect. 2.5) and the vector net flux $\mathbf{F}_\nu$ (*cf.* Sect. 2.2) is obtained

from the (time-independent) equation of transfer in the form appropriate to LTE (see (5.13)):

$$\frac{1}{\rho}\mathbf{n}\cdot\nabla I_\nu' + \frac{1}{\rho}\frac{\mathrm{d}\mathbf{n}}{\mathrm{d}s}\cdot\nabla_n I_\nu' = \kappa_\nu S_\nu' - \kappa_\nu I_\nu' , \tag{8.1a}$$

where (as in Chaps. 6 and 7) *throughout this section* a prime means division by μ_ν^2 (unless specified otherwise), where μ_ν is the (real) refractive index. The meaning of the mass absorption coefficient κ_ν and of the source function S_ν for the LTE case was explained in Sect. 4.1.* The unit vector $\mathbf{n}$ specifies the (arbitrary) direction of observation at the spatial point $\mathbf{r}$ under consideration ($\mathbf{n}$ is also the unit tangent vector of the ray path followed by the photons passing through $\mathbf{r}$ in direction $\mathbf{n}$); $\mathrm{d}\mathbf{n}/\mathrm{d}s$ is the spatial rate of change of direction of the photons (brought about by spatial variation of μ_ν) following the ray path specified by $\mathbf{r}$ and $\mathbf{n}$; ∇ is the ordinary spatial gradient vector operator; and ∇_n is the "directional" gradient vector operator (*cf.* Sect. 2.11). The interpretation of the two terms on the left side of (8.1a) was explained in Sect. 2.9.

We first multiply (8.1a) through by $\mathbf{n}$ and note that, since ∇ implies only *spatial* differentiation, with direction (say $\mathbf{n}$) held fixed, we may write $\mathbf{n}(\mathbf{n}\cdot\nabla I_\nu') = \nabla\cdot(\mathbf{n}\mathbf{n}I_\nu')$. Next, we multiply (8.1a) through by $\mathrm{d}\omega_\mathbf{n}$, the element of solid angle about $\mathbf{n}$, and integrate over all solid angles. We obtain

$$\frac{1}{\rho}\nabla\cdot\int_{4\pi} I_\nu'\mathbf{n}\mathbf{n}\mathrm{d}\omega_\mathbf{n} + \frac{1}{\rho}\int_{4\pi}\mathbf{n}\left(\frac{\mathrm{d}\mathbf{n}}{\mathrm{d}s}\cdot\nabla_n I_\nu'\right)\mathrm{d}\omega_\mathbf{n} = \kappa_\nu\int_{4\pi} S_\nu'\mathbf{n}\mathrm{d}\omega_\mathbf{n} - \kappa_\nu\int_{4\pi} I_\nu'\mathbf{n}\mathrm{d}\omega_\mathbf{n} . \tag{8.1b}$$

The first term on the right side of (8.1b) vanishes if S_ν is isotropic (recall (*cf.* Chap. 2) that in this book we consider only isotropic media, for which μ_ν is not a function of direction), since $\int_{4\pi}\mathbf{n}\mathrm{d}\omega_\mathbf{n} = 0$. We then have, assuming that S_ν is isotropic and referring to the definitions of $\mathsf{p}_{r,\nu}$ (see (2.34′)) andf $\mathbf{F}_\nu$ (see (2.16)),

$$\nabla\cdot\left(\frac{\mathsf{p}_{r,\nu}'}{\mu_\nu}\right) + c\int_{4\pi}\mathbf{n}\left(\frac{\mathrm{d}\mathbf{n}}{\mathrm{d}s}\cdot\nabla_n I_\nu'\right)\mathrm{d}\omega_\mathbf{n} = -\frac{\kappa_\nu\rho}{c}\mathbf{F}_\nu' . \tag{8.1c}$$

Consider now the value of the integral on the left side of (8.1c). This integral obviously vanishes identically if μ_ν is constant in space (since then $\mathrm{d}\mathbf{n}/\mathrm{d}s = 0$, *cf.* Sect. 2.9). We shall show here that this integral, which we shall

* Actually, the formal developments in this section do not depend at all on the LTE assumption, since the (time-independent) equation of transfer can always be written in the form (8.1a), where the detailed meaning of κ_ν and S_ν is then not necessarily the same as in the LTE case. All formal developments in this section following (8.1b) do, however, depend on the assumption of *isotropy* of S_ν.

for the time being call $\mathscr{J}$, also vanishes identically when μ_ν is *not* constant in space. Integrating by parts, we obtain

$$\mathscr{J} \equiv \int_{4\pi} \mathbf{n}\left(\frac{\mathrm{d}\mathbf{n}}{\mathrm{d}s}\cdot\nabla_n I'_\nu\right)\mathrm{d}\omega_{\mathbf{n}} = \int_{4\pi} \nabla_n\cdot\left(\mathbf{n}\frac{\mathrm{d}\mathbf{n}}{\mathrm{d}s}I'_\nu\right)\mathrm{d}\omega_{\mathbf{n}} - \int_{4\pi} I'_\nu \nabla_n\cdot\left(\mathbf{n}\frac{\mathrm{d}\mathbf{n}}{\mathrm{d}s}\right)\mathrm{d}\omega_{\mathbf{n}} \qquad (8.1\mathrm{d})$$

$$\equiv \mathscr{J}_1 - \mathscr{J}_2,$$

which defines the two integrals $\mathscr{J}_1$ and $\mathscr{J}_2$. However, noting that the vectors $(\mathrm{d}\mathbf{n}/\mathrm{d}s)$ and $(I'_\nu \mathrm{d}\mathbf{n}/\mathrm{d}s)$ are both perpendicular to $\mathbf{n}$ (*cf.* Sect. 2.9), we may use a result established in Sect. 2.11. This result states that $\nabla_n\cdot(\mathbf{nv})\equiv 2\mathbf{v}$ for any vector $\mathbf{v}$ which is always perpendicular to $\mathbf{n}$. Hence the integrands of $\mathscr{J}_1$ and $\mathscr{J}_2$ are both equal to $2\,I'_\nu \mathrm{d}\mathbf{n}/\mathrm{d}s$; $\mathscr{J}_1$ and $\mathscr{J}_2$ are thus identically equal, and $\mathscr{J}\equiv 0$, Q.E.D.

The desired general relation between $\mathsf{p}_{r,\nu}$ and $\mathbf{F}_\nu$ is then

$$\nabla\cdot\left(\frac{\mathsf{p}'_{r,\nu}}{\mu_\nu}\right) = -\frac{\kappa_\nu\rho}{c}\mathbf{F}'_\nu,^{*} \qquad (8.2\mathrm{a})$$

where $\mathsf{p}'_{r,\nu}\equiv \mathsf{p}_{r,\nu}/\mu_\nu^2$ and $\mathbf{F}'_\nu\equiv\mathbf{F}_\nu/\mu_\nu^2$. (The physical interpretation of (8.2a) (or of later versions of this equation) in terms of "quasi-photons" (*cf.* Sect. 2.10) will be discussed later in this section.) In the case where $\mu_\nu = 1$ everywhere, (8.2a) reduces to the more usual form of this equation:

$$\nabla\cdot\mathsf{p}_{r,\nu} = -\frac{\kappa_\nu\rho}{c}\mathbf{F}_\nu. \qquad (8.2\mathrm{b})$$

It can easily be shown that (8.2a) and (8.2b) are also valid if the source function is non-isotropic but symmetric as regards forward and backward scattering. A particular example of this is Thomson scattering, which depends on direction through the factor $1+\cos^2\theta$, where θ is the angle between the radius vector and the direction of scattering. (For a discussion of a general non-isotropic source function, *cf.* Frank-Kamenetskii [Fr62, pp.88–90].)

For the case of spherical symmetry with axial symmetry about every radial direction (which is the case we are interested in), (8.2a) and (8.2b) take simpler forms. Consider an orthogonal cartesian coordinate system with origin at the point of interest and with the positive z (or x_3) axis along the (outward) radius vector (as in Fig. 2.9, where the polar and aximuthal angles θ and ϕ are defined). Using the relations (2.40′) between the direction cosines α_i and the angles θ and ϕ, using the expression (2.35) giving the components p_{ij}

* This relation between $\mathsf{p}_{r,\nu}$ and $\mathbf{F}_\nu$ for the case of a spatially variable refractive index μ_ν has not, to the best of our knowledge, previously appeared in the published literature and is, we believe, a new result.

(omitting temporarily the subscripts r,ν) of p, and noting that in this case I_ν does not depend on ϕ, it is easy to see that, in this coordinate system, only the diagonal components of p or of (p/μ_ν^3) are non-zero.* Consider now the vector $\nabla \cdot (\mathsf{p}/\mu_\nu^3)$, the k^{th} component of which is

$$[\nabla \cdot (\mathsf{p}/\mu_\nu^3)]_k = \sum_i \frac{\partial}{\partial x_i}\left(\frac{p_{ik}}{\mu_\nu^3}\right) = \frac{\partial}{\partial x_k}\left(\frac{p_{kk}}{\mu_\nu^3}\right)$$

since, for axial symmetry, $p_{ik} = 0$ for $i \neq k$. Because of the spherical symmetry, however, the components p_{ij} and the quantities (p_{ij}/μ_ν^3) depend only on radial distance r (or only on x_3, not on x_1 or x_2) in the neighborhood of the point under consideration (the origin of coordinates). Consequently, at this point we have

$$[\nabla \cdot (\mathsf{p}/\mu_\nu^3)]_{1,2} = 0,$$

$$[\nabla \cdot (\mathsf{p}/\mu_\nu^3)]_3 = \frac{\partial}{\partial x_3}\left(\frac{p_{33}}{\mu_\nu^3}\right).$$

But

$$p_{33}/\mu_\nu = \frac{1}{c}\int_{4\pi} I_\nu(\theta)\alpha_3^2 \mathrm{d}\omega = \frac{2\pi}{c}\int_{-1}^{1} I_\nu(\mu)\mu^2 \mathrm{d}\mu \equiv p_{r,\nu}/\mu_\nu \tag{8.3}$$

($\mu \equiv \cos\theta$), where $p_{r,\nu}$, the component along the outward radial direction of the force exerted by radiation on an element of unit area whose inward normal is along $(-\mathbf{r})$, is what we have been calling the "radiation pressure" (*cf.* Sect. 2.5). Hence, with this kind of symmetry the vector $[\nabla \cdot (\mathsf{p}_{r,\nu}/\mu_\nu^3)]$ is directed radially (parallel to $\pm\,\mathbf{r}, \mathbf{r}$ = radius vector from stellar center) and its magnitude is equal to that of $\nabla(p_{r,\nu}/\mu_\nu^3)$; (8.2a) and (8.2b) may then be written in the forms

$$\nabla\left(\frac{p_{r,\nu}}{\mu_\nu^3}\right) = -\frac{\kappa_\nu \rho}{c}\frac{\mathbf{F}_\nu}{\mu_\nu^2}, \tag{8.4a}$$

$$\nabla p_{r,\nu} = -\frac{\kappa_\nu \rho}{c}\mathbf{F}_\nu; \tag{8.4b}$$

or

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{p_{r,\nu}}{\mu_\nu^3}\right) = -\frac{\kappa_\nu \rho}{c}\frac{F_\nu}{\mu_\nu^2}, \tag{8.5a}$$

$$\frac{\mathrm{d}p_{r,\nu}}{\mathrm{d}r} = -\frac{\kappa_\nu \rho}{c}F_\nu, \tag{8.5b}$$

* Recall that μ_ν is assumed isotropic, *cf.* Chap. 2.

where F_ν, the "scalar" net flux, is defined in Sect. 2.2. We note that a shorter derivation of (8.5b) (for $\mu_\nu = 1$) is to assume axial symmetry about every radial direction from the outset and write (8.1a) in the form

$$\frac{1}{\rho}\mu \mathrm{d}I_\nu/\mathrm{d}r = \kappa_\nu S_\nu - \kappa_\nu I_\nu \ ,$$

where $\mu = \cos\theta$. Multiplying by $\mu \mathrm{d}\mu$ and integrating from $\mu = -1$ to $\mu = +1$, we immediately obtain (8.5b) for an isotropic source function and for unity refractive index.

We note that a more restricted case in which (8.2a) and (8.2b) reduce to the forms (8.4a) and (8.4b) is that of a pure hydrostatic radiation pressure (as would result from an isotropic radiation field, for example), where $\mathsf{p}_{r,\nu} = p_{r,\nu}\mathsf{I}$ (I = unit tensor, $p_{r,\nu}$ given by (8.3)); for, then $\nabla \cdot [(p_{r,\nu}/\mu_\nu{}^3)\mathsf{I}] = \nabla(p_{r,\nu}/\mu_\nu{}^3)$. It is interesting to note that, as can easily be shown, the radiation pressure tensor in a star reduces to this "pure hydrostatic" form if only the first two terms in the expansions for I_ν (*cf.* Sect. 6.3) are retained, *i.e.*, if I_ν is of the form $I_\nu = A + B\cos\theta$, where A and B are independent of θ.

The interpretation of (8.5b) is that $\kappa_\nu \rho\, F_\nu/c$ is the net amount of momentum absorbed, per unit frequency interval, by a unit volume of matter in unit time from a radiation field whose net flux is F_ν. This gain of momentum by the matter represents a force, and this force per unit volume per unit frequency interval exerted by the radiation must be just the gradient of the monochromatic radiation pressure. (The emission from the element, having been assumed isotropic or symmetric as regards the forward and backward directions, contributes nothing to the momentum balance and so does not appear in (8.5b).)

Consider now the interpretation of (8.4a), which can be written as follows:

$$\nabla p_{r,\nu} = -\mu_\nu \frac{\kappa_\nu \rho}{c}\mathbf{F}_\nu + 3p_r \nabla \ln \mu_\nu \, . \tag{8.4c}$$

We shall base our qualitative interpretation of (8.4c) on the concept of "quasi-photons" (see Sects. 2.10a, 2.10d, and 10.2). The first term on the right side of (8.4c) is the same, aside from the factor μ_ν, as the right side of (8.4b). This factor μ_ν, it will be recalled, is the factor by which the momentum of a "bare" photon (a photon in a vacuum) of energy $\hbar\omega$ and momentum $\hbar\omega/c$ ($\omega = 2\pi\nu$ = angular frequency, $2\pi\hbar$ = Planck's constant) must be multiplied to yield the momentum of the corresponding quasi-photon (the photon in a material, dispersive medium) of the same energy (*cf.* Sect. 2.10a). In the case of a dispersion relation of the form (2.163) (see also (2.157)), μ_ν is also equal to v_g/c (v_g = group velocity of the quasi-photons, *cf.* Sect. 2.10a).

Since (*cf.* Sect. 2.5) $p_{r,\nu}$ was defined in terms of momentum transfer by quasi-photons, then the presence of this factor μ_ν (which will be present even if μ_ν is constant in space) in (8.4c) is at least not unreasonable.

Consider now the last term in (8.4c), which can be non-zero even if $\mathbf{F}_\nu = 0$ (as would be the case for an isotropic radiation field, for example) and which arises from the spatial variation of refractive index μ_ν. Since $(-\nabla p_{r,\nu})$ is equal to the vector force per unit volume per unit frequency interval (normally directed radially outward in a star) due to radiation of frequency ν, this last term is seen to represent a force per unit volume per unit frequency interval having the same direction as $(-\nabla\mu_\nu)$. In the case of a star in which complete ionization may be assumed, μ_ν is given (*cf.* Sect. 2.10d) approximately by a relation of the form $\mu_\nu = \sqrt{1 - \text{const.}\, N_e}$, where N_e is the electron density (proportional to mass density ρ in the case of complete ionization, if all electrons present are ionization electrons, *cf.* Chap. 15). Since ρ normally decreases outward in a star (*cf.* Part II of this book), it follows that μ_ν normally *increases* outward in a star, at least in those regions where the ionization is essentially complete. Hence $(-\nabla\mu_\nu)$ is normally directed *toward* the stellar center, whereas $(-\nabla p_{r,\nu})$ is normally directed *away from* the stellar center. In stars, then, this last term in (8.4c) normally *diminishes* the (outward) force due to radiation pressure.

The effect of this last term in (8.4c) can be understood qualitatively in terms of the properties of quasi-photons. Consider, purely for the sake of argument, the case where $\mathbf{F}_\nu = 0$, but where a gradient of μ_ν exists. To simplify the discussion still further, suppose that the entire radiation field is made up of only *two* quasi-photons, each having the same energy $\hbar\omega$ but moving in opposite directions and crossing a mathematical surface element perpendicularly (hence the net flux at the surface element is zero). Let the value of μ_ν for the quasi-photon lying, say, to the left of the surface element (and moving toward the right) be smaller than the value of μ_ν for the other quasi-photon. The magnitudes of the momenta of the two quasi-photons will then be different, and a spatial gradient of momentum will be present (momentum increasing from left to right). Because $p_{r,\nu}$ is defined in terms of momentum transfer by quasi-photons (*cf.* Sect. 2.5), it is clear that the spatial gradient of momentum referred to above is equivalent to a spatial gradient of $p_{r,\nu}$ ($p_{r,\nu}$ increasing from left to right), and hence to a force (directed from right to left). From this argument we see that a spatial variation in μ_ν should lead to a contribution to the force due to the radiation field having a direction parallel to $(-\nabla\mu_\nu)$, in agreement with (8.4c).

The above argument may be more clear if the two quasi-photons are regarded as material particles of equal total energy (including rest energy) but of different rest masses. If we associate a momentum $p = (\hbar\omega/c)\mu_\nu$ (see

Sect. 2.10) with a quasi-photon of total energy $E = \hbar\omega$, then the equivalent rest mass m_0 of a quasi-photon can be obtained by using these values of E and p in the relativistic relation $E^2 = p^2c^2 + m_0{}^2c^4$. We obtain

$$m_0 = \frac{\hbar\omega}{c^2}\sqrt{1-\mu_v{}^2}. \tag{8.5'}$$

(We note from (8.5') that for a dispersion relation of the form (2.163), we obtain $m_0 = \hbar\omega_P/c^2$, where ω_P is the plasma frequency, *cf.* Sect. 2.10d. We note, also, that (8.5') yields real values of m_0 only for $\mu_v \leq 1$, which is the only case we consider in the present argument.)

Since we have assumed that μ_v is smaller for the quasi-photon on the left side of the surface element (and moving toward the right), then this quasi-photon has a greater equivalent rest mass than the other quasi-photon. Since both quasi-photons have the same total energy, then the quasi-photon with the larger equivalent rest mass (the one on the left side of the surface element) has the smaller momentum (because of the above relativistic relation between E, p, and m_0), and we again conclude that a momentum gradient (momentum increasing from left to right) exists.

In all this discussion of effects of a non-unity μ_v, the reader is advised to keep in mind the way we have defined $\mathsf{p}_{r,v}$ (*cf.* Sect. 2.5), and to re-read the precautionary paragraph following (2.28').

8.2 Rosseland Mean Mass Absorption Coefficient

Because of the importance of the Rosseland mean mass absorption coefficient as conventionally defined for work in stellar interiors, and also for heuristic reasons, we divide the present discussion into two parts. In the first part (Sect. 8.2a) we base the discussion on the usual assumption of unity refractive index μ_v everywhere. In the second part (Sect. 8.2b) we discuss the modifications introduced by a non-unity and spatially variable μ_v, based on the work in Chaps. 2 through 7, and suggest a "modified" Rosseland mean mass absorption coefficient to deal with this case. This second case has not, to the best of our knowledge, been heretofore discussed in the published literature. The reader who is not interested in effects of dispersion can simply skip Sect. 8.2b.

8.2a Rosseland Mean for Unity Refractive Index Everywhere

We assume that $\mu_v = 1$ everywhere throughout this subsection. Then, we wish to define a *mean* mass absorption coefficient, such that an equation

exactly similar to (8.5b) is valid for the frequency-integrated quantities; *i.e.*, such that

$$\frac{dp_r}{dr} = \frac{d}{dr}\int_0^\infty p_{r,\nu}\,d\nu = -\frac{\kappa\rho}{c}F. \tag{8.6}$$

Using (8.5b) in the second equality in (8.6), we have

$$\frac{1}{\kappa} = \frac{F}{\int_0^\infty \kappa_\nu F_\nu d\nu} = \frac{\int_0^\infty F_\nu d\nu}{\int_0^\infty \kappa_\nu F_\nu d\nu} \tag{8.7}$$

as the definition of the required mean (known as the *flux* mean). We may also use (8.5b) to obtain

$$\frac{1}{\kappa} = \frac{\int_0^\infty \frac{1}{\kappa_\nu}\frac{dp_{r,\nu}}{dr}d\nu}{\int_0^\infty \frac{dp_{r,\nu}}{dr}d\nu}, \tag{8.8}$$

which shows explicitly that the flux mean may also be regarded as a *radiation pressure* mean if the source function is isotropic.

Although (8.7) and (8.8) are exact equations (for an isotropic source function and for $\mu_\nu = 1$), they cannot be used, in general, for the evaluation of κ because I_ν, which is obtained from the solution of the equation of transfer, must be known first in order to calculate F_ν and $p_{r,\nu}$. We may describe the situation by saying that κ, as defined by (8.7) or (8.8), is *not* a function of the local physical properties of the medium at the point of interest but depends, rather, on the complete run of the source function S_ν throughout the entire emitting region. We note, however, from the expansions given in Sect. 6.3 that, at large (normal) optical depths ($\tau_\nu \gg 1$), F_ν and $p_{r,\nu}$ approach the first terms ($(4\pi/3)dS_\nu/d\tau_\nu$ and $(4\pi/3c)S_\nu$, respectively) in their expansions; moreover, it was shown in Chap. 7 that, at the same time, S_ν rapidly approaches $B_\nu(T)$ as τ_ν becomes large compared with unity (*i.e.*, LTE sets in), even if scattering predominates over absorption (see also Sect. 4.2). As τ_ν becomes large, then, we see that

$$F_\nu \to -\frac{4\pi}{3}\frac{1}{\kappa_\nu\rho}\frac{dB_\nu(T)}{dr}, \tag{8.9a}$$

$$p_{r,\nu} \to \frac{4\pi}{3c}B_\nu(T), \tag{8.9b}$$

so that κ as given by (8.7) or (8.8) approaches κ_R as τ_ν becomes large, where (writing $dB_\nu(T)/dr = (\partial B_\nu(T)/\partial T) \cdot dT/dr$)

$$\frac{1}{\kappa_R} \equiv \frac{\int_0^\infty \frac{1}{\kappa_\nu}\frac{\partial B_\nu(T)}{\partial T}d\nu}{\int_0^\infty \frac{\partial B_\nu(T)}{\partial T}d\nu} = \frac{\int_0^\infty \frac{1}{\kappa_\nu}\frac{\partial B_\nu(T)}{\partial T}d\nu}{dB(T)/dT} = \frac{\pi}{4\sigma T^3}\int_0^\infty \frac{1}{\kappa_\nu}\frac{\partial B_\nu(T)}{\partial T}d\nu; \quad (8.10)$$

κ_R is known as the *Rosseland mean mass absorption coefficient*, the *Rosseland mean*, or, often, simply the *opacity*. We see that κ_R (toward which κ approaches for $\tau_\nu \gg 1$) has the important and useful property that it is a function *only* of the local physical properties at the point of interest and does *not* depend on the run of S_ν throughout the emitting region. For a given chemical composition, then, one may compute κ_R once and for all as a function of ρ and T, for example (*cf.* Chap. 16).

With κ defined as in (8.8), then, we have as the (exact) relation connecting the integrated net flux with the integrated radiation pressure

$$\frac{dp_r}{dr} = -\frac{\kappa\rho}{c}F(r). \quad (8.11)$$

Since $\kappa \simeq \kappa_R$ for $\tau_\nu \gg 1$, it is customary for work in stellar interiors always to replace κ by κ_R in (8.11); the resulting equation is then no longer exact but is an excellent approximation at least at large optical depths. One also uses the approximation (valid at great optical depths and for $\mu_\nu = 1$, see Sect. 6.4)

$$p_r \simeq \frac{1}{3}u \simeq \frac{1}{3}aT^4, \quad (8.12)$$

where a is the radiation constant. Also, the luminosity interior to r (the "interior luminosity") is defined by

$$L(r) \equiv 4\pi r^2 F(r). \quad (8.13)$$

Using (8.12) and (8.13) in (8.11), and henceforth simply writing κ for κ_R, we obtain

$$\frac{d}{dr}\left(\frac{1}{3}aT^4\right) = -\frac{\kappa\rho}{c}\cdot\frac{L(r)}{4\pi r^2} \quad (8.14)$$

or

$$\frac{dT}{dr} = -\frac{3\kappa\rho}{4acT^3}\frac{L(r)}{4\pi r^2}. \quad (8.15)$$

Equation (8.14) is the basic equation for computing the *radiative temperature gradient* in a star in terms of κ, ρ, T, and $L(r)$, and is one of the basic equations of stellar structure. Note that (8.14) is strictly valid only where the diffusion approximation is valid (*cf.* Sect. 6.4), *i.e.*, only at large optical depths (say $\tau \gtrsim 10$), and for $\mu_\nu = 1$ (*cf.* Sect. 8.2b). Moreover, the temperature gradient will be given by (8.14) only as long as the stellar material is in *radiative equilibrium*, *i.e.*, as long as all the (non-neutrinic) energy is being transported by radiation.

We shall investigate in Chap. 13 the stability of the radiative gradient, *i.e.*, the conditions under which (8.14) will actually describe the temperature gradient in a star.

8.2b "Modified" Rosseland Mean for Non-Unity Refractive Index

To deal with the case of a non-unity and spatially variable refractive index μ_ν, we must start with the general identity (8.5a), valid for an isotropic source function and for the kind of symmetry we have assumed (see remarks immediately preceding (8.4a)). Integrating (8.5a) over all frequencies, we obtain

$$\int_0^\infty \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{p_{r,\nu}}{\mu_\nu{}^3}\right)\mathrm{d}\nu = -\frac{\rho}{c}\int_0^\infty \frac{\kappa_\nu F_\nu}{\mu_\nu{}^2}\mathrm{d}\nu \equiv -\frac{\kappa\rho}{c}F, \tag{8.16}$$

where

$$F \equiv \int_0^\infty F_\nu \mathrm{d}\nu \tag{8.17}$$

is the integrated net flux and where the last equality in (8.16) defines the mean mass absorption coefficient κ (reasons for this choice for κ will be discussed in the next paragraph). We have, specifically,

$$\frac{1}{\kappa} \equiv \frac{\int_0^\infty F_\nu \mathrm{d}\nu}{\int_0^\infty \frac{\kappa_\nu F_\nu}{\mu_\nu{}^2}\mathrm{d}\nu} \equiv \frac{\int_0^\infty \frac{\mu_\nu{}^2}{\kappa_\nu}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{p_{r,\nu}}{\mu_\nu{}^3}\right)\mathrm{d}\nu}{\int_0^\infty \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{p_{r,\nu}}{\mu_\nu{}^3}\right)\mathrm{d}\nu}, \tag{8.18}$$

where the second equality is valid for an isotropic source function. Hence κ is a flux-weighted average of the quantity $(\kappa_\nu/\mu_\nu{}^2)$ over all frequencies.

Consider now the case of large (normal) optical depths ($\tau_\nu \gg 1$). We see from Sect. 6.4 that in this case F_ν reduces very nearly to the first term of the expansion given there, *i.e.*,

$$F_\nu \to -\mu_\nu^2 \frac{4\pi}{3} \frac{1}{\kappa_\nu \rho} \frac{dB_\nu(T)}{dr} \quad (\tau_\nu \to \infty). \tag{8.19}$$

*

Using (8.19) for F_ν in (8.5a), we see that

$$\frac{d}{dr}\left(\frac{p_{r,\nu}}{\mu_\nu^3}\right) \to \frac{4\pi}{3c} \frac{dB_\nu(T)}{dr} \quad (\tau_\nu \to \infty), \tag{8.20}$$

and hence that the first term in (8.16) becomes

$$\int_0^\infty \frac{d}{dr}\left(\frac{p_{r,\nu}}{\mu_\nu^3}\right) d\nu \to \frac{4\pi}{3c} \frac{dB(T)}{dr} = \frac{d}{dr}\left(\frac{1}{3} aT^4\right), \tag{8.21}$$

where $B(T) = (ac/4\pi)T^4$ is the integrated Planck function (*cf.* Sect. 3.7). We see, then, that with our present definition of κ the usual expression for the radiative gradient,

$$\frac{d}{dr}\left(\frac{1}{3} aT^4\right) = -\frac{\kappa\rho}{c} F \tag{8.22a}$$

(*cf.* (8.14)), becomes valid at large optical depths. In this way no alteration in the usual formulae is entailed, as all effects of dispersion have been absorbed (at large optical depths) into the mean absorption coefficient κ.

The explicit expression for κ at large optical depths is immediately obtained from (8.18) and (8.20) (writing $dB_\nu(T)/dr = (dT/dr)\partial B_\nu(T)/\partial T$):

$$\frac{1}{\kappa} = \frac{\int_0^\infty \frac{\mu_\nu^2}{\kappa_\nu} \frac{\partial B_\nu(T)}{\partial T} d\nu}{\int_0^\infty \frac{\partial B_\nu(T)}{\partial T} d\nu} = \frac{\pi}{4\sigma T^3} \int_0^\infty \frac{\mu_\nu^2}{\kappa_\nu} \frac{\partial B_\nu(T)}{\partial T} d\nu, \tag{8.22b}$$

which shows that at large optical depths ($1/\kappa$) is a kind of average of the quantity (μ_ν^2/κ_ν) over frequency, whereas (see (8.10)) the reciprocal of the usual Rosseland mean is an average of ($1/\kappa_\nu$).

* Recall (*cf.* Sect. 6.4) that this first term of the expansion is not affected at great optical depths by the presence of curvature in the ray paths of the light rays.

One final remark concerning (8.22b) is in order. It was mentioned in Sect. 2.9 that effects of dispersion appear not only explicitly, as indicated, for example, in (8.22b), but also implicitly, in κ_ν. According to Sect. 2.10b, the relation between κ_ν (with dispersion effects present) and κ_ν°, say (with dispersion effects ignored, *i.e.*, as computed from atomic cross sections), is given by

$$\kappa_\nu = \kappa_\nu^\circ/\mu_\nu \,.^* \tag{8.23}$$

If this relation is valid, then at large optical depths κ is to be computed from the formula

$$\frac{1}{\kappa} = \frac{\pi}{4\sigma T^3} \int_0^\infty \frac{\mu_\nu^3}{\kappa_\nu^\circ} \frac{\partial B_\nu(T)}{\partial T} \mathrm{d}\nu, \tag{8.24}$$

which shows a stronger effect of μ_ν on κ than is suggested by (8.22b) (with effects of dispersion on κ_ν ignored). The effects of a non-unity μ_ν on the numerical values of κ have not yet been investigated in detail.

* However, see the precautionary remarks following (2.141).

CHAPTER 9

Some Thermodynamic Relations

The purpose of this chapter is to summarize and collect together some of the basic thermodynamic laws. These principles are of great importance for work in stellar interiors and will find numerous applications throughout much of our work. Certain results already obtained in earlier chapters will be rederived on the basis of the principles to be summarized in this chapter.

It is important to note that "classical" thermodynamics is not strictly applicable to stars, for this discipline is concerned, for the most part, with systems which are "homogeneous" (uniform temperature and composition throughout, for example), in mechanical (*i.e.*, hydrostatic) equilibrium and in either thermal* or thermodynamic equilibrium (see Chap. 3 for definitions of these kinds of equilibrium); and we know that stars are not, in general, examples of such systems. Formally, a "non-equilibrium thermodynamics" such as is expounded by deGroot and Mazur [de62] would perhaps be more appropriate for applications to stellar interiors (and to astrophysics generally). However, we may assume that classical thermodynamics can be applied with sufficient accuracy to most relevant problems of interest in stellar interiors, since a star may be conceptually divided into a great many small parts, each part of which is taken small enough so that it approximately satisfies the various equilibrium and homogeneity requirements of classical thermodynamics, and therefore obeys the basic laws of this discipline. For example, we have seen that *local thermodynamic equilibrium* (LTE, *cf.* Chaps. 4 and 7) obtains throughout practically the entire stellar interior in essentially all cases. We are therefore assuming that the principles of classical thermodynamics may be applied, locally, to a star, even though the star as a whole (or even an appreciable part of it) is, thermodynamically, a highly non-equilibrium system. When thermodynamic concepts are applied to a

* The use of this term in this chapter must not be confused with our use of the term in Chap. 5.

star, then, it is important to keep in mind that the "homogeneous systems in thermal or thermodynamic equilibrium" (*cf.* Sect. 9.1), which form the subject matter of classical thermodynamics, must really be regarded as infinitesimal parts of a larger, inhomogeneous, *non-equilibrium* system (*i.e.*, the star). This assumption that the principles of classical thermodynamics may be applied, *locally*, to a non-equilibrium system is in fact the basic physical assumption underlying the new "non-equilibrium thermodynamics" [de62].

Sections 9.1 through 9.11 will be primarily devoted to a summary, based to some extent on the Caratheodory approach as presented in Chap. 1 of Chandrasekhar [Ch39], of the basic thermodynamic laws. We shall have in mind, specifically, a system (or phase) consisting of a fixed amount of a single homogeneous component or of a homogeneous mixture of several "chemically" non-interacting substances (see Sects. 3.1 and 9.1 for definitions of terms), although our treatment of the basic laws will be kept essentially quite general. We make no attempt to present a completely logical, self-contained exposition of the basic laws of thermodynamics; such an exposition is provided, for example, by the Caratheodory approach (*cf.* Chandrasekhar [Ch39, Chap. 1]). For example, we assume from the outset (as we have assumed in previous chapters) that the concept of temperature is familiar to the reader. Section 9.12 will be devoted to a consideration of homogeneous multi-component systems (or phases) in which "chemical" reactions may take place, and of conditions for chemical equilibrium. Besides the reference cited above, another excellent treatise on classical thermodynamics is Epstein [Ep37]; *cf.* also Landau and Lifshitz [La58]. The remaining sections are largely concerned with specific heats and adiabatic exponents (gammas) for various astrophysically interesting systems.

9.1 Definitions

It is assumed that the reader has some familiarity with the usual terminology employed in classical thermodynamics. Consequently, we shall here only summarize some of the more basic terms.

Several types of equilibrium are contemplated in thermodynamics, four of which are *mechanical* (or *hydrostatic*), *thermal*, *chemical*, and *thermodynamic*. These types of equilibrium have been defined and discussed in Sect. 3.1 and these definitions will not be repeated here. We only wish to emphasize that, in the absence of "chemical" reactions, *thermal* and *thermodynamic* equilibrium are synonymous. Classical thermodynamics is concerned, by and large, only with systems which are in either thermal or thermodynamic equilibrium.

A *homogeneous system* is one in mechanical and thermal equilibrium (*i.e.*, uniform temperature) whose chemical composition is uniform throughout. It is normally assumed that the pressure is also uniform throughout a homogeneous system, so that such a system would not be in the presence of any external or internal force fields. A system is generally regarded as consisting of a fixed amount of material, for example, a fixed *mass* of material if mass is conserved. If mass is not an appropriate term to use in describing the amount of a substance (as in the case of photons, for example), one might regard the entire system as "everything" contained within an adiabatically sealed enclosure (see a few paragraphs further on for the definition of this term) with completely rigid and impermeable walls that completely seal the system off from the rest of the universe.

If no "chemical" reactions may occur among the constituents of the system, then the thermodynamic state of the system can be described completely by three *thermodynamic state variables* (or simply *variables*), generally taken to be *pressure* P,* *volume* V, and *absolute temperature* T. Such homogeneous systems are often referred to as *simple* homogeneous systems, or, briefly, *simple systems*. Note that the very fact that P and T can be used as state variables implies that the system is in thermal equilibrium. When mass is assumed to be conserved in the system, one often speaks of *specific* quantities, usually defined with respect to unit mass of material; for example, the *specific volume* would be the *volume* per unit mass of material: $V = 1/\rho$ (ρ = mass density). If mass conservation is not assumed, but if *baryon*† number conservation is, then specific quantities could be defined unambiguously with respect to a single average baryon or with respect to a single average *nucleus*, for example. In this case the specific volume, for example, would be the total volume of the system divided by the total number of baryons (a constant) in the system, *i.e.*, the volume per average baryon. In Sections 9.1 through 9.11 we shall be concerned primarily (but not exclusively) with *simple* thermodynamic systems.

A heterogeneous system is one composed of a number of homogeneous systems (called *phases*) separated spatially from one another but possibly in contact with one another. For example, the phases may be separated by partitions but are regarded as not being mixed with one another.

Two idealized types of *walls*, or *partitions*, each type being the opposite extreme of the other, are contemplated in thermodynamics; these are

* In general, the *pressure tensor* (see, for example, Sects. 2.5 and 10.1), rather than the *pressure*, would have to be specified for the system; we shall, however, assume in most cases that the pressure tensor reduces simply to a pure hydrostatic pressure (see Sect. 9.7).

† By *baryon* is meant the "heavy" fundamental particles of matter (proton, neutron, etc.).

adiabatic and *diathermal* walls. An *adiabatic* wall may be defined by the following statement: The equilibrium of a system within an adiabatic enclosure but in the absence of external force fields can be disturbed only by actual movements of parts of the wall. We may say that an adiabatic wall is impervious to "heat flow." An *adiabatic process* is one that is carried out in an enclosure surrounded by adiabatic walls. A *diathermal* wall may be defined by the following statement: Two phases within an adiabatic enclosure but separated by a diathermal wall can be in thermal equilibrium only if the temperature of the two phases are the same:

$$T_1 = T_2 . \tag{9.1}$$

This equation defines *thermal contact* or *thermal equilibrium* between two phases. We may say that a diathermal wall offers no resistance to "heat flow."

For every simple thermodynamic system (or phase) there exists a function relating the three state variables P,V,T:

$$f(P,V,T) = 0 ; \tag{9.2}$$

this is known as the *equation of state* of the system. For example, the equation of state for one mole of a simple,* nondegenerate perfect gas (the mass of a mole is numerically equal to the mean molecular weight μ, *cf.* Chap. 15) is

$$PV = \mathscr{R}T , \tag{9.3}$$

where V is the volume per mole and $\mathscr{R}$ is the universal gas constant per mole. Whenever we wish to regard V as the volume per *unit mass* (as would be appropriate if μ were not constant) and not per *mole*, all we need do is replace $\mathscr{R}$ in any equation in which it occurs by $\mathscr{R}/\mu$. Any pair of the state variables P,V,T may then be taken as specifying the *state* of a simple system. Note that specification of the state of *any* thermodynamic system necessarily implies that the system is in *thermal* (but not necessarily *thermodynamic*) equilibrium, for otherwise the concept of state would have no meaning.† In the case of *non-simple* systems, further variables, such as relative concentrations of the various constituents of the system, would have to be specified in addition to two variables from the set P,V,T; see Sect. 9.12.

* The term "simple" is used here in the same sense as in the case of a simple thermodynamic system.

† Note that the *statistical mechanical* analogues of thermodynamic quantities such as pressure, internal energy, and entropy *can* be defined without reference to any equilibrium concepts, *cf.* Sect. 10.3.

9.2 "Zeroth" Law of Thermodynamics

The "Zeroth" Law of thermodynamics may be stated as follows: *If two phases of a system are separately in thermal equilibrium with a third phase, then they are in thermal equilibrium with each other.* This principle can be shown (*cf.*, for example, Chandrasekhar [Ch39, Chap. 1]) to provide the basis for a rigorous definition of an *empirical temperature* (which we do not distinguish here from the *absolute* temperature), and is the first step in the Caratheodory approach to classical thermodynamics.

9.3 First Law of Thermodynamics

The first law of thermodynamics may be stated as follows: *In bringing a thermodynamic system from a certain initial state to a certain final state* adiabatically, *the net amount of work (mechanical, electrical, etc.) required is independent of the manner in which the transition is carried out.* The initial and final states must both necessarily be *thermal* equilibrium states (*cf.* Sect. 9.1), but not necessarily thermodynamic equilibrium states (recall that these two kinds of equilibrium are identical for simple systems).

Let the initial state of the system be specified by, say, P_0, V_0, plus other necessary variables, and the final state by P, V, plus other necessary variables, and the amount of work required to bring the system from the initial to the final state *adiabatically* be W. Then the first law states that

$$W = E - E_0 , \tag{9.4}$$

where the E's are functions of the state variables of the system in the two states. If the initial state is held fixed, then W clearly depends only on the final state, and so is independent of the manner in which the transition from the initial to the final state is carried out. The function E is called the *internal energy* of the system. It is always determined to within an additive constant.

Note that the form (9.4) of the first law is valid whether the system is simple or non-simple. Consider, for example, a *non*-simple system for which $W = 0$ in the adiabatic process under consideration. Then, according to (9.4), there can be *no* change in the internal energy of the system. If a change of state of the system has nevertheless occurred, brought about by a change, due to a "chemical" reaction, for example, in the relative concentrations of the constituents of which the system is composed, then the final values of temperature and pressure may differ from the initial ones. This example illustrates the fact that in non-simple systems the internal energy E must include all relevant kinds of energy, such as latent energy of reaction; such

latent energy may, through the action of a "chemical" reaction, alter the state of a system even in an adiabatic process for which $W = 0$.

It is clear physically that the internal energy is *additive*; *i.e.*, the total internal energy of a heterogeneous system is equal to the sum of the internal energies of each of its phases. This principle is assumed to be valid whether each phase is adiabatically enclosed or whether all the phases are in thermal equilibrium with each other. The additivity principle is also assumed to apply to a *mixture* of different substances.*

9.4 Quantity of Heat

In an arbitrary *non-adiabatic* process connecting two states, suppose that the internal energy of the system increases by $(E-E_0)$ and an amount of work W is performed on the system during the process. Then the *difference*, Q, is defined as the *amount of heat* which has been supplied to the system:

$$Q = E-E_0-W\,; \tag{9.5}$$

Q is thus seen to be a derived quantity. Equation (9.5) may be regarded as an alternate statement of the first law.

9.5 Quasi-Static (or Reversible) Process

A quasi-static process is one carried out in such a manner that the system is always separated from a state of *thermodynamic equilibrium* by an infinitesimal amount. Since all variables of the system thus have their thermodynamic equilibrium values at each stage of the process, it is clear that such a process could be carried out in the reverse direction as well as in the original one, the "reverse" process carrying the system through precisely the same thermodynamic equilibrium states as the "direct" process, only in reverse order; a *quasi-static* process is therefore also a *reversible* process. For a process to be considered reversible, then, it must be carried out at a rate very slow compared to the rate at which the system can "relax" to a thermodynamic equilibrium state at each stage of the process (see Sect. 3.1).

9.6 Infinitesimal Changes

An *infinitesimal change* in a thermodynamic system may be taken as a change which carries the system from one state to another state differing only infinitesimally from the initial one. Since a system in a definite thermo-

* However, see paragraph at the end of Sect. 9.9.

dynamic state is necessarily in *thermal* (but not necessarily *thermodynamic*) *equilibrium*, then it is clear that such an infinitesimal change must be carried out at a rate slow compared to the rate at which the system can "relax" to thermal equilibrium under the given conditions. If the change is carried out *reversibly*, *i.e.*, if the two infinitesimally differing states are both in *thermodynamic equilibrium*, then the change is called an *infinitesimal quasi-static* (or *infinitesimal reversible*) *change*. (Clearly, an *infinitesimal quasi-static* change includes an *infinitesimal* change as a special case, since thermodynamic equilibrium necessarily implies thermal equilibrium.) We now write the first law, (9.5), in a form appropriate to an infinitesimal change.

Consider the expression for the work $\mathrm{d}W$ done on a system in an infinitesimal process. We exclude here and henceforth throughout this book, unless we state otherwise, other forms of work (such as electrical) than mechanical. In an infinitesimal change of volume, $\mathrm{d}V$, the work $\mathrm{d}W$ done on the system which does not result in mass motions of the system or of some part thereof is clearly given by

$$\mathrm{d}W = -P\mathrm{d}V\,, \tag{9.6}$$

provided that P, the equilibrium pressure, is constant throughout the system. If $\mathrm{d}E$ represents the change in internal energy and $\mathrm{d}Q$ the heat added to the system during the process, the first law (9.5) becomes

$$\mathrm{d}Q = \mathrm{d}E + P\mathrm{d}V \tag{9.7}$$

for infinitesimal processes. If we regard E as an explicit function of V and T for a simple system, we may write

$$\mathrm{d}Q = \left[\left(\frac{\partial E}{\partial V}\right)_T + P\right]\mathrm{d}V + \left(\frac{\partial E}{\partial T}\right)_V \mathrm{d}T\,. \tag{9.8}$$

An infinitesimal *adiabatic* process is one in which

$$\mathrm{d}Q = 0\,. \tag{9.9}$$

It should be noted that in applications of thermodynamics to hydrodynamics the differential operator d means differentiation following the motion of a given fluid element; thus

$$\mathrm{d} \equiv (\mathrm{d}t)\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right), \tag{9.9'}$$

where $\mathbf{v}$ is the local value of the fluid velocity (assumed very small in a quasi-

static process) and $\mathbf{V}$ is the gradient operator (see any treatise on vector analysis). In the case of a *static* fluid ($\partial/\partial t \equiv 0$) we have

$$\mathrm{d} \equiv \mathrm{d}\mathbf{r} \cdot \mathbf{V}\,, \tag{9.9''}$$

where $\mathrm{d}\mathbf{r}$ is a small vector displacement of the *point of observation*.

9.7 Case in Which the Pressure Tensor Does Not Reduce to a Pure Hydrostatic Pressure

If the pressure tensor P for the system does not correspond to a pure hydrostatic pressure, (9.6) for $\mathrm{d}W$ and (9.7) for the first law must be generalized.

We do this by regarding the system as a continuous fluid of fixed total mass M, enclosed within a closed surface S which may move with the fluid. The forces which act on the system are the *stresses* exerted on the surface S and *body forces* exerted throughout the body of the fluid by either external or internal force fields. The force exerted on an element of surface $\mathrm{d}\mathbf{S} = \mathbf{n}\mathrm{d}S$ with outward normal $\mathbf{n}$ is $-\mathrm{d}\mathbf{S} \cdot \mathsf{P}$ (see Sect. 2.5) and the force exerted on an element of matter of mass $\mathrm{d}m$ by the body forces will be written as $\mathbf{G}\mathrm{d}m$. The rates at which these forces do work on the elements of area $\mathrm{d}\mathbf{S}$ and of mass $\mathrm{d}m$ are clearly given by $-\mathrm{d}\mathbf{S} \cdot \mathsf{P} \cdot \mathbf{v}$ and $\mathbf{G} \cdot \mathbf{v}\mathrm{d}m$, where $\mathbf{v}$ is the local fluid velocity. The total rate at which all the forces do work on the whole system is then clearly $-\oint_S \mathrm{d}\mathbf{S} \cdot \mathsf{P} \cdot \mathbf{v} + \int_M \mathbf{G} \cdot \mathbf{v}\mathrm{d}m$, where the integrations are carried out over, respectively, the entire surface S and the entire mass M.

Some of this work will be used to compress the system or to change its shape and some will be used to impart mass motions (*i.e.*, kinetic energy) to the system or some part of it. Since we are concerned only with that part of the total work done on the system which does not result in mass motions, we must subtract off the rate at which kinetic energy of mass motions appears in the system. If $\mathrm{d}W/\mathrm{d}t$ denotes the rate at which work is done on the system which does not result in mass motions, we have

$$\frac{\mathrm{d}W}{\mathrm{d}t} = -\oint_S \mathrm{d}\mathbf{S} \cdot \mathsf{P} \cdot \mathbf{v} + \int_M \mathbf{G} \cdot \mathbf{v}\mathrm{d}m - \frac{\mathrm{d}}{\mathrm{d}t}\int_M (1/2)v^2 \mathrm{d}m\,, \tag{9.10}$$

where the last integral in (9.10) gives the total kinetic energy of mass motions (non-relativistic) in the system. The first integral in (9.10) can be transformed to a volume integral by use of the generalized divergence theorem (*cf.*, for example, Phillips [Ph33, Chap. 10]):

$$\oint_S \mathrm{d}\mathbf{S} \cdot \mathsf{P} \cdot \mathbf{v} = \int_V \mathbf{V} \cdot (\mathsf{P} \cdot \mathbf{v})\mathrm{d}V\,, \tag{9.11}$$

where the volume integration is extended over the entire (instantaneous) volume of the system. But the vector identity

$$\nabla \cdot (\mathsf{P} \cdot \mathbf{v}) \equiv \mathbf{v} \cdot (\nabla \cdot \mathsf{P}) + \mathsf{P} : \nabla \mathbf{v} \tag{9.12}$$

can be established, where the "double dot product" $\mathsf{A}:\mathsf{B}$ of two second rank tensors A and B is given in terms of their components A_{ij} and $B_{k\ell}$ in a rectangular Cartesian coordinate system by

$$\mathsf{A}:\mathsf{B} = \sum_{ij} A_{ij} B_{ij} \; ; \tag{9.13}$$

thus

$$\mathsf{P}:\nabla \mathbf{v} = \sum_{ij} P_{ij} \frac{\partial v_j}{\partial x_i} . \tag{9.14}$$

Equation (9.11) thus becomes

$$\oint_S \mathrm{d}\mathbf{S} \cdot \mathsf{P} \cdot \mathbf{v} = \int_V \mathbf{v} \cdot (\nabla \cdot \mathsf{P}) \mathrm{d}V + \int_V \mathsf{P} : \nabla \mathbf{v} \mathrm{d}V . \tag{9.15}$$

Consider now the last term in (9.10). Regarding $\mathrm{d}/\mathrm{d}t$ as a *Lagrangian* time derivative (taken following the motion of a *particular* element of matter whose mass does not change during the motion), we may write

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_M (1/2) v^2 \mathrm{d}m = \int_M \frac{\mathrm{d}}{\mathrm{d}t} ((1/2) v^2) \mathrm{d}m = \int_M \mathbf{v} \cdot \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} \mathrm{d}m . \tag{9.16}$$

However, we have the general equation of motion for unit mass of the fluid (see, for example, Landau and Lifshitz [La59, Chap. 2])

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = -\frac{1}{\rho} \nabla \cdot \mathsf{P} + \mathbf{G} , \tag{9.17}$$

where ρ is the local mass density, whence (noting that $\mathrm{d}m = \rho \mathrm{d}V$)

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_M (1/2) v^2 \mathrm{d}m = -\int_V \mathbf{v} \cdot \nabla \cdot \mathsf{P} \mathrm{d}V + \int_M \mathbf{G} \cdot \mathbf{v} \mathrm{d}m . \tag{9.18}$$

Substituting (9.15) and (9.18) into (9.10), we are left simply with

$$\frac{\mathrm{d}W}{\mathrm{d}t} = -\int_V \mathsf{P} : \nabla \mathbf{v} \mathrm{d}V , \tag{9.19}$$

and the first law may be written in the form

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \frac{\mathrm{d}E}{\mathrm{d}t} + \int_V \mathsf{P} : \nabla \mathbf{v} \mathrm{d}V . \tag{9.20}$$

Equations (9.19) and (9.20) are the appropriate generalizations of (9.6) and (9.7) for the case in which the pressure tensor does not reduce to a pure hydrostatic pressure.

Suppose now that the pressure tensor P *does* reduce to a pure hydrostatic pressure, *i.e.*, that $\mathsf{P} = P\mathsf{I}$, where I is the unit tensor (with components δ_{ij}, the Kronecker delta, *cf.* Sect. 2.5) and P is the hydrostatic pressure. In this case it is clear from (9.14) that (9.19) becomes

$$\frac{dW}{dt} = -\int_V P\nabla \cdot \mathbf{v} dV = -P\int_V \nabla \cdot \mathbf{v} dV, \tag{9.21}$$

where the second equality in (9.21) is valid if P is constant throughout the system. Transforming the second integral in (9.21) to a surface integral by use of the divergence theorem, we obtain

$$\frac{dW}{dt} = -P\oint_S \mathbf{v}\cdot d\mathbf{S} = -P\frac{dV}{dt}, \tag{9.22}$$

since the surface integral is clearly equal to the time rate of change of the entire volume of the system. Equation (9.22) is the same as (9.6), and (9.20) becomes the same as (9.7) (aside from the factor $(dt)^{-1}$).

9.8 Exactness

Equation (9.8) is seen to be of the form

$$dQ = X(x,y)dx + Y(x,y)dy, \tag{9.23}$$

where X and Y are considered now to be arbitrary functions of the two variables x and y, and dQ is now considered to be some *arbitrary* differential. The right side of (9.23) is a special case of a *Pfaffian differential expression* (*cf.* Chandrasekhar [Ch39, Chap. 1]). The integral of dQ between the two points $1(x_1,y_1)$ and $2(x_2,y_2)$ in general depends on the path of integration, or the manner in which the process is carried out. Hence $\int_1^2 dQ$ cannot in general be written only as a function of the initial and final states of the system. When this is the case, the differential dQ is said to be *not integrable*, or *not exact*. If dQ is not exact, then dQ cannot be equated to the differential $d\sigma(x,y)$ of a function $\sigma(x,y)$ of x and y. (If dQ represents the element of heat added to a system, then dQ is easily seen from the first law (9.7) to be inexact in general because PdV is in general not exact; this fact may be understood by forming the integral $\int_1^2 PdV$ between two points $1(P_1,V_1)$

and $2(P_2,V_2)$ on the P-V plane. The value of the integral obviously depends on the path over which the integration is carried out.)

If, however, dQ *is* exact, *i.e.*, if $dQ = d\sigma(x,y)$, where $\sigma(x,y)$ is a function of x and y, we can clearly write

$$dQ = d\sigma(x,y) = \frac{\partial\sigma}{\partial x}dx + \frac{\partial\sigma}{\partial y}dy. \tag{9.24}$$

Comparison with (9.23) shows that

$$X(x,y) = \partial\sigma/\partial x, \quad Y(x,y) = \partial\sigma/\partial y,$$

whence

$$\frac{\partial X}{\partial y} = \frac{\partial Y}{\partial x}. \tag{9.25}$$

Equation (9.25) is a *necessary* condition for exactness. It can also be shown to be sufficient (see, for example, Conkwright [Co34, Sect. 14], or any book on differential equations). The important property of an exact differential $d\sigma(x,y)$ is that the integral of $d\sigma(x,y)$ between any two points $1(x_1,y_1)$ and $2(x_2,y_2)$ is independent of the path of integration, or the manner in which the process is carried out. Thus the value of

$$\int_1^2 d\sigma(x,y) = \sigma(x_2,y_2) - \sigma(x_1,y_1) \tag{9.26}$$

depends only on the values of $\sigma(x,y)$ at the two points, not on the path connecting them.

It is clear that all the above conclusions also apply to a Pfaffian differential expression in more than two variables. In particular, the necessary and sufficient condition (9.25) applies to *every pair* of the variables x,y (and the corresponding functions X,Y).

The differential of any quantity which depends only on the state variables of the system is exact. For example, the internal energy E is a function only of the state variables of the system, so that dE is an exact differential.

9.9 Second Law of Thermodynamics

A concise formulation of the second law of thermodynamics is the following: (a) *There exists a thermodynamic function, S, a function of the state variables of the system, called the entropy, such that, for an infinitesimal quasi-static change of the system,*

$$dS = \frac{dQ}{T}, \tag{9.27}$$

where dQ *is the heat added to the system and* T *is the absolute temperature of the system during the change.* (b) *For an adiabatically enclosed system, the entropy can never decrease:*

$$\Delta S \geq 0\,, \tag{9.28}$$

where ΔS *is the change in entropy during an adiabatic process and where the inequality sign applies to irreversible processes, the equality sign to reversible (or quasi-static) processes.*

Since S is a function of the state variables of the system, it follows that dS is an exact differential. Moreover, because S can never decrease for an adiabatically enclosed system (for which $\Delta Q = 0$ for all possible processes), it must be true that

$$dS \geq dQ/T \tag{9.29}$$

for any conceivable infinitesimal change in a system, the inequality sign applying to an irreversible change, the equality sign to a reversible change. The entropy of a system is determined in thermodynamics to within an additive constant. However, the zero point for entropy is fixed by the statistical mechanical interpretation of entropy (*cf.* Sect. 10.3).

The entropy is assumed to be *additive*. Thus, the total entropy of a heterogeneous system is equal to the sum of the entropies of its respective phases. The additivity follows directly from the defining equation (9.27) if the phases are in thermal equilibrium with one another (*i.e.*, if they all have the same temperature T), since heat increments (dQ) are clearly additive. If the phases are not in thermal equilibrium with each other, the additivity follows from the possibility that the phases may (at least conceptually) be brought into thermal equilibrium with one another. The entropies of a mixture of different substances in a given phase will also be assumed here to be additive; however, it is important to note that the entropy of a phase formed by *mixing* two phases together is generally *greater* than the sum of the entropies of the two phases before mixing, because mixing is, in general, an irreversible process.

The additivity of entropy is actually subject to the following important qualification. The entropy of a system is equal to the sum of the entropies of its separate parts *only* if the interaction between these separate parts can be neglected. Thus the entropy of a system composed of particles whose interactions with one another cannot be neglected (as in the case of solids and liquids or, perhaps, a *plasma*) is not simply proportional to the number of such particles in the system but depends also on other factors such as the size and configuration of the system. Also, if the different substances in a mixture have a significant interaction with one another, the entropy of the

mixture is not simply equal to the sum of the separate entropies of the various substances, computed as if each substance alone occupied the same volume as is occupied by the mixture itself. The principle of the additivity of entropy is ultimately based, in the statistical interpretation, on the concept of "statistical independence" of the individual parts making up a system (see Landau and Lifshitz [La58, Sect. 2]). In this book we shall usually neglect interactions among the parts of a system and assume that the entropy is additive.

9.10 Conditions for Thermodynamic Equilibrium

Of equal importance with the second law is the *principle of the increase of entropy*, sometimes regarded as part of the second law. This principle states that any system completely isolated from the rest of the universe (for example, enclosed in an adiabatic enclosure with rigid walls and not in the presence of any time-varying force fields) will spontaneously change its condition with time in such a way that its entropy *increases* whenever the constraints on the system permit a change in entropy to occur. (Note that the second law only states that the entropy of such a system cannot decrease.) In the limit when no further increases in entropy can occur under the given constraints, the system is evidently in a condition of *thermodynamic equilibrium* and the entropy of the system then has its maximum possible value. Any infinitesimal change in such a system in thermodynamic equilibrium must then be a *reversible* (or *quasi-static*) change, for which, in general, $\mathrm{d}S = \mathrm{d}Q/T$. Since $\mathrm{d}Q = 0$ in an adiabatic infinitesimal change, it follows that

$$\mathrm{d}S = 0 \tag{9.30}$$

for every such change in a system in thermodynamic equilibrium. (A reversible adiabatic process is therefore an *isentropic* process.) Since (9.30) must be valid for every possible infinitesimal adiabatic change in a system in thermodynamic equilibrium, this equation may be used as a condition for thermodynamic equilibrium for the system; $\mathrm{d}S$ may then be regarded as a *variation* (*i.e.*, a *virtual* infinitesimal change), which includes the actual contemplated change, in the system. If the system is *mechanically* as well as *thermally* isolated from the rest of the universe, then no work $\mathrm{d}W$ will be done on the system by the rest of the universe. We consider here only systems for which the pressure tensor reduces to a pure hydrostatic pressure (*cf.* Sect. 9.7), so that $\mathrm{d}W = -P\mathrm{d}V$. Mechanical isolation can then be realized by requiring constancy of volume for the system. Because $\mathrm{d}Q = \mathrm{d}E + P\mathrm{d}V$ (first law), it follows that constancy of volume also implies constancy of internal energy. Hence, for a system mechanically and thermally isolated

from the rest of the universe, we have, in addition to (9.30), the further conditions

$$dV = 0, \; dE = 0 \tag{9.31}$$

which apply when the isolated system is in thermodynamic equilibrium.

The principle of the increase of entropy is also embodied in the statement that *all isolated systems tend toward thermodynamic equilibrium.*

Since, by the second law, any increase in the entropy of an adiabatically enclosed system must be the result of an irreversible process, then an increase in the entropy of any such system can be used as a *criterion* of irreversibility for a process.

For systems which are not isolated but for which the *temperature* and *volume* are held fixed, it is appropriate to introduce a new thermodynamic function, F, called the *free energy*, which is defined by the relation

$$F \equiv E - TS \,. \tag{9.32}$$

Since E and S are assumed to be additive, it follows that F is also additive for systems or phases in thermal equilibrium with one another (same value of T). Considering some arbitrary change in the system in which $\Delta V = 0$, $\Delta T = 0$, we have

$$\Delta F = \Delta E - T\Delta S = \Delta Q - T\Delta S$$

(since $\Delta E = \Delta Q$ if $\Delta V = 0$, by the first law). Since, however, $T\Delta S \geq \Delta Q$ by the second law, we have that

$$\Delta F \leq 0 \tag{9.33}$$

in any such change, the inequality sign applying to irreversible processes, the equality sign to reversible processes. For systems maintained at constant temperature and volume, then, the condition of thermodynamic equilibrium is that F be a *minimum, i.e.*, that

$$dF = 0 \,, \tag{9.34}$$

where dF is any possible infinitesimal variation in the state of the system maintained at constant T and V ($dT = 0$, $dV = 0$).

Finally, for non-isolated systems maintained at constant *temperature* and *pressure*, the appropriate thermodynamic function is customarily taken to be the *thermodynamic potential*, Φ, defined by the relations

$$\Phi \equiv E - TS + PV \tag{9.35a}$$

$$= F + PV \,. \tag{9.35b}$$

Since E, S, and V are additive, it follows that Φ is also additive for the phases of a heterogeneous system all of which have the same temperature (thermal

equilibrium) and the same pressure. (If the pressures of the various phases are not all the same, the term PV in (9.35b) must be replaced by $\sum_i P^{(i)}V^{(i)}$, where $P^{(i)}$ and $V^{(i)}$ are the pressure and volume of the i^{th} phase, and the summation is carried out over all the phases in the system. In the case of a mixture of substances (assumed non-interacting or only weakly interacting; see paragraph at the end of the previous section) in a given phase, the volume V is common to all substances, and the term PV must be replaced by $V\sum_i P_i$, where P_i is the "partial pressure" exerted by the i^{th} substance in the mixture.) Considering some arbitrary change in the system effected with $\Delta T = 0$, $\Delta P = 0$, we have

$$\begin{aligned}\Delta\Phi &= \Delta E - T\Delta S + P\Delta V \\ &= \Delta Q - T\Delta S\end{aligned}$$

by the first law. Since by the second law $\Delta Q \leq T\Delta S$, we have that

$$\Delta\Phi \leq 0\,, \tag{9.36}$$

the inequality sign applying to irreversible processes, the equality sign to reversible processes. For systems maintained at constant pressure and temperature, then, the condition for thermodynamic equilibrium is that Φ be a minimum, *i.e.*, that

$$\mathrm{d}\Phi = 0\,, \tag{9.37}$$

where $\mathrm{d}\Phi$ is any possible infinitesimal variation in the state of a system maintained at constant T and $P(\mathrm{d}T = 0, \mathrm{d}P = 0)$.

9.11 Reciprocity Relation

Let us regard the volume V and temperature T as the independent variables of a simple thermodynamic system. We then have (9.8) for an infinitesimal process:

$$\mathrm{d}Q = \left[\left(\frac{\partial E}{\partial V}\right)_T + P\right]\mathrm{d}V + \left(\frac{\partial E}{\partial T}\right)_V \mathrm{d}T.$$

For a quasi-static, or reversible, process we have from the second law

$$\mathrm{d}S = \frac{\mathrm{d}Q}{T} = \frac{1}{T}\left[\left(\frac{\partial E}{\partial V}\right)_T + P\right]\mathrm{d}V + \frac{1}{T}\left(\frac{\partial E}{\partial T}\right)_V \mathrm{d}T.$$

Since $\mathrm{d}S$ is exact, we must have from (9.25)

$$\frac{\partial}{\partial T}\left\{\frac{1}{T}\left[\left(\frac{\partial E}{\partial V}\right)_T + P\right]\right\} = \frac{\partial}{\partial V}\left[\frac{1}{T}\left(\frac{\partial E}{\partial T}\right)_V\right].$$

Carrying out the differentiations, we obtain the *reciprocity relation*:

$$\left(\frac{\partial E}{\partial V}\right)_T = T\left(\frac{\partial P}{\partial T}\right)_V - P, \tag{9.38}$$

which gives the dependence of the internal energy E on volume V for constant temperature T.

The applicability of the reciprocity relation to systems which are *not* simple (possibility of occurrence of "chemical" reactions) will be discussed at the end of the next section (Sect. 9.12).

As an application of (9.38), consider the case of one mole of a simple perfect gas, for which the equation of state is $PV = \mathscr{R}T$. Using this equation of state in (9.38), we obtain

$$\left(\frac{\partial E}{\partial V}\right)_T = 0,$$

which shows that the internal energy of a fixed mass of a simple perfect gas depends *only* on temperature and not on volume: $E = E(T)$.

As another application of (9.38), we consider black body radiation. We showed in Chap. 3 that the intensity B_ν of this kind of radiation was a function only of the frequency ν and the temperature T of the enclosure containing the radiation.

Hence the integrated intensity $B = \int_0^\infty B_\nu \mathrm{d}\nu$ is a function only of T. Since the energy density $u = (1/c)\int_{4\pi} B \mathrm{d}\omega$, then u is a function only of the temperature T of the radiation: $u = u(T)$. If the volume of the enclosure containing the radiation is V, then the internal energy of the radiation is $E = uV$. Further, we have shown that the radiation pressure of black body radiation is given by $p_r = (1/3)u$, since this radiation is isotropic (*cf.* Sect. 2.5). Substituting these relations into (9.38) gives $u = (1/3)T\mathrm{d}u/\mathrm{d}T - (1/3)u$ Thus, upon integration, we obtain $u = aT^4$, where a is the *radiation constant*. Hence we have $p_r = (1/3)u = (1/3)aT^4$, which agrees with the results we obtained in Section 3.7d. Also, we know that the integrated intensity $B(T)$ for black body radiation is homogeneous and isotropic. Thus $u = (4\pi/c)B(T) = aT^4$, or $B(T) = (ac/4\pi)T^4 = (\sigma/\pi)T^4$, in agreement with the result obtained in Sect. 3.7a, where $a = 4\sigma/c$ is the radiation constant and σ is the Stefan-Boltzmann constant.

9.12 Chemical Equilibrium

Consider a homogeneous system or phase whose thermodynamic state cannot be completely specified by giving only two of the state variables

(such as P,V,T), but for which certain additional parameters must be specified. We suppose, in particular, that the system consists of a mixture of various substances, whose "amounts" are not fixed: these "amounts" may change either as a result of "chemical" reactions (in which case our system is no longer simple, *cf.* Sect. 9.1) or by the addition or removal of substances to or from the system. We shall specify these "amounts" by giving the numbers $N_1, N_2, \ldots$ of "particles" (molecules, atoms, ions, electrons, nuclei, photons, etc.) of the various kinds of substances in the system. We assume that the system is always in *thermal* equilibrium (uniform temperature T) with uniform pressure P (no external or internal force fields), so that the system always has a definite, computable internal energy E, entropy S, etc.; however, we do *not* assume the system to be necessarily in *thermodynamic* equilibrium.

If we contemplate an infinitesimal virtual (*i.e.*, imaginary) change in the system in which we always preserve *thermal* equilibrium and do not allow any of the N_i to change, then we are clearly dealing with the case of an infinitesimal change in a *simple* system (fixed total and relative "amounts" of substances). Since thermal and thermodynamic equilibrium are one and the same in such a simple system, it follows that such an infinitesimal change would clearly be *quasi-static* (or *reversible*), and we would then have, from the second law,

$$\mathrm{d}S = \mathrm{d}Q/T,$$

or

$$T\mathrm{d}S = \mathrm{d}E + P\mathrm{d}V\,. \tag{9.39}$$

Consider now an *actual* infinitesimal change in the system in which we again always preserve thermal equilibrium but now allow some (or all) of the N_i to change. We now no longer have *thermodynamic* equilibrium, so that (9.39) is now not valid. To obtain the appropriate generalization of (9.39), we must regard the thermodynamic functions such as E and S as functions not only of some two of the thermodynamic variables, but also of all of the N_i (in general), since the values of these thermodynamic functions will obviously depend on these N_i. Regarding E as a function of S,V, and the N_i, for example, we have for any virtual infinitesimal change in the system

$$\mathrm{d}E = \left(\frac{\partial E}{\partial S}\right)_{V,N} \mathrm{d}S + \left(\frac{\partial E}{\partial V}\right)_{S,N} \mathrm{d}V + \sum_i \left(\frac{\partial E}{\partial N_i}\right)_{S,V} \mathrm{d}N_i\,, \tag{9.40}$$

where the subscript N means that *all* the N_i are kept constant during the differentiation and where in each of the partial derivatives $(\partial E/\partial N_i)_{S,V}$ all N's except the i^{th} are held constant, as well as S and V. However, for the

case in which all the N_i are held fixed, (9.39) is also valid, and all the terms in (9.40) involving the dN_i are zero. Comparison of (9.39) and (9.40) then yields the important general relations

$$\left(\frac{\partial E}{\partial S}\right)_{V,N} = T, \quad \left(\frac{\partial E}{\partial V}\right)_{S,N} = -P. \tag{9.41}$$

Using (9.41) in (9.40), the latter may also be written in the form

$$TdS = dE + PdV - \sum_i \mu_i dN_i , \tag{9.42}$$

where

$$\mu_i \equiv \left(\frac{\partial E}{\partial N_i}\right)_{S,V} \tag{9.43}$$

is called the *chemical potential* of the i^{th} substance. Equation (9.42) is sometimes referred to as the "fundamental thermodynamic identity" and is the appropriate generalization of (9.39). We state again, for further emphasis, that the two infinitesimally separated states connected by (9.42) must be in *thermal* equilibrium, but need not be in *thermodynamic* equilibrium; hence the infinitesimal change described by (9.42) is in general *irreversible*. If we regard S as a function of E,V, and the N_i, we may also form its variation; comparison with (9.42) then shows that we may also write

$$\mu_i \equiv -T\left(\frac{\partial S}{\partial N_i}\right)_{E,V}. \tag{9.44}$$

We may also consider the differentials of free energy F and thermodynamic potential Φ during an infinitesimal change in the system, making use of the defining equations (9.32) and (9.35). We have

$$dF = dE - TdS - SdT,$$

$$d\Phi = dE - TdS - SdT + PdV + VdP.$$

Using (9.42) in these equations, we obtain

$$dF = -PdV - SdT + \sum_i \mu_i dN_i , \tag{9.45}$$

$$d\Phi = -SdT + VdP + \sum_i \mu_i dN_i . \tag{9.46}$$

The forms of (9.45) and (9.46) suggest that F is to be considered a function of V,T, and the N_i, and Φ a function of T,P, and the N_i. Regarding F and

Φ thusly, we may form their differentials; comparing these differentials with (9.45) and (9.46), we obtain the following further relations:

$$\left(\frac{\partial F}{\partial V}\right)_{T,N} = -P, \quad \left(\frac{\partial F}{\partial T}\right)_{V,N} = -S, \quad \mu_i = \left(\frac{\partial F}{\partial N_i}\right)_{V,T}; \tag{9.47}$$

$$\left(\frac{\partial \Phi}{\partial T}\right)_{P,N} = -S, \quad \left(\frac{\partial \Phi}{\partial P}\right)_{T,N} = V, \quad \mu_i = \left(\frac{\partial \Phi}{\partial N_i}\right)_{P,T}. \tag{9.48}$$

It should be noted that each μ_i is, in general, a function of all the N_i as well as of some two of the state variables. Hence μ_i need not be zero even if no "particles" of type i are actually present in the mixture.

It is useful to note that all additive thermodynamic functions (such as E, S, Φ, etc.) must (in general) be linear, homogeneous functions of all of the N_i (*cf.* paragraph at the end of Sect. 9.9). For example, the thermodynamic potential Φ must be of the form

$$\Phi = \sum_i N_i \phi_i(P,T), \tag{9.49}$$

where ϕ_i is the thermodynamic potential per "particle" of type i. Comparing this expression with the third equation of (9.48), we see that we must have $\mu_i = \phi_i(P,T)$, so that the chemical potential μ_i is just the thermodynamic potential per "particle" of type i for constant pressure and temperature. We then have from (9.49) the useful relation

$$\Phi = \sum_i \mu_i N_i. \tag{9.50}$$

We note that it is not possible, in general, to write equations of the form (9.50) for the other additive functions such as E and F. In the case of F, for example, the expression analogous to (9.49) would be (*cf.* Landau and Lifshitz [La58, Sect. 24]) $F = \sum_i N_i F_i(V_i, T)$, where $V_i = V/N_i$ is the average volume per particle of type i (recall that F is regarded as a function of V, T, and the N's). Differentiation of this expression for F with respect to N_i, as in the third equation of (9.47), does not result in the simple expression $\mu_i = F_i$, because of the dependence of V_i on N_i. The result (9.50) is a consequence of the dependence of $\phi_i(P,T)$ on P and T, which are not directly related, in the present context, to the particle numbers N_i.

Before proceeding to write down the conditions for chemical equilibrium, we note two other useful relations. To obtain the first relation, we use the second equation of (9.47) in the defining equation for F (solved for E):

$$E = F + TS = F - T\left(\frac{\partial F}{\partial T}\right)_{V,N} = -T^2\left[\frac{\partial (F/T)}{\partial T}\right]_{V,N} = \left[\frac{\partial (F/T)}{\partial (1/T)}\right]_{V,N}. \tag{9.51}$$

The second relation is obtained by using the first equation of (9.47) in the defining equation for Φ:

$$\Phi = F + PV = F - V\left(\frac{\partial F}{\partial V}\right)_{T,N} = -V^2\left[\frac{\partial (F/V)}{\partial V}\right]_{T,N} = \left[\frac{\partial (F/V)}{\partial (1/V)}\right]_{T,N}. \quad (9.52)$$

We see then that, given an expression for the free energy F as a function of V,T, and the N_i, all the other thermodynamic functions (E, Φ, PV, and S) can be obtained from (9.51) and (9.52) as functions of the same independent variables.

In considering the conditions for chemical equilibrium, it is important to realize that the properties of a system in chemical equilibrium are completely independent of the particular process by which the system reached this state (*cf.* the discussion in Sect. 3.1 on detailed balancing). Hence, in deriving the conditions for chemical equilibrium we may consider any kind of process that proves convenient. In particular, let us consider a system or phase containing a mixture of several substances which is always maintained in thermal (but not necessarily thermodynamic) equilibrium at *constant* temperature T and pressure P. In an infinitesimal change involving virtual changes in the N_i, the corresponding change in the thermodynamic potential Φ is given (*cf.* (9.46)) by

$$\mathrm{d}\Phi = \sum_i \mu_i \mathrm{d}N_i\,, \quad (9.53)$$

where the μ_i are the chemical potentials of the substances in the mixture. If the system is in *chemical* as well as in *thermal* equilibrium, then, it is in *thermodynamic* equilibrium (*cf.* Sect. 3.1), for which the condition is that Φ be a minimum with respect to every conceivable change in the system (subject to the appropriate constraints, of course, *cf.* Sect. 9.10); the virtual changes in the N_i can then lead to no change in Φ, *i.e.*, $\mathrm{d}\Phi = 0$. The condition for chemical equilibrium is, then, from (9.53)

$$\sum_i \mu_i \mathrm{d}N_i = 0\,. \quad (9.54)$$

(Note that (9.54) also applies to the special case where only *one* constituent is present in the system. Note also that (9.54) could as well have been derived from (9.42) or (9.45).)

For example, if no "chemical" reactions may occur among the constituents of the system, then the $\mathrm{d}N_i$ are all independent of one another, and the only way (9.54) can be satisfied is for each individual term $\mu_i \mathrm{d}N_i$ (all i) to vanish. If all $\mu_i \neq 0$ (which is normally the case with material particles, see

Sect. 10.4), then we must have $dN_i = 0$ for all i; *i.e.*, there can be no change in the numbers of the various kinds of "particles" (and hence no change in the total number of "particles" of all types) in the system. This is consistent with the unchanging nature of a system of this kind (no "chemical" reactions) in thermodynamic or thermal equilibrium at constant P and T. If, on the other hand, it is possible to have $dN_j \neq 0$ under the given conditions for, say, "particles" of type j (as is the case with photons, whose number does not remain constant, except in a statistical sense, even in a closed system in thermodynamic equilibrium, because of interactions with matter), then the condition $\mu_j dN_j = 0$ requires that $\mu_j = 0$ for such particles. *Photons* are "particles" of this type, whence we have

$$\mu = 0 \qquad (9.55)$$

for photons in thermodynamic equilibrium (*i.e.*, for black body radiation, *cf.* Sect. 3.5).

(The correctness of this result (9.55) may also be seen by means of the following considerations. It is known (*cf.* Sect. 3.7d and the preceding section) that the internal energy of black body (or "equilibrium") radiation depends only on the temperature T of the radiation and the volume V of the container in which the radiation is supposed confined, and the radiation pressure depends only on T. Consequently, all the thermodynamic functions, such as S, F, and Φ, can also depend only on T and V for such radiation, and not on the "number of photons"; the very concept of "number of photons," in fact, enters nowhere in the purely thermodynamic treatment of black body radiation. Hence the various expressions (9.43), (9.44), (9.47), (9.48) giving μ as derivatives of the thermodynamic functions with respect to the "number of photons," can only give $\mu = 0$ for equilibrium radiation.)

Consider now a "chemical" reaction, which may be written symbolically in the form

$$\sum_i \nu_i A_i = 0\,, \qquad (9.56)$$

where A_i is the symbol for the i^{th} reactant and the ν_i, the *stoichiometric coefficients*, are positive or negative integers (see Sect. 3.4c). The ratios of these coefficients to one another give the ratios by which the corresponding numbers of "particles" of the various kinds which take part in the reaction must change whenever the reaction proceeds. For example, if N_1 changes by dN_1, then N_i must change by dN_i (assuming the i^{th} kind of "particle" to be a reactant in the reaction (9.56)), where

$$\frac{dN_i}{dN_1} = \frac{\nu_i}{\nu_1} \quad \text{or} \quad dN_i = \frac{\nu_i}{\nu_1}\, dN_1\,. \qquad (9.57)$$

Considering, then, an arbitrary virtual change dN_1 in N_1, say, the condition (9.54) for chemical equilibrium becomes, using (9.57),

$$\sum_i \mu_i dN_i = \frac{dN_1}{\nu_1} \sum_i \nu_i \mu_i = 0 .$$

Since dN_1 is arbitrary, it follows that the condition for chemical equilibrium is

$$\sum_i \nu_i \mu_i = 0 , \tag{9.58}$$

which is a relation involving the chemical potentials and the stoichiometric coefficients of the reactants of the "chemical" reaction (9.56).

It is clear that an equation of the form (9.58) applies to each type of reaction that may occur in the system (assuming that each such reaction is in equilibrium).

Because $\sum_i \mu_i dN_i = 0$ for states which are in chemical (*i.e.*, thermodynamic) equilibrium, it follows that the fundamental thermodynamic relation (9.42) becomes simply

$$T dS = dE + P dV \tag{9.59}$$

if the two infinitesimally differing states connected by the change (9.59) are *both* in chemical equilibrium.

If the μ_i in (9.58) are expressed as functions of N_i, V, and T, for example, then (9.58) provides a relation among the relative numbers of the reactants and the temperature T under conditions of chemical equilibrium. For example, if the reactants are all perfect, non-relativistic gases, the chemical potentials may be expressed as follows (*cf.* Sect. 10.5):

$$\mu_i = I_i + kT \ln \left[\frac{n_i h^3}{B_i(T)(2\pi m_i kT)^{(3/2)}} \right], \tag{9.60}$$

where the I_i are the "zero point" energies of the reactants, $B_i(T)$ are the partition functions for internal states (electronic or spin, for example), the m_i are the masses, and the n_i are the number densities. Using (9.60) in (9.58), we obtain

$$\prod_i n_i^{\nu_i} = \left[\exp\left(- \sum_i \nu_i I_i / kT \right) \right] : \prod_i \left[B_i(T)(2\pi m_i)^{(3/2)} (kT)^{(3/2)} / h^3 \right]^{\nu_i} . \tag{9.61}$$

A relation of the form (9.61) clearly applies to each reaction that is in equilibrium in the system and for which (9.60) is appropriate. It is easily seen that the Saha ionization equation (3.29) is a special case of (9.61).

We conclude this section by considering the applicability of the reciprocity relation (9.38) to non-simple systems. We start with the fundamental thermodynamic identity (9.42):

$$T\mathrm{d}S = \mathrm{d}E + P\mathrm{d}V - \sum_i \mu_i \mathrm{d}N_i \,. \tag{9.62}$$

Regarding E as a function of V, T, and the N_i, we have

$$\mathrm{d}S = \frac{1}{T}\left[\left(\frac{\partial E}{\partial V}\right)_{T,N} + P\right]\mathrm{d}V + \frac{1}{T}\left(\frac{\partial E}{\partial T}\right)_{V,N}\mathrm{d}T + \frac{1}{T}\sum_i\left[\left(\frac{\partial E}{\partial N_i}\right)_{V,T} - \mu_i\right]\mathrm{d}N_i . \tag{9.63}$$

Recalling (*cf.* Sect. 9.8) that the condition for exactness applies to every pair of variables (and corresponding functions) in a Pfaffian differential expression and that $\mathrm{d}S$ is exact, we see that the method of deriving (9.38) in the case of a simple system can also be used in the case of a non-simple system. We therefore have

$$\left(\frac{\partial E}{\partial V}\right)_{T,N} = T\left(\frac{\partial P}{\partial T}\right)_{V,N} - P \tag{9.64}$$

as the reciprocity relation for a non-simple system. Note that (9.64), in which all the N_i are held constant during the differentiations, is the same as (9.38). Note also that (9.64) is valid even if the system is not in chemical equilibrium.

Finally, we consider the case where the system is non-simple but in chemical equilibrium. In this case we have the relations

$$\sum_i \mu_i \mathrm{d}N_i = 0 \quad \text{or} \quad \sum_i \nu_i \mu_i = 0$$

(one of these relations applying to each type of reaction that is in equilibrium in the system), which determine the relative numbers of "particles" of the reactants under conditions of chemical equilibrium for the given mixture at the given volume and temperature. The N_i which partake in the reactions are then functions of V and T, and (9.63) becomes

$$\mathrm{d}S = \frac{1}{T}\left[\left(\frac{\partial E}{\partial V}\right)_{T,N} + \sum_i \left(\frac{\partial E}{\partial N_i}\right)_{V,T}\left(\frac{\partial N_i}{\partial V}\right)_T + P\right]\mathrm{d}V +$$

$$+ \frac{1}{T}\left[\left(\frac{\partial E}{\partial T}\right)_{V,N} + \sum_i \left(\frac{\partial E}{\partial N_i}\right)_{V,T}\left(\frac{\partial N_i}{\partial T}\right)_V\right]\mathrm{d}T . \tag{9.65}$$

But the sums of partial derivatives in the square brackets above are simply $(\mathrm{d}E/\mathrm{d}V)_T$ and $(\mathrm{d}E/\mathrm{d}T)_V$, respectively, which are derivatives holding either T or V fixed but allowing the N_i to change in accordance with the require-

ments of chemical equilibrium. If we regard the functions $N_i(V,T)$ as substituted (formally) in the function $E = f(V,T,N_i)$, then we can regard $E = g(V,T)$ as another function only of V and T. Then (9.65) becomes

$$dS = \frac{1}{T}\left[\left(\frac{\partial E}{\partial V}\right)_T + P\right]dV + \frac{1}{T}\left(\frac{\partial E}{\partial T}\right)_V dT,$$

and the reciprocity relation in its original form (9.38) is valid:

$$\left(\frac{\partial E}{\partial V}\right)_T = T\left(\frac{\partial P}{\partial T}\right)_V - P, \quad E = E(V,T). \tag{9.66}$$

Hence the reciprocity relation for a non-simple system in which the "reactants" are always in chemical equilibrium is formally the same as for a simple system.

9.13 Specific Heats

We define a generalized specific heat of a system as follows: The *specific heat at constant* α, c_α, is the amount of heat dQ that must be added to a system, *keeping* α *constant*, in order to raise the temperature T of the system by unit amount (for example, by one degree Kelvin). Thus

$$c_\alpha \equiv (dQ/dT)_\alpha , \tag{9.67}$$

where α (held constant in the differentiation) may or may not be a function of the state variables of a system and may even represent more than one quantity in the case of non-simple systems. The infinitesimal gain of heat dQ is given by the first law:

$$dQ = dE + P dV . \tag{9.68}$$

The most commonly used specific heats are the *specific heat at constant volume*, c_V, and the *specific heat at constant pressure*, c_P:

$$c_V \equiv (dQ/dT)_V , \tag{9.69}$$

$$c_P \equiv (dQ/dT)_P . \tag{9.70}$$

In the case of simple systems (no "chemical" reactions, fixed amount of material) the definitions (9.69) and (9.70) are unambiguous. In the case of non-simple systems (possibility of occurrence of "chemical" reactions, variable amounts of the constituents of a mixture) the definitions (9.69) and (9.70) must be supplemented by further conditions before c_V and c_P have definite meanings. For non-simple systems which are *not* in chemical

equilibrium, we shall always regard the derivatives in (9.69) and (9.70) as being performed *holding all the numbers N_i of "particles" of each kind of substance in the mixture constant.* (We henceforth omit the subscript N which has been used in previous sections to denote this practice, for reasons which will become clear presently.) When we hold all the N_i constant, our system is then effectively a simple one. For non-simple systems which are always in chemical equilibrium with respect to all relevant reactions, the N_i are functions of V and T (say). Consequently, all thermodynamic functions which depend on the N_i can be written, after (formal) substitution of the expression $N_i(V,T)$ into these functions, as functions only of V and T (or of any two of the set P,V,T). In this case, then, the derivatives in (9.69) and (9.70) are treated (formally) exactly as for a simple system, and these definitions are thus unambiguous. In our formal manipulations in this section and in the next (Sect. 9.14), we shall thus treat all systems as simple systems whose states are completely determined by any two of the three variables P,V,T. The formulae we derive will then apply not only to simple systems, but also to non-simple systems under the particular condition stated above.

It is often convenient to work in terms of *specific* quantities (*cf.* Sect. 9.1). When we do this we shall generally not use any special notation, but the notation will be explained in each case. We shall often regard specific quantities as defined with respect to *unit mass* of material; in this case V, the specific volume, would be the volume per unit mass:

$$V = 1/\rho\,, \tag{9.71}$$

where ρ is the mass density. In cases where departures from mass conservation cannot be ignored, ρ could be interpreted as the baryon number density (assuming the validity of baryon number conservation), for example, or as the nucleon or nuclear number density. In such cases V would then be interpreted as the volume per average baryon, nucleon, or nucleus, for example. Nevertheless, in all general formulae which we develop in this section and in the next one, (9.71) will be formally assumed, and the interpretation of the symbols will then depend on the physical nature of the particular system of interest. Thus c_V may be defined by the equivalent relations

$$c_V = (\mathrm{d}Q/\mathrm{d}T)_V = (\partial E/\partial T)_V \tag{9.72a}$$

$$= (\mathrm{d}Q/\mathrm{d}T)_\rho = (\partial E/\partial T)_\rho\,, \tag{9.72b}$$

and the first law for unit "mass" of material may also be written in the forms

$$\begin{aligned} \mathrm{d}Q &= \mathrm{d}E + P\mathrm{d}(1/\rho) \\ &= \mathrm{d}E - (P/\rho^2)\mathrm{d}\rho\,. \end{aligned} \tag{9.73}$$

Consider, for example, the case of a simple perfect gas. The equation of state for one mole of such a gas will be assumed to be

$$PV = \mathscr{R}T\,. \tag{9.74}$$

(It will be shown in Sects. 10.5 and 10.6 that (9.74) is valid, provided that the gas is nondegenerate, even if the gas is relativistic.) Moreover, it was shown in Sect. 9.11 that E, the internal energy per mole of such a gas, is a function only of T. Hence the specific heat at constant volume per mole is

$$c_V = (\mathrm{d}Q/\mathrm{d}T)_V = (\partial E/\partial T)_V = \mathrm{d}E/\mathrm{d}T\,. \tag{9.75}$$

It is generally assumed that c_V is constant if the gas is nonrelativistic as well as nondegenerate (this constancy follows from kinetic theory at least for the case of a simple monatomic gas (see Sect. 10.7a)). In this case we may write

$$E = c_V T\,, \tag{9.76}$$

where the constant of integration has been set equal to zero so that E will be zero at the absolute zero of temperature.

To obtain c_P, the specific heat per mole at constant pressure for our gas, we must regard T and P as the independent variables in the first law (9.68). From the equation of state (9.74) we have $\mathrm{d}(PV) = P\mathrm{d}V + V\mathrm{d}P = \mathscr{R}\mathrm{d}T$ so that, using (9.76), (9.68) becomes

$$\mathrm{d}Q = (c_V + \mathscr{R})\mathrm{d}T - V\mathrm{d}P\,.$$

We then obtain the well-known result

$$c_P = (\mathrm{d}Q/\mathrm{d}T)_P = c_V + \mathscr{R}\,. \tag{9.77}$$

It follows from (9.77) that c_P is also constant for the kind of gas we are considering. (If c_P and c_V are regarded as defined with respect to *unit mass* of material, rather than a *mole*, then $\mathscr{R}$ in (9.77) is to be replaced by $\mathscr{R}/\mu$, where μ is the mean molecular weight (*cf.* Chap. 15).)

We now derive a general relation between c_P and c_V, where we now regard them as defined with respect to unit "mass" of material. Regarding the specific internal energy $E(P,T)$ as a function of P and T, we may write the first law (9.73) in the form

$$\mathrm{d}Q = \mathrm{d}E - (P/\rho^2)\mathrm{d}\rho = \left(\frac{\partial E}{\partial P}\right)_T \mathrm{d}P + \left(\frac{\partial E}{\partial T}\right)_P \mathrm{d}T - \frac{P}{\rho^2}\mathrm{d}\rho\,. \tag{9.78}$$

However, from the equation of state $\rho = \rho(P,T)$ we have $\mathrm{d}\rho = (\partial\rho/\partial P)_T \mathrm{d}P + (\partial\rho/\partial T)_P \mathrm{d}T$, so that (9.78) becomes

$$\mathrm{d}Q = \left[\left(\frac{\partial E}{\partial P}\right)_T - \frac{P}{\rho^2}\left(\frac{\partial \rho}{\partial P}\right)_T\right]\mathrm{d}P + \left[\left(\frac{\partial E}{\partial T}\right)_P - \frac{P}{\rho^2}\left(\frac{\partial \rho}{\partial T}\right)_P\right]\mathrm{d}T.$$

By definition of c_P we have, then,

$$c_P = \left(\frac{dQ}{dT}\right)_P = \left(\frac{\partial E}{\partial T}\right)_P - \frac{P}{\rho^2}\left(\frac{\partial \rho}{\partial T}\right)_P. \tag{9.79}$$

We wish to express the two derivatives in (9.79) in alternative forms in order to relate c_P to c_V. For P = constant, we have $dP = (\partial P/\partial \rho)_T d\rho + (\partial P/\partial T)_\rho dT = 0$, whence

$$\left(\frac{\partial \rho}{\partial T}\right)_P = -\frac{\rho}{T}\frac{\chi_T}{\chi_\rho}, \tag{9.80}$$

where

$$\chi_T \equiv \left(\frac{\partial \ln P}{\partial \ln T}\right)_\rho = \left(\frac{\partial \ln P}{\partial \ln T}\right)_V, \tag{9.81}$$

$$\chi_\rho \equiv \left(\frac{\partial \ln P}{\partial \ln \rho}\right)_T = -\left(\frac{\partial \ln P}{\partial \ln V}\right)_T \tag{9.82}$$

may be called, respectively, the "temperature and density exponents in the pressure equation of state."

To transform the derivative $(\partial E/\partial T)_P$ in (9.79), we regard $E(P(\rho,T),T)$ as an explicit function of P (which is in turn regarded as an explicit function of ρ and T) and T. Thus we have

$$dE = \left(\frac{\partial E}{\partial P}\right)_T \left[\left(\frac{\partial P}{\partial \rho}\right)_T d\rho + \left(\frac{\partial P}{\partial T}\right)_\rho dT\right] + \left(\frac{\partial E}{\partial T}\right)_P dT,$$

from which it follows that

$$\left(\frac{\partial E}{\partial T}\right)_\rho = \left(\frac{\partial E}{\partial P}\right)_T \left(\frac{\partial P}{\partial T}\right)_\rho + \left(\frac{\partial E}{\partial T}\right)_P, \tag{9.83}$$

which can be solved for $(\partial E/\partial T)_P$. Using (9.80) and (9.83) in (9.79) and recalling that $c_V = (\partial E/\partial T)_\rho$, we have

$$c_P = c_V - \left(\frac{\partial E}{\partial \rho}\right)_T \left(\frac{\partial \rho}{\partial P}\right)_T \left(\frac{\partial P}{\partial T}\right)_\rho + \frac{P}{\rho T}\frac{\chi_T}{\chi_\rho} = c_V - \frac{E}{T}\left(\frac{\partial \ln E}{\partial \ln \rho}\right)_T \frac{\chi_T}{\chi_\rho} + \frac{P}{\rho T}\frac{\chi_T}{\chi_\rho}. \tag{9.84}$$

The quantity $(\partial \ln E/\partial \ln \rho)_T$ can be evaluated by use of the reciprocity relation (9.38), which can be written in the form

$$\left(\frac{\partial \ln E}{\partial \ln \rho}\right)_T = -\frac{P}{\rho E}(\chi_T - 1). \tag{9.85}$$

Using (9.85) in (9.84), we finally obtain the important and useful general result that

$$c_P - c_V = \frac{P}{\rho T}\frac{\chi_T^2}{\chi_\rho}, \tag{9.86}$$

which clearly reduces to (9.77) in the simple example cited above (noting that $\rho = \mu/V$, where V is in this example the volume per *mole* and μ is the mean molecular weight).

If the pressure P is positive, (9.86) shows that $c_P > c_V$ always, since it can be shown (*cf.* Landau and Lifshitz [La58, Sect. 21]) that χ_ρ must be positive for all real systems.

It is clear physically why c_P is always greater than c_V. If the pressure P is held constant while an amount of heat dQ is added to the system, some of this heat must go into increasing the internal energy E, and some must go into doing work against the external forces acting on the system during the expansion. Hence, if the temperature of the system is to be raised by a fixed amount dT, more heat must be added at constant pressure than at constant volume, whence $c_P > c_V$.

In terms of the gammas to be defined and discussed in the next section (Sect. 9.14), we have for the ratio of specific heats, using (9.93) and (9.97),

$$\gamma \equiv \frac{c_P}{c_V} = 1 + \frac{\chi_T}{\chi_\rho}(\Gamma_3 - 1) = \frac{\Gamma_1}{\chi_\rho} = \frac{\Gamma_3 - 1}{\chi_\rho}\frac{\Gamma_2}{\Gamma_2 - 1}; \tag{9.87}$$

this relation will be derived in the next section.

9.14 Quasi-Static Infinitesimal Adiabatic Changes

In this section we shall be concerned with the behavior of general thermodynamic systems under infinitesimal *adiabatic* changes ($dQ = 0$). In order to be able to derive general relations, we shall restrict consideration only to simple systems or to non-simple systems which are in chemical equilibrium (see the preceding section). All systems are assumed to be in thermal equilibrium (which is necessary for them to admit of a thermodynamic description, *cf.* the introduction to this chapter). All the systems we consider are therefore assumed to be always in thermodynamic equilibrium. We therefore have by the second law $dS = dQ/T = 0$ if $dQ = 0$. Hence the infinitesimal adiabatic changes we consider are isentropic (S = constant) and hence *quasi-static* (or *reversible*).

Application to some specific physical systems of astrophysical interest will be made in Sects. 9.15–9.18 and in several other places throughout the remainder of the book.

9.14a *The Three Adiabatic Exponents (Gammas)*

The requirement that $dQ = 0$ along an adiabat imposes, through the first law, one relation between the differentials dP, dT, and $d\rho$ (or dV). The relations between the various differentials along an adiabat may be described by the three *adiabatic exponents* (or *gammas*), defined as follows:

$$\left.\begin{aligned} \Gamma_1 &\equiv -\left(\frac{d\ln P}{d\ln V}\right)_{ad} = \left(\frac{d\ln P}{d\ln \rho}\right)_{ad}, \\ \frac{\Gamma_2}{\Gamma_2-1} &\equiv \left(\frac{d\ln P}{d\ln T}\right)_{ad}, \\ \Gamma_3 - 1 &\equiv -\left(\frac{d\ln T}{d\ln V}\right)_{ad} = \left(\frac{d\ln T}{d\ln \rho}\right)_{ad}, \end{aligned}\right\} \tag{9.88}$$

where we have used the relation $V = 1/\rho$ connecting the specific volume V and the "mass" density ρ (note that, if mass is not conserved, ρ can be interpreted as a *number* density, *cf.* Sect. 9.1) and where the subscript ad means that the derivatives are to be evaluated along an adiabat. Since we are always considering quasi-static, or reversible, processes, the subscript S ("entropy") could also be used in place of ad. We see from (9.88) that only two of the three gammas are independent; we have, in fact, the general identity

$$\boxed{\frac{\Gamma_1}{\Gamma_3-1} \equiv \frac{\Gamma_2}{\Gamma_2-1}.} \tag{9.89}$$

Hence, given any two of the gammas, the third can be computed from (9.89).

Another relation between the differentials is provided by the equation of state, say $P = P(\rho,T)$, of the system. Along an adiabat, then, only one of the three variables can be varied independently. Moreover, of the three gammas, only one is independent along an adiabat when the equation of state is specified (*i.e.*, when values of χ_ρ and χ_T are known, *cf.* (9.81) and (9.82)). By this we mean that, given the value of any one gamma, values of the other two may be computed from this one, when the equation state is specified. This fact may be seen from the identity (9.89) and the independent identity (9.97) which will be obtained presently from the equation of state.

We note that none of the gammas is in general equal to the ratio of specific heats

$$\gamma \equiv c_P/c_V \; ; \tag{9.90}$$

this ratio can be used to describe the adiabats only for sufficiently simple physical systems (see the next subsection for a necessary and sufficient condition that γ may be used to describe the adiabats).

The gammas play a very important role in various questions concerning stellar interiors. As we shall see (*cf.* Chaps. 13, 14, 20, 21, 23, and 26), Γ_2 is important in connection with the question of *convective instability* in stars; Γ_1 is important primarily in connection with *dynamical instability* of stars and, to a lesser extent, with properties of *pulsating* stars (*cf.* Chap. 27); and Γ_3 is important in connection with *pulsational instability* in stars (*cf.* Chap. 27).

9.14b Relations Between the Gammas and Specific Heats

To derive a general relation between the specific heat (per unit "mass") at constant volume, c_V, and Γ_3-1, we regard the specific internal energy $E(\rho,T)$ as a function of ρ and T. Thus we have from the first law, along an adiabat,

$$dQ = \left[\left(\frac{\partial E}{\partial \ln \rho}\right)_T - \frac{P}{\rho}\right]\frac{d\rho}{\rho} + \left(\frac{\partial E}{\partial \ln T}\right)_\rho \frac{dT}{T} = 0,$$

whence

$$\Gamma_3 - 1 \equiv \left(\frac{d \ln T}{d \ln \rho}\right)_{ad} = \frac{P/\rho - (\partial E/\partial \ln \rho)_T}{c_V T}, \tag{9.91}$$

since

$$c_V \equiv (\partial E/\partial T)_\rho . \tag{9.92}$$

The quantity $(\partial E/\partial \ln \rho)_T = E(\partial \ln E/\partial \ln \rho)_T$ can be eliminated by use of the reciprocity relation in the form (9.85): $(\partial \ln E/\partial \ln \rho)_T = -(P/\rho E)(\chi_T - 1)$, where χ_T is defined by (9.81). Carrying out this elimination and solving (9.91) for c_V, we obtain

$$c_V = \frac{P}{\rho T}\frac{\chi_T}{\Gamma_3 - 1}, \tag{9.93}$$

which is the desired relation. We see from this equation that for given values of P, ρ, and T a value of Γ_3 near (but greater than) unity usually implies a large value of c_V, since values of χ_T are generally of order unity for most systems of interest (see end of Sect. 9.18; see Sect. 24.9c for exceptions to this last statement).

In order to derive an expression for the ratio c_P/c_V, we shall need an identity relating Γ_1, $\Gamma_3 - 1$, χ_ρ, and χ_T, which we shall now obtain. From the equation of state in the form $P = P(\rho,T)$, we have for an arbitrary infinitesimal change

$$d \ln P = \chi_T \, d \ln T + \chi_\rho \, d \ln \rho , \tag{9.94}$$

where χ_T and χ_ρ are defined by (9.81) and (9.82). Dividing through by $\mathrm{d}\ln\rho$, we have

$$\mathrm{d}\ln P/\mathrm{d}\ln\rho = \chi_T \mathrm{d}\ln T/\mathrm{d}\ln\rho + \chi_\rho, \tag{9.95}$$

which is a general relation between the two derivatives $\mathrm{d}\ln P/\mathrm{d}\ln\rho$ and $\mathrm{d}\ln T/\mathrm{d}\ln\rho$ for the arbitrary process we are considering. Equation (9.95) must clearly be valid also for an *adiabatic* process. Recalling that

$$(\mathrm{d}\ln P/\mathrm{d}\ln\rho)_{\mathrm{ad}} \equiv \Gamma_1, \quad (\mathrm{d}\ln T/\mathrm{d}\ln\rho)_{\mathrm{ad}} \equiv \Gamma_3 - 1, \tag{9.96}$$

we immediately obtain the result

$$\Gamma_1 = \chi_T(\Gamma_3 - 1) + \chi_\rho, \tag{9.97}$$

which is the required identity.

Using (9.93) for c_V and (9.97), the expression for c_P/c_V presented previously by (9.87) follows at once from the expression (9.86) for $c_P - c_V$:

$$\gamma \equiv \frac{c_P}{c_V} = 1 + \frac{\chi_T}{\chi_\rho}(\Gamma_3 - 1) = \frac{\Gamma_1}{\chi_\rho} = \frac{\Gamma_3 - 1}{\chi_\rho}\frac{\Gamma_2}{\Gamma_2 - 1}. \tag{9.98}$$

It is seen from (9.89), (9.97), and (9.98) that a necessary and sufficient condition that $\Gamma_1 = \Gamma_2 = \Gamma_3 = \gamma = c_P/c_V$ is that $\chi_\rho = \chi_T = 1$, *i.e.*, that the equation of state be of the perfect gas law form:

$$P = \text{const.}\,\rho T.$$

It is possible, however, to have $\Gamma_1 = \Gamma_2 = \Gamma_3$ even if $\chi_\rho \neq 1$ and $\chi_T \neq 1$; but, in this case, none of the gammas is equal to $\gamma = c_P/c_V$. This second case occurs, for example, in the case of black body radiation (*cf.* Sect. 9.16), in a highly degenerate electron gas in both the non-relativistic and extreme relativistic limits (see Sect. 24.6), and in the case of electron-position pairs in both the non-relativistic and extreme relativistic limits (see Sect. 24.9c).

We note another important special case. Let us write (9.93) in the form

$$\Gamma_3 - 1 = \frac{P}{\rho T}\frac{\chi_T}{c_V} = \frac{1}{\rho}\left(\frac{\partial P}{\partial E}\right)_\rho = \left(\frac{\partial P}{\partial u}\right)_\rho, \tag{9.93'}$$

where $u = \rho E$ is the internal energy per unit *volume*. If $P = const.\ u$, then

$$\Gamma_3 - 1 = \text{const.} = P/\rho E = P/u. \tag{9.98'}$$

Hence, if $P = const.\ u$, then the value of the constant is $\Gamma_3 - 1$, which is therefore itself constant in this case. Examples of this important special case are given in Sects. 10.2, 10.7, 24.6, and 26.5a.

9.14c Computation of the Gammas

We shall describe here two methods for computing values of the gammas in general cases. Of course, one may always work out explicit expressions for the gammas directly from the first law in each individual case; the methods we shall describe here, however, are quite general (for the types of systems we consider, see the introduction to this section), simple, and straightforward.

The first method is based on the constancy of the entropy S along an adiabat for a reversible process. Suppose, for example, that S is known as a function of ρ and T, either in the form of a formula or in the form of a table, for a particular system (given chemical composition, for example). Then, along an adiabat $\mathrm{d}\ln S = 0$ whence, for example,

$$\Gamma_3 - 1 \equiv \left(\frac{\mathrm{d}\ln T}{\mathrm{d}\ln \rho}\right)_S = -\frac{(\partial \ln S/\partial \ln \rho)_T}{(\partial \ln S/\partial \ln T)_\rho}, \tag{9.99}$$

where the derivatives in (9.99) can be evaluated analytically in sufficiently simple cases and either graphically or numerically (from tables) in more complicated cases.

The second method is convenient when formulae or tables are available giving values of the specific internal energy $E(\rho,T)$ and the total pressure $P(\rho,T)$ as functions of density ρ (or specific volume V) and temperature T for the system of interest (these are the independent variables most commonly used by astrophysicists in constructing such tables). The following three derivatives are needed: $c_V \equiv (\partial E/\partial T)_\rho$ (specific heat per unit "mass" at constant volume), $\chi_\rho \equiv (\partial \ln P/\partial \ln \rho)_T$, $\chi_T \equiv (\partial \ln P/\partial \ln T)_\rho$, which can be evaluated numerically if only tables are available, or analytically in sufficiently simple cases.

First, $\Gamma_3 - 1$ may be computed from (9.93). Second, Γ_1 may be computed from $\Gamma_3 - 1$ by use of (9.97). Finally, $\Gamma_2/(\Gamma_2 - 1)$ may be computed by use of the identity (9.89)

$$\frac{\Gamma_2}{\Gamma_2 - 1} = \frac{\Gamma_1}{\Gamma_3 - 1} = \frac{\chi_\rho}{\Gamma_3 - 1} + \chi_T. \tag{9.105}$$

9.15 Gammas and Specific Heats for a Simple Perfect Gas

We shall assume our simple perfect gas to obey the perfect gas equation of state

$$P = (\mathscr{R}/\mu)\rho T, \tag{9.106}$$

where ρ is the mass density and μ, the mean molecular weight (*cf.* Chap. 15), is assumed here to be constant.

We see from (9.106) that

$$\chi_\rho = \chi_T = 1 \tag{9.108}$$

(where the χ's are defined by (9.81)). Hence (*cf.* (9.86))

$$c_P - c_V = \frac{P}{\rho T}\frac{\chi_T^2}{\chi_\rho} = \mathcal{R}/\mu\,,$$

so that γ, the ratio of specific heats, is given by

$$\gamma - 1 = \frac{\mathcal{R}/\mu}{c_V}. \tag{9.109}$$

We also have (*cf.* (9.93) and (9.97))

$$\Gamma_3 - 1 = \frac{P\chi_T}{c_V \rho T} = \gamma - 1$$

and

$$\Gamma_1 = \chi_\rho + \chi_T(\Gamma_3 - 1) = \Gamma_3 = \Gamma_2$$

since $\Gamma_2/(\Gamma_2 - 1) = \Gamma_1/(\Gamma_3 - 1)$. We therefore have

$$\Gamma_{1,2,3} = \gamma\,, \tag{9.110}$$

so that all the gammas are equal to one another and equal to c_P/c_V for this kind of gas. This result (9.110) was established in the previous section (Sect. 9.14b) for any equation of state of the perfect gas law form. If c_V is constant (as in a non-relativistic, simple perfect gas, for example), so that $E = c_V T$, then $\Gamma_{1,2,3}$ and γ are also constant. In this case of constant $\Gamma_{1,2,3}$ and γ the adiabats for the gas may be easily integrated and we obtain, from the definitions of the gammas (*cf.* (9.88)),

$$\begin{aligned} P/\rho^\gamma &= PV^\gamma = \text{const.}\,, \\ T/\rho^{\gamma-1} &= TV^{\gamma-1} = \text{const.}\,, \\ PT^{-\gamma/(\gamma-1)} &= \text{const.}\,. \end{aligned} \tag{9.111}$$

9.16 Gammas and Specific Heats for Black Body Radiation

We consider here an adiabatically sealed enclosure of volume V containing black body radiation but no matter.* Using the results established

* There must be some matter with which the radiation may interact if it is ever to come to equilibrium, *i.e.*, to become black body radiation. This matter could be the walls of the enclosure unless they are perfectly reflecting. In the latter case a mere speck of absorbing matter, whose effects on the thermodynamic properities of the radiation could be neglected, would be sufficient to provide the necessary interaction.

in Sects. 3.7c and 3.7d, we have for the internal energy E and the pressure P of the radiation

$$E = aT^4V, \qquad P = (1/3)aT^4, \tag{9.112}$$

where a is the radiation constant. The specific heat at constant volume is then

$$c_V = (\partial E/\partial T)_V = 4aT^3V. \tag{9.113}$$

From the equation of state (second equation of (9.112)) we see immediately that

$$\chi_\rho = -(\partial \ln P/\partial \ln V)_T = 0,$$

$$\chi_T = (\partial \ln P/\partial \ln T)_V = 4. \tag{9.114}$$

We then have

$$\Gamma_3 - 1 = \frac{PV\chi_T}{c_V T} = (1/3),$$

$$\Gamma_1 = \chi_\rho + \chi_T(\Gamma_3 - 1) = (4/3),$$

$$\Gamma_2/(\Gamma_2 - 1) = \Gamma_1/(\Gamma_3 - 1) = 4.$$

Hence

$$\Gamma_{1,2,3} = (4/3) = \text{const.} \tag{9.115}$$

and the integrated adiabats for black body radiation are given by

$$\begin{aligned} PV^{(4/3)} &= \text{const.}, \\ TV^{(1/3)} &= \text{const.}, \\ PT^{-4} &= \text{const.}. \end{aligned} \tag{9.116}$$

Comparison of (9.116) with the adiabats (9.111) of a simple perfect gas shows that black body radiation behaves, adiabatically, like a simple perfect gas with $\Gamma_1 = \Gamma_2 = \Gamma_3 = (4/3)$.

We note, however, that $\Gamma_{1,2,3} = (4/3)$ are *not* equal to $c_P/c_V = \gamma$ for black body radiation. We have, in fact, from (9.98),

$$\gamma = c_P/c_V = \Gamma_1/\chi_\rho = \infty \tag{9.117}$$

unless $T = 0$. Hence c_P is infinite for black body radiation at a finite temperature (this result is just a consequence of the fact that P does not depend on V in this case.)

9.17 Gammas and Specific Heats for a Mixture of Black Body Radiation and a Simple Perfect Gas

We consider now a mixture of black body radiation and a simple perfect gas with $c_V = \text{const}$. We neglect any interactions between the two constituents (gas and radiation) of the mixture (other than the negligibly small interaction required to permit the two constituents to coexist in equilibrium at the same temperature T) and assume that each constituent can be treated as if the other did not exist. We can then simply add the internal energies and pressures of the two constituents to get the total internal energy and pressure of the mixture. We consider an amount of the mixture which contains unit mass of the gas (we neglect here the relativistic mass associated with the radiation); we then have for the total pressure and the total internal energy per unit mass, using (9.109) and the relation $E_g = c_V T$,

$$P = (1/3)aT^4 + (\mathscr{R}/\mu)\rho T, \tag{9.118}$$

$$E = \frac{aT^4}{\rho} + \frac{(\mathscr{R}/\mu)T}{\gamma_g - 1}, \tag{9.119}$$

where γ_g and ρ denote, respectively, the ratio of specific heats and the mass density of the gas, and the other symbols have the same meaning as in Sects. 9.15 and 9.16.*

We then have

$$\chi_\rho = \frac{\rho}{P}\left(\frac{\partial P}{\partial \rho}\right)_T = \frac{\rho}{P}\left(\frac{\mathscr{R}T}{\mu}\right) = \frac{P_g}{P},$$

where P_g is the gas pressure. We define β to be the ratio of gas pressure to total pressure:

$$\beta \equiv P_g/P. \tag{9.120}$$

Then we have

$$\chi_\rho = \beta, \tag{9.121}$$

and

$$\chi_T = \frac{T}{P}\left(\frac{\partial P}{\partial T}\right)_\rho = \frac{T}{P}\left(\frac{4aT^4/3}{T} + \frac{\mathscr{R}\rho T/\mu}{T}\right) = \frac{4P_r}{P} + \frac{P_g}{P},$$

or

$$\chi_T = 4 - 3\beta. \tag{9.122}$$

* If the gas were dense enough so that the refractive index is not equal to unity (a non-unity refractive index is a result of an interaction between the gas and the radiation, *cf.* Sect. 2.10), then the contributions to both the pressure and internal energy of the radiation would have to be corrected as indicated in Sects. 2.3 and 2.5.

Finally, we have

$$c_V = (\partial E/\partial T)_\rho = \frac{4aT^3}{\rho} + \frac{\mathscr{R}/\mu}{\gamma_g - 1} = \frac{\mathscr{R}/\mu}{\gamma_g - 1}\left(\frac{12P_r(\gamma_g - 1)}{P_g} + 1\right) =$$

$$= \frac{\mathscr{R}/\mu}{(\gamma_g - 1)}\left(\frac{12(1-\beta)(\gamma_g - 1) + \beta}{\beta}\right). \tag{9.123}$$

It is seen that c_V approaches its perfect gas value (9.109) as $\beta \to 1$ and its black body radiation value (9.113) as $\beta \to 0$. We then have (*cf.* (9.93))

$$\Gamma_3 - 1 = \frac{P\chi_T}{c_V \rho T} = \frac{P_g}{\beta \rho T(\mathscr{R}/\mu)} \frac{\beta(4-3\beta)(\gamma_g - 1)}{[12(1-\beta)(\gamma_g - 1) + \beta]} \tag{9.124a}$$

$$= \frac{(4-3\beta)(\gamma_g - 1)}{12(1-\beta)(\gamma_g - 1) + \beta}, \tag{9.124b}$$

$$\Gamma_1 = \chi_\rho + \chi_T(\Gamma_3 - 1) = \beta + \frac{(4-3\beta)^2(\gamma_g - 1)}{12(1-\beta)(\gamma_g - 1) + \beta}, \tag{9.125}$$

and

$$\frac{\Gamma_2}{\Gamma_2 - 1} = \frac{\Gamma_1}{\Gamma_3 - 1} = \frac{12\beta(1-\beta)(\gamma_g - 1) + \beta^2 + (4-3\beta)^2(\gamma_g - 1)}{(4-3\beta)(\gamma_g - 1)}. \tag{9.126}$$

These expressions for the gammas agree with those given by Chandrasekhar [Ch39, pp. 57–58].

Since all three gammas involve β, the ratio of gas to total pressure, which depends on ρ and T, then expressions for the *integrated* adiabats are in general very complicated. For the important special case where $\gamma_g = (5/3)$, however, the expressions for the integrated adiabats become relatively simple. Such expressions have been derived by direct integration of the adiabatic equations by Chandrasekhar [Ch51, p. 667] and by Menzel, Bhatnagar, and Sen [Me63, Sect. 5.7]. A simpler method is to make use of the constancy of the entropy in a reversible adiabatic process (see (10.45), (10.68a), and Sect. 20.8). The expressions are as follows, letting $Z \equiv (1-\beta)/\beta$:

$$P = \text{const.}\ (1+Z)Z^{5/3}e^{32Z/3}, \tag{9.127a}$$

$$\rho = \text{const.}\ Ze^{8Z}, \tag{9.127b}$$

$$T = \text{const.}\ Z^{2/3}e^{8Z/3}. \tag{9.127c}$$

Using (10.68b) for an extremely relativistic, non-degenerate perfect gas and proceeding as above, one may easily show that the integrated adiabats for a mixture of this kind of gas and black body radiation are simply

$$\beta = \text{const.}\ ; \tag{9.127d}$$

i.e., the ratio of gas to total pressure is constant along an adiabat in this case. Since $Z = (1-\beta)/\beta$ is accordingly constant along an adiabat, then Z can no longer be used as a parameter along an adiabat, as in (9.127a,b,c). However, noting that $1-\beta = P_r/P \propto T^4/P$, we have that $P = \text{const.}\ T^4$ (noting that β is constant) along an adiabat. Also, since $\beta = P_g/P \propto \rho T/T^4$, we have that $\rho = \text{const.}\ T^3$. Finally, it follows from the $(P-T)$ and $(\rho-T)$ relations above that $P = \text{const.}\ \rho^{4/3}$ along an adiabat in this case.

We note that, as $\beta \to 1$, $\Gamma_{1,2,3} \to \gamma_g$; and, as $\beta \to 0$, $\Gamma_{1,2,3} \to (4/3)$. Hence the gammas for a mixture of an ideal gas and black body radiation are expected to be intermediate in value between the gamma for a simple perfect gas (γ_g) and that for black body radiation alone (4/3).

From the first equality in (9.124a) we see that for given values of P,ρ, and T, c_V will be *increased* over its simple perfect gas value as a result of the presence of the radiation, since χ_T is larger than the simple perfect gas value (unity) and Γ_3 is smaller than the simple perfect gas value (γ_g). (This conclusion also follows immediately from (9.123), since the quantity in square brackets exceeds unity for $\beta < 1$, provided that $\gamma_g > 1$.)

Finally, we may compute the ratio of specific heats for the mixture (*cf.* (9.98)):

$$\gamma = \frac{c_P}{c_V} = \frac{\Gamma_1}{\beta} = 1 + \frac{(4-3\beta)^2(\gamma_g - 1)}{12\beta(1-\beta)(\gamma_g - 1) + \beta^2}. \tag{9.129}$$

It is seen that $\gamma \to \gamma_g$ for $\beta \to 1$, and that $\gamma \to (1 + 4\beta/3) \to \infty$ as $\beta \to 0$, in agreement with the conclusions reached in Sects. 9.15 and 9.16.

9.18 Gammas and Specific Heats for a Mixture of Perfect Gases Undergoing Ionization, with Radiation Pressure Included

Here we consider a mixture of perfect gases, each of which obeys the perfect gas equation of state $P_i = n_i kT$ (n_i = number density of type i particles), but in which the n_i may change as a result of ionization; for completeness, we also include the effects of black body radiation pressure.* The mixture is assumed to be always in chemical equilibrium, which implies in this particular example that the Saha ionization equation is always instantaneously valid. We neglect the effects of any interactions between the

* The expressions for Γ_2 for this case have been worked out also by Underhill [Un49] and by Krishna-Swami [Kr61]; see also Baker and Kippenhahn [Ba62a, appendix A]. Expressions for Γ_2 for the case of negligible radiation pressure have been given by Vardya [Va60a]. Vardya [Va65] has also given values of Γ_2 for cases of multiple ionization, with effects of radiation pressure included.

gases and the radiation on the thermodynamic properties (see footnote in preceding section). We also assume that no negative ions are present (*cf.* Sect. 15.4).

We first define the relative abundance, by numbers, of element j:

$$v_j = \frac{\text{Number of atoms and ions of element } j}{\text{Total number of atoms and ions}}, \tag{9.130a}$$

where

$$\sum_j v_j = 1. \tag{9.130b}$$

Also let y_k^j be the degree of k^{th} ionization of the j^{th} element:

$$y_k^j = \frac{\text{Number of } k\text{-times ionized atoms of element } j}{\text{Number of atoms and ions of element } j}, \tag{9.131}$$

where

$$\sum_k y_k^j = 1. \tag{9.132}$$

(A "k-times ionized atom" is one which has lost k electrons.) We now define the mean degree of ionization $\bar{y}$:

$$\bar{y} = \frac{\text{Number of free ionization electrons*}}{\text{Total number of atoms and ions}}$$

$$= \sum_j v_j \left(\sum_k k y_k^j \right), \tag{9.133}$$

where the two summations are extended over all stages of ionization for each element and over all elements present.

Next we let n be the number of atoms and ions per unit volume; then the equation of state is

$$P = (1+\bar{y})nkT + (1/3)aT^4, \tag{9.134}$$

where k is Boltzmann's constant and a is the radiation constant. If X_j and A_j are the relative mass abundance and atomic mass number, respectively, of element j, then the number of atoms and ions of element j per unit volume is

$$n_j = \frac{\rho}{H} \frac{X_j}{A_j}, \tag{9.135}$$

where ρ is the mass density of the stellar material (not including the relativistic mass associated with the radiation) and $1/H$ is Avogadro's number (the number of particles in a mole). Thus

$$n = \frac{\rho}{H} \sum_j \frac{X_j}{A_j}, \tag{9.136}$$

* See Sect. 15.1 for the meaning of this term.

or the number of atoms and ions per unit mass is

$$N = \frac{n}{\rho} = \frac{1}{H}\sum_j \frac{X_j}{A_j}. \tag{9.137}$$

Thus the equation of state (9.134) may be written as

$$P = (1+\bar{y})N\rho kT + (1/3)aT^4. \tag{9.138}$$

We may also write

$$P = \frac{k}{\mu H}\rho T + (1/3)aT^4,$$

whence the mean molecular weight is expressed, in the present notation, by

$$\frac{1}{\mu} = (1+\bar{y})\sum_j \frac{X_j}{A_j} \tag{9.139}$$

(*cf.* Chap. 15).

In writing down an equation for the internal energy per unit mass of the mixture we shall make use of a result that is not formally established until the next chapter (Chap. 10), but which is no doubt well known to the reader. This is the principle of the equipartition of energy, according to which the average translational kinetic energy per free particle (assuming the gas to be nondegenerate and nonrelativistic) is $(3/2)kT$. The total translational kinetic energy per unit mass is then $N(1+\bar{y}) \cdot (3/2)kT$; and the radiant energy per unit mass of gas is aT^4/ρ. Finally, we must include the ionization energy, but we neglect excitation energy and assume that molecules are not present, so that our gas is monatomic. Let χ^j_{k-1} be the k^{th} ionization potential of element j, *i.e.*, the energy required to remove the k^{th} electron from the ground state of a $(k-1)$-times ionized "atom". Then the internal energy per unit mass is (neglecting excitation energy)

$$E = N(1+\bar{y})(3/2)kT + N\left\{\sum_j v_j\left[\sum_k y^j_k\left(\sum_{m=1}^{k}\chi^j_{m-1}\right)\right]\right\} + \frac{aT^4}{\rho}, \tag{9.140}$$

since $\sum_{m=1}^{k}\chi^j_{m-1}$ is the total energy released by each k-times ionized atom of element j. The second term on the right side of (9.140) is the *ionization* energy per unit mass. Methods of evaluating this term are given in Sect. 15.5. Numerical values of this energy plus the (here neglected) excitation energy will also be given in Sect. 15.6 as a function of ρ and T, for a particular chemical compositon.

The Saha ionization equation is (*cf.* (3.29))

$$\frac{y^j_k}{y^j_{k-1}}P_e = \frac{2B_{j,k}(T)}{B_{j,k-1}(T)}\frac{(2\pi m)^{(3/2)}}{h^3}(kT)^{(5/2)}e^{-\chi^j_{k-1}/kT}, \tag{9.141}$$

where $P_e = n_e kT$ is the electron pressure (n_e = ionization electron number density) and $B_{j,k}(T)$ is the partition function for a k-times ionized atom of element j (*cf.* (3.25)), and m is the electron rest mass. Since

$$P_e = \frac{\bar{y}}{1+\bar{y}} P_g = \frac{\bar{y}}{1+\bar{y}}(P-(1/3)aT^4) \tag{9.142}$$

is the relation between electron pressure and gas or total pressure, we may also write

$$\frac{y_k^j}{y_{k-1}^j}\frac{\bar{y}}{1+\bar{y}}(P-(1/3)aT^4) = \frac{2B_{j,k}(T)}{B_{j,k-1}(T)}\frac{(2\pi m)^{(3/2)}}{h^3}(kT)^{(5/2)}e^{-\chi_{k-1}^j/kT}. \tag{9.143}$$

We now assume that only one element, say the i^{th}, is undergoing (partial) ionization at a time (*i.e.*, $y_k^j = 0$ or 1 for each stage of ionization k for all the other elements j); furthermore, we assume that the various ionization potentials of this element are far enough apart so that only two stages of ionization, say the r^{th} and the $(r-1)^{\text{st}}$, are simultaneously in progress under the given conditions of density and temperature. Thus (9.132) becomes

$$y_r^i + y_{r-1}^i \approx 1. \tag{9.144}$$

Expressing y_{r-1}^i in terms of y_r^i from (9.144), we may write the ionization equation in the form

$$\frac{y_r^i}{1-y_r^i}\frac{\bar{y}}{1+\bar{y}}(P-(1/3)aT^4) = \frac{2B_{i,r}(T)}{B_{i,r-1}(T)}\frac{(2\pi m)^{(3/2)}}{h^3}(kT)^{(5/2)}e^{-\chi_{r-1}^i/kT}. \tag{9.145}$$

We now take the differential of E from (9.140):

$$\mathrm{d}E = (3/2)Nk(1+\bar{y})\mathrm{d}T + (3/2)NkT\mathrm{d}\bar{y} + Nv_i\left[\left(\sum_{m=1}^{r-1}\chi_{m-1}^i\right)\mathrm{d}y_{r-1}^i + \right.$$
$$\left. + \left(\sum_{m=1}^{r}\chi_{m-1}^i\right)\mathrm{d}y_r^i\right] + \frac{4aT^4}{\rho}\frac{\mathrm{d}T}{T} - \frac{aT^4}{\rho}\frac{\mathrm{d}\rho}{\rho}. \tag{9.146}$$

Because of (9.144) we have

$$\mathrm{d}y_{r-1}^i = -\mathrm{d}y_r^i,$$

whence the square bracket in (9.146) reduces to

$$\left(\sum_{m=1}^{r}\chi_{m-1}^i - \sum_{m=1}^{r-1}\chi_{m-1}^i\right)\mathrm{d}y_r^i = \chi_{r-1}^i\,\mathrm{d}y_r^i. \tag{9.147}$$

Form (9.133) we have

$$\begin{aligned}\mathrm{d}\bar{y} &= v_i[(r-1)\mathrm{d}y_{r-1}^i + r\mathrm{d}y_r^i] \\ &= v_i[-(r-1)\mathrm{d}y_r^i + r\mathrm{d}y_r^i] \\ &= v_i\mathrm{d}y_r^i.\end{aligned} \tag{9.148}$$

We now use (9.147) and (9.148) in (9.146):

$$dE = (3/2)NkT\bar{y}\left\{\left(\frac{1+\bar{y}}{\bar{y}}+\frac{4aT^4}{(3/2)NkT\rho\bar{y}}\right)\frac{dT}{T}+\frac{\nu_i dy_r^i}{\bar{y}}+\frac{\nu_i\chi_{r-1}^i dy_r^i}{(3/2)kT\bar{y}}\right.$$

$$\left.-\frac{1}{(3/2)NkT\bar{y}}\frac{aT^4}{\rho}\frac{d\rho}{\rho}\right\}. \tag{9.149}$$

We now make use of the quantity β:

$$\beta = \frac{P_g}{P} = \frac{(1+\bar{y})N\rho kT}{P}, \quad 1-\beta = \frac{P_r}{P} = \frac{(1/3)aT^4}{P} \tag{9.150}$$

so that (9.149) finally becomes

$$dE = (3/2)NkT(1+\bar{y})\left\{\left(\frac{8-7\beta}{\beta}\right)\frac{dT}{T}\right.$$

$$\left.+(2/3)\left((3/2)+\frac{\chi_{r-1}^i}{kT}\right)\frac{\nu_i dy_r^i}{1+\bar{y}}-2\left(\frac{1-\beta}{\beta}\right)\frac{d\rho}{\rho}\right\}. \tag{9.151}$$

We consider now the ionization equation (9.145). We assume that the partition functions are constant and take the logarithmic differential of this equation:

$$\frac{dy_r^i}{y_r^i}+\frac{dy_r^i}{1-y_r^i}+\frac{\nu_i dy_r^i}{\bar{y}}-\frac{\nu_i dy_r^i}{1+\bar{y}}+\frac{1}{\beta}\frac{dP}{P}-4\frac{1-\beta}{\beta}\frac{dT}{T} = (5/2)\frac{dT}{T}+\frac{\chi_{r-1}^i}{kT}\frac{dT}{T},$$

where we have used the relation $d\bar{y} = \nu_i dy_r^i$; or

$$\left[\frac{1}{y_r^i(1-y_r^i)}+\frac{\nu_i}{\bar{y}(1+\bar{y})}\right]dy_r^i+\frac{1}{\beta}\frac{dP}{P} = \left[4\frac{(1-\beta)}{\beta}+(5/2)+\frac{\chi_{r-1}^i}{kT}\right]\frac{dT}{T}. \tag{9.152}$$

We consider, finally, the equation of state (9.138). Taking the logarithmic differential, we have

$$\frac{dP}{P} = \frac{N\rho kT d\bar{y}+(1+\bar{y})NkT d\rho+(1+\bar{y})N\rho k dT+4((1/3)aT^4)dT/T}{P} =$$

$$= \beta\frac{d\bar{y}}{1+\bar{y}}+\beta\frac{d\rho}{\rho}+\beta\frac{dT}{T}+4(1-\beta)\frac{dT}{T}.$$

Using the relation $d\bar{y} = \nu_i dy_r^i$, we have finally

$$\frac{dP}{P} = \beta\frac{\nu_i dy_r^i}{1+\bar{y}}+\beta\frac{d\rho}{\rho}+(4-3\beta)\frac{dT}{T}. \tag{9.153}$$

We now drop the sub- and superscripts, remembering that $y \equiv y_r^i$, $\chi \equiv \chi_{r-1}^i$, and $v \equiv v_i$. We also define a dimensionless quantity Ξ:

$$\Xi \equiv \frac{2\bar{y}y(1-y)v}{\bar{y}(1+\bar{y})+y(1-y)v}, \tag{9.154a}$$

which may also be written in the form

$$\frac{1}{\Xi} = (1/2)\left[\frac{1+\bar{y}}{y(1-y)v}+\frac{1}{\bar{y}}\right]. \tag{9.154b}$$

Thus (9.151)–(9.153), *i.e.*, the differentials of the internal energy equation, the ionization equation, and the equation of state, may be written as follows:
Internal energy equation:

$$dE = (3/2)NkT(1+\bar{y})\left\{\left(\frac{8-7\beta}{\beta}\right)\frac{dT}{T} + (2/3)\left((3/2)+\frac{\chi}{kT}\right)\frac{v\,dy}{1+\bar{y}} - 2\left(\frac{1-\beta}{\beta}\right)\frac{d\rho}{\rho}\right\}. \tag{9.155}$$

Ionization equation:

$$\frac{2}{\Xi}\frac{v\,dy}{1+\bar{y}}+\frac{1}{\beta}\frac{dP}{P} = \left[\frac{4(1-\beta)}{\beta}+(5/2)+\frac{\chi}{kT}\right]\frac{dT}{T}. \tag{9.156}$$

Equation of state:

$$\frac{1}{\beta}\frac{dP}{P} = \frac{v\,dy}{1+\bar{y}}+\frac{d\rho}{\rho}+\left(\frac{4-3\beta}{\beta}\right)\frac{dT}{T}. \tag{9.157}$$

Eliminating $\beta^{-1}dP/P$ between (9.156) and (9.157), we obtain

$$\frac{v\,dy}{1+\bar{y}} = \frac{\Xi}{2+\Xi}\left(\overline{(3/2)+\frac{\chi}{kT}}\right)\frac{dT}{T}-\frac{\Xi}{2+\Xi}\frac{d\rho}{\rho}. \tag{9.158}$$

Using (9.158) in (9.155) and (9.156), the latter two equations can be expressed, finally, in terms of $d\rho$ and dT alone:

$$dE = (3/2)NkT(1+\bar{y})\left\{\left[\frac{8-7\beta}{\beta}+(2/3)\frac{\Xi}{2+\Xi}\left((3/2)+\frac{\chi}{kT}\right)^2\right]\frac{dT}{T} - \left[2\left(\frac{1-\beta}{\beta}\right)+(2/3)\frac{\Xi}{2+\Xi}\left((3/2)+\frac{\chi}{kT}\right)\right]\frac{d\rho}{\rho}\right\}, \tag{9.159}$$

$$\frac{1}{\beta}\frac{dP}{P} = \left[\frac{4-3\beta}{\beta}+\frac{\Xi}{2+\Xi}\left((3/2)+\frac{\chi}{kT}\right)\right]\frac{dT}{T}+\frac{2}{2+\Xi}\frac{d\rho}{\rho}. \tag{9.160}$$

From (9.159) and (9.160), we readily obtain the three relevant quantities

$$\bar{c_V} = \left(\frac{\partial E}{\partial T}\right)_\rho = (3/2)Nk(1+\bar{y})\left[\frac{8-7\beta}{\beta} + (2/3)\frac{\Xi}{2+\Xi}\left((3/2)+\frac{\chi}{kT}\right)^2\right], \tag{9.161}$$

$$\chi_\rho = \left(\frac{\partial \ln P}{\partial \ln \rho}\right)_T = \frac{2\beta}{2+\Xi}, \tag{9.162}$$

$$\chi_T = \left(\frac{\partial \ln P}{\partial \ln T}\right)_\rho = 4-3\beta+\frac{\beta\Xi}{2+\Xi}\left((3/2)+\frac{\chi}{kT}\right). \tag{9.163}$$

Using these values of the above quantities and the relations (9.93), (9.97), and (9.89) connecting them with the gammas, we obtain after some reduction

$$\Gamma_1 = \frac{(4-3\beta)(8-3\beta)+12\beta(1-\beta)+\beta^2\Xi\left[4\left(\frac{1-\beta}{\beta}\right)+\left((5/2)+\frac{\chi}{kT}\right)\right]^2}{3(8-7\beta)+\beta\Xi\left\{12\left(\frac{1-\beta}{\beta}\right)+\left[\left((3/2)+\frac{\chi}{kT}\right)\left((5/2)+\frac{\chi}{kT}\right)-\frac{\chi}{kT}\right]\right\}}, \tag{9.164}$$

$$\frac{\Gamma_2}{\Gamma_2-1} = \frac{(4-3\beta)(8-3\beta)+12\beta(1-\beta)+\beta^2\Xi\left[4\left(\frac{1-\beta}{\beta}\right)+\left((5/2)+\frac{\chi}{kT}\right)\right]^2}{2(4-3\beta)+\beta\Xi\left[4\left(\frac{1-\beta}{\beta}\right)+\left((5/2)+\frac{\chi}{kT}\right)\right]}, \tag{9.165}$$

$$\Gamma_3-1 = \frac{2(4-3\beta)+\beta\Xi\left[4\left(\frac{1-\beta}{\beta}\right)+\left((5/2)+\frac{\chi}{kT}\right)\right]}{3(8-7\beta)+\beta\Xi\left\{12\left(\frac{1-\beta}{\beta}\right)+\left[\left((3/2)+\frac{\chi}{kT}\right)\left((5/2)+\frac{\chi}{kT}\right)-\frac{\chi}{kT}\right]\right\}}, \tag{9.166}$$

where Ξ is defined by (9.154).

For $\beta = 1$ (negligible radiation pressure), we obtain

$$\Gamma_1 = \frac{5+\Xi\left((5/2)+\frac{\chi}{kT}\right)^2}{3+\Xi\left[\left((3/2)+\frac{\chi}{kT}\right)\left((5/2)+\frac{\chi}{kT}\right)-\frac{\chi}{kT}\right]}, \tag{9.167}$$

$$\frac{\Gamma_2}{\Gamma_2-1}=\frac{5+\Xi\left((5/2)+\dfrac{\chi}{kT}\right)^2}{2+\Xi\left((5/2)+\dfrac{\chi}{kT}\right)}, \tag{9.168}$$

$$\Gamma_3-1=\frac{2+\Xi\left((5/2)+\dfrac{\chi}{kT}\right)}{3+\Xi\left[\left((3/2)+\dfrac{\chi}{kT}\right)\left((5/2)+\dfrac{\chi}{kT}\right)-\dfrac{\chi}{kT}\right]}. \tag{9.169}$$

For $\beta = 0$ (negligible gas pressure), we obtain $\Gamma_{1,2,3} = (4/3)$, just as we expect.

For $\Xi = 0$, which corresponds to $y = 0$ or 1 (zero or complete ionization) or $\nu = 0$ (zero abundance of the ionizing element), (9.164)–(9.166) reduce to

$$\Gamma_1=\frac{(4-3\beta)(8-3\beta)+12\beta(1-\beta)}{3(8-7\beta)}, \tag{9.170}$$

$$\frac{\Gamma_2}{\Gamma_2-1}=\frac{(4-3\beta)(8-3\beta)+12\beta(1-\beta)}{2(4-3\beta)}, \tag{9.171}$$

$$\Gamma_3-1=\frac{2(4-3\beta)}{3(8-7\beta)}, \tag{9.172}$$

which are the same as (9.124b)–(9.126) derived in Sect. 9.17 for $\gamma_g = (5/3)$. For $\beta = 1$ in (9.170)–(9.172), we have $\Gamma_{1,2,3} = (5/3)$, which is appropriate to a perfect monatomic gas with negligible radiation pressure (see Sect. 9.15).

The present case therefore embraces all the foregoing ones.

We now discuss the gammas for the case when the radiation pressure is negligible, so that the effects of ionization on the gammas may be clearly brought out without the complicating effects of radiation pressure, which we considered in Sect. 9.17. We consider only Γ_3-1 here, to illustrate the general qualitative features of the gammas, since all three gammas are numerically approximately equal under most circumstances. The appropriate equation is (9.169), where Ξ is given by (9.154).

We note that for a mixture consisting of only one element, *i.e.*, $\nu = 1$, we have from (9.133) and (9.144), for the transition from the $(k-1)^{\text{st}}$ to the k^{th} stage of ionization,

$$\bar{y} = k-1+y \tag{9.175}$$

(where y is the degree of k^{th} ionization) so that in this case

$$\Xi = \frac{2y(1-y)(k-1+y)}{k(k-1+2y)}.$$

For $k = 1$ (*i.e.*, first ionization),

$$\Xi = y(1-y). \tag{9.176}$$

We consider first the dependence of $\Gamma_3 - 1$ on y (degree of ionization of the ionizing element). For this purpose we consider the dependence of Ξ on y, since this function depends more strongly on y than the functions explicitly containing T in the expression (9.169) for $\Gamma_3 - 1$ (because of the exponential temperature dependence of the Saha equation, T is always a slowly varying function of y when y is neither very close to zero nor to unity, see footnote near end of this section). We write the expression for Ξ in the form

$$\Xi = y(1-y)v', \tag{9.181}$$

$$v' \equiv v \frac{2}{1+\bar{y}+\dfrac{y(1-y)v}{\bar{y}}}. \tag{9.182}$$

Note that

$$v' \leq v \cdot \frac{2}{1+\bar{y}}, \tag{9.183}$$

since all terms in (9.182) are intrinsically positive. Also, since the last term in the denominator of (9.182) is always ≤ 1, then, to a fair approximation,

$$v' \approx v \cdot \frac{2}{1+\bar{y}}. \tag{9.184}$$

In a star composed predominantly of hydrogen and helium $\bar{y} \approx 1$, so that to a rough approximation $v' \approx v$. We may refer to v' as the "effective" relative number abundance of the ionizing-element.

(For $v = 1$, we have $\bar{y} = k-1+y$, so that

$$v' = \frac{2}{k+y+\dfrac{y(1-y)}{k-1+y}} \tag{9.185a}$$

$$= \frac{2k-1}{k^2} \quad \text{for } y = (1/2), \text{ say.}) \tag{9.185b}$$

Since ν' is a slowly varying function of y, it follows that Ξ attains a maximum for $y \approx (1/2)$ (or for $y = (1/2)$ exactly if ν' is strictly constant) and vanishes both for $y = 0$ and $y = 1$. Inspection of (9.169) for $\Gamma_3 - 1$ shows that if T remained constant as y was varied, $\Gamma_3 - 1$ would attain its minimum value at the value of y for which Ξ attains its maximum, *i.e.*, at $y \approx (1/2)$. In reality, T does not vary very much with y for y in the vicinity of (1/2), because of the exponential factor in the Saha equation; thus $\Gamma_3 - 1$ should attain its minimum also for $y \approx (1/2)$. Since $[\partial(\Gamma_3 - 1)/\partial T]_\Xi > 0$ (from (9.169)) and since, generally, $dT/dy > 0$, the minimum value of $\Gamma_3 - 1$ actually occurs for y slightly less than (1/2) (see Fig. 9.1).

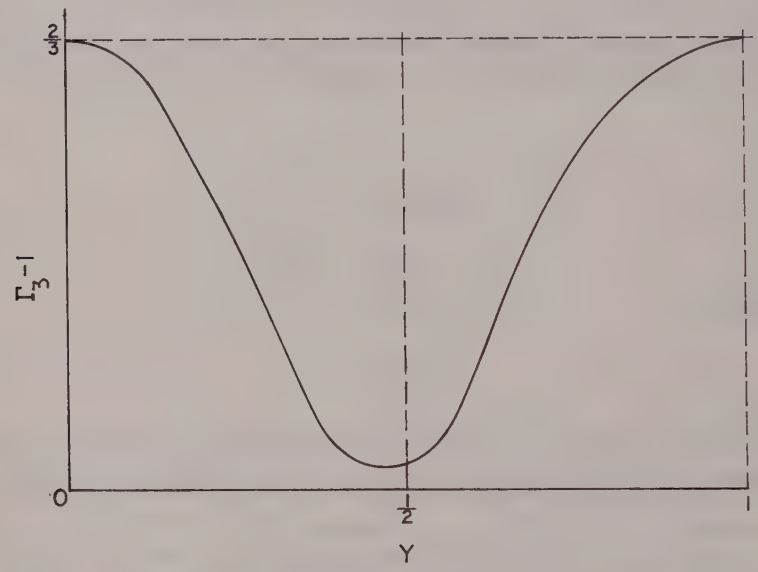


Fig. 9.1 Schematic dependence of $\Gamma_3 - 1$ on y.

We now consider the dependence of the *minimum* value $(\Gamma_3 - 1)_{\min}$ on ν' and T. To sufficient accuracy we can suppose that the minimum occurs at $y = (1/2)$. We then have, from (9.169) and (9.181),

$$(\Gamma_3 - 1)_{\min} \approx \frac{2 + (1/4)\nu' \left((5/2) + \dfrac{\chi}{kT} \right)}{3 + (1/4)\nu' \left[\left((3/2) + \dfrac{\chi}{kT} \right) \left((5/2) + \dfrac{\chi}{kT} \right) - \dfrac{\chi}{kT} \right]} =$$

$$= \frac{2+(1/4)\nu'\left[1+\left((3/2)+\frac{\chi}{kT}\right)\right]}{3+(1/4)\nu'\left[(3/2)+\left((3/2)+\frac{\chi}{kT}\right)^2\right]}. \tag{9.186}$$

We note from (9.186) the following features:

(1) $(\Gamma_3-1)_{\min}$ increases as ν' decreases. Since $\nu' \approx \nu \cdot 2/(1+\bar{y})$, then increasing $\bar{y}$ and decreasing ν both cause $(\Gamma_3-1)_{\min}$ to increase. Physically, this is because both effects essentially represent a "dilution" of the ionizing constituents of the material (small ν means only a small abundance of the ionizing element, while large $\bar{y}$ means a large number of free electrons, so that the large amount of energy needed to add each new free electron to the mixture via ionization is nevertheless small compared with the kinetic energy of thermal motions, *cf.* (10.85)).

(2) $(\Gamma_3-1)_{\min}$ increases as T increases. In fact, $(\Gamma_{1,2,3})_{\min} \to 1$ as $T \to 0$ and $(\Gamma_{1,2,3})_{\min} \to (5/3)$ as $T \to \infty$. Physically, this behavior can also be understood as a sort of "dilution" effect: Since $\chi/kT \approx$ ionization energy/thermal kinetic energy, large χ/kT means that the energy required to ionize the gas is large compared to the thermal energy of the gas, so that the effects of ionization on the gas are large. Small χ/kT clearly means that the ionization energy is small compared to the thermal energy, so that the effects of ionization are small. Since, for given y, P (or ρ) increases with increasing T, it follows that $(\Gamma_3-1)_{\min}$ also increases with increasing P (or ρ).

Physically, the small values of Γ_3-1 in the vicinity of $y = (1/2)$ simply reflect the fact that the work of adiabatic compression, for example, done on a partially ionized gas goes largely into energy required to ionize the gas rather than into kinetic energy of thermal motion, so that T does not increase as much during the compression for a partially ionized gas as for a neutral or totally ionized gas. We note that molecular dissociation, for example, could also produce a decrease in the values of the gammas.

We now present some rough numerical estimates of the value of $(\Gamma_3-1)_{\min}$ under typical stellar conditions. It is a property of the Saha equation that, as long as the electron gas is nondegenerate, the quantity $((3/2)+\chi/kT)$ has a characteristic value of, say, about 15 or so (at any rate large compared with unity) when $y \sim (1/2)$, under a wide range of stellar conditions, roughly independently of the particular ionizing element or the stage of ionization being considered.* Thus we have from (9.186), setting $((3/2)+\chi/kT) = 15$,

$$(\Gamma_3-1)_{\min} \approx \frac{2+4\nu'}{3+56\nu'}, \tag{9.189}$$

* See footnote page 230.

which should be accurate to within, say, 10 to 20 percent over a very wide range of elements and physical conditions. Equation (9.189) shows that the value of $(\Gamma_3-1)_{\min}$ is determined primarily by the effective relative number abundance ν' of the ionizing element, and is fairly insensitive to the ionization potential of the ionizing element and to the prevailing physical conditions of density and temperature. For example, for $\nu' = 1$, we have $(\Gamma_3-1)_{\min}\approx 0.10$; for $\nu' = 1/7$, $(\Gamma_3-1)_{\min}\approx 0.23$. The first example would apply to the case where the mixture consists mostly of only the ionizing element and if this ionizing element is undergoing first ionization. This case would apply, for example, to the region of hydrogen ionization in a star composed predominantly of hydrogen. The second example would apply to the case where the relative number abundance of the ionizing element is about 0.15 and the ionizing element is in a predominantly ionized hydrogen environment ($\bar{y}\approx 1$). This case would apply, for example, to the regions of first or second helium ionization in a star in which the relative number abundances of hydrogen and helium are close to 0.85 and 0.15, respectively. These examples show that $\Gamma_{3,\min}$ can be close to unity in ionization zones of astrophysically interesting mixtures.

Next, we note from the relation (*cf.* (9.93)) $c_V = (\mathscr{R}/\mu)\chi_T/(\Gamma_3-1)$ that the specific heat per unit mass at constant volume c_V is increased over its "monatomic" value $(3/2)(\mathscr{R}/\mu)$ by the factor $(2/3)\chi_T(\Gamma_3-1)^{-1}$. Since, typically, $\chi_T\approx 2-3$ in the midstages of ionization of an abundant element (see end of this section) and $(\Gamma_3-1)\approx 0.1-0.25$, this factor may amount to, say, 20–30 for pure hydrogen undergoing ionization and to, say, 6–10 for helium undergoing ionization when the abundance of helium is about 15 per cent by numbers.

* The reason for this is the following: We recall a general form of the Saha equation that is valid when the *ions* are nondegenerate and nonrelativistic (*cf.* (3.27)):

$$\frac{n_{r+1}}{n_r} = e^{-\eta}\frac{B_{r+1}(T)}{B_r(T)}\left(\frac{m_{r+1}}{m_r}\right)^{3/2}\exp(-\chi_r/kT), \tag{9.187}$$

where η is the degeneracy parameter for the electron gas. The criterion for nondegeneracy of the electron gas is that $(-\eta)\gg 1$, so that in this approximation

$$e^{-\eta}\approx\frac{2(2\pi mkT)^{(3/2)}}{n_e h^3}\gg 1 \tag{9.188}$$

(m = electron rest mass). Hence, when $n_{r+1}/n_r\sim 1$ or $y\sim(1/2)$ (partial ionization), we must have $\exp(-\chi_r/kT)\ll 1$, or $\chi_r/kT\gg 1$, regardless of the particular stage of ionization under consideration, if the electron gas is nondegenerate. Since χ_r/kT is a logarithm, its weak dependence on physical conditions is explained.

Finally, we recall that

$$\gamma = c_P/c_V = \Gamma_1/\chi_\rho , \tag{9.190}$$

where in the present example $\chi_\rho = 2\beta/(2+\Xi)$ (*cf.* (9.162)) and Γ_1 is given by (9.164). Referring to (9.181)−(9.183), it is easily seen that an upper limit to the value of Ξ is $2y(1-y)$, since $v \leq 1$ and $\bar{y} \geq 0$. Since the maximum value of $y(1-y)$ is (1/4), it follows that, in all cases,

$$0 \leq \Xi \leq (1/2), \tag{9.191}$$

where the upper limit of (1/2) generally considerably exceeds the actual values of Ξ in most astrophysically interesting situations. Hence the values of χ_ρ are limited to the range

$$\beta \geq \chi_\rho \geq (4/5)\beta \text{ or } 1 \geq \chi_\rho \geq 0.8 \text{ for } \beta = 1 . \tag{9.192}$$

Hence, for $\beta \approx 1$ (negligible radiation pressure), the value of γ is only slightly larger than that of Γ_1. Hence the ratio of specific heats is approximately equal, numerically, to the other gammas (which are also approximately equal to one another), provided that the radiation pressure is negligible.

Correspondingly, the values of χ_T(*cf.* (9.163)) are such that

$$4-3\beta \leq \chi_T \leq 4-3\beta+(1/5)\beta\left((3/2)+\frac{\chi}{kT}\right). \tag{9.193}$$

If $\beta = 1$, we have

$$1 \leq \chi_T \leq 1+(1/5)\left((3/2)+\frac{\chi}{kT}\right). \tag{9.194}$$

If we use a representative value of, say, 15 for $((3/2)+\chi/kT)$, we have $1 \lesssim \chi_T \lesssim 4$, where the upper limit is a fairly generous one.

Integrated adiabats in ionization zones are considered for some special cases in Sect. 20.8.

CHAPTER 10

Some Results of Kinetic Theory and Statistical Mechanics

In this chapter we shall summarize some of the more useful results of kinetic theory and statistical mechanics. We shall neglect throughout this chapter interactions between the "particles" making up the system. Some discussion of such interactions is given in Sect. 15.5. In the first two sections we shall develop, on the basis of kinetic theory, a general relation between the pressure and the internal translational kinetic energy per unit volume for a system of non-interacting particles of unspecified nature. The next two sections will be devoted to summarizing some fundamental results of statistical mechanics and to collecting together some useful explicit formulae for various thermodynamic quantities of interest; certain results presented in earlier chapters will be rederived in the course of the development. Application of these formulae to the special but important case of non-degenerate (or Maxwell-Boltzmann) systems will be made in the following section. Finally, the last two sections are devoted to the development of the principle of the equipartition of energy and its application to some simple physical systems.

10.1 Pressure in a System of Non-Interacting "Particles"

Here we consider a system of identical, non-interacting "particles" of unspecified nature and compute the pressure arising from the transfer of momentum across a mathematical surface element exposed to the particles in the system. If the particles are non-interacting, this contribution to the pressure will of course be the only contribution.

Consider an infinitesimal element of area $d\sigma$ whose outward normal is in the direction of the unit vector $\mathbf{n}$, located at some arbitrary point in the

system. We consider particles having momenta $\mathbf{p}$ within the "volume" element $d^3\mathbf{p} \equiv dp_1 dp_2 dp_3$ in momentum space (representing a given magnitude and direction of momentum) and velocity $\mathbf{v}$ impinging on $d\sigma$. If θ denotes the angle between $\mathbf{n}$ and $\mathbf{p}$ (or $\mathbf{v}$), then we have, using an argument exactly analogous to that used in Sect. 2.5, that the normal component of momentum transferred across $d\sigma$ in time dt by particles whose momenta are $\mathbf{p}$ within $d^3\mathbf{p}$ is

$$n(\mathbf{p}) d^3\mathbf{p} p v \cos^2\theta \, d\sigma dt,$$

where $n(\mathbf{p})d^3\mathbf{p}$ is the number of particles per unit volume having momenta $\mathbf{p}$ within $d^3\mathbf{p}$ and where $p \equiv |\mathbf{p}|$ and $v \equiv |\mathbf{v}|$ denote the *magnitudes* of $\mathbf{p}$ and $\mathbf{v}$ ($n(\mathbf{p})$ may in general also depend on position, but all the considerations in this section and the next are understood as applying at any given point, so we do not consider explicitly the space dependence). Since the normal force exerted on $d\sigma$ is just the rate of transfer of normal momentum across $d\sigma$, we have that the normal force per unit area exerted on $d\sigma$ by particles whose momenta are $\mathbf{p}$ within $d^3\mathbf{p}$ is

$$dP(\mathbf{n}) = n(\mathbf{p}) p v \cos^2\theta \, d^3\mathbf{p} = \mathbf{n} \cdot [n(\mathbf{p}) p v \mathbf{n}' \mathbf{n}' d^3\mathbf{p}] \cdot \mathbf{n},$$

where we have written $\cos\theta = \mathbf{n} \cdot \mathbf{n}'$, $\mathbf{n}'$ being a unit vector in the direction of $\mathbf{p}$ (or $\mathbf{v}$). Integrating over all momentum space gives for the total normal force per unit area exerted on a surface element whose outward normal is in the direction $\mathbf{n}$:

$$P(\mathbf{n}) = \mathbf{n} \cdot \mathsf{P} \cdot \mathbf{n}, \tag{10.1}$$

where the *pressure tensor* P is defined by the relation

$$\mathsf{P} \equiv \int n(\mathbf{p}) \mathbf{p} \mathbf{v} d^3\mathbf{p}, \tag{10.2}$$

since $\mathbf{p} = p\mathbf{n}'$ and $\mathbf{v} = v\mathbf{n}'$. The components of P in a rectangular Cartesian coordinate system are given by

$$P_{ij} = \int_{-\infty}^{\infty} \int \int n(p_1, p_2, p_3) p_i v_j dp_1 dp_2 dp_3 (i,j = 1,2,3). \tag{10.3}$$

We now define the *average* pressure by the relation

$$\bar{P} = \frac{1}{4\pi} \int_{4\pi} P(\mathbf{n}) d\omega_{\mathbf{n}}, \tag{10.4}$$

which represents an average of $P(\mathbf{n})$ over all possible orientations of the surface element. We have, using (10.2),

$$\bar{P} = \frac{1}{4\pi}\int_{4\pi} \mathbf{n}\cdot\mathsf{P}\cdot\mathbf{n}\,\mathrm{d}\omega_{\mathbf{n}} = \frac{1}{4\pi}\sum_{ij} P_{ij}\int_{4\pi}\alpha_i\alpha_j\,\mathrm{d}\omega_{\mathbf{n}}$$

$$= \frac{1}{4\pi}\sum_{ij} P_{ij}\cdot\frac{4\pi}{3}\delta_{ij} = (1/3)\sum_{i} P_{ii} = (1/3)Tr\mathsf{P}\,, \tag{10.5}$$

where the α's denote the direction cosines of the unit vector $\mathbf{n}$, and where we have used some results established in Sect. 2.5b. Thus we may write

$$\bar{P} = (1/3)Tr\,\mathsf{P} = (1/3)\int n(\mathbf{p})(\mathbf{p}\cdot\mathbf{v})\mathrm{d}^3\mathbf{p}$$

$$= (1/3)\int_{-\infty}^{\infty}\int\int n(p_1,p_2,p_3)(p_1v_1+p_2v_2+p_3v_3)\mathrm{d}p_1\mathrm{d}p_2\mathrm{d}p_3 \tag{10.6}$$

as the expression for the mean pressure in a system of identical, non-interacting particles of unspecified nature. Note that (10.6) is quite general and gives the contribution to the mean pressure arising from momentum transfer by the particles. In (10.6) the velocity $\mathbf{v} = \mathbf{v}(\mathbf{p})$ must be regarded as a function of momentum $\mathbf{p}$.

10.2 Internal Translational Kinetic Energy per Unit Volume

We let $\varepsilon(\mathbf{p})$ be the *translational kinetic* energy of a particle whose momentum is $\mathbf{p}$ at some arbitrary point. Then the total translational kinetic energy per unit volume is

$$u_{\mathrm{kin}} = \int n(\mathbf{p})\varepsilon(\mathbf{p})\mathrm{d}^3\mathbf{p} = \int_{-\infty}^{\infty}\int\int n(p_1,p_2,p_3)\varepsilon(p_1,p_2,p_3)\mathrm{d}p_1\mathrm{d}p_2\mathrm{d}p_3\,. \tag{10.7}$$

Let us first apply (10.6) and (10.7) to a system whose particles are identical Newtonian mass points each of mass m. In this case the momentum and velocity of a particle are related by $\mathbf{p} = m\mathbf{v}$ and the translational kinetic energy of a particle is $\varepsilon = p^2/2m$. The average pressure exerted by the particles is then

$$\bar{P} = (1/3)\int n(\mathbf{p})(p^2/m)\mathrm{d}^3\mathbf{p} = (2/3)\int n(\mathbf{p})\varepsilon(\mathbf{p})\mathrm{d}^3\mathbf{p}$$

or, by (10.7),

$$\bar{P} = (2/3)u_{\mathrm{kin}}\,. \tag{10.8}$$

From this we see that, for this kind of particle, the pressure is a measure of the amount of translational kinetic energy per unit volume of the system.

Next, we may apply (10.6) and (10.7) to the case of a system whose "particles" are photons in a vacuum or in a nondispersive medium of unity refractive index. Here $\mathbf{v} = \mathbf{c}$, where $\mathbf{c}$ is the velocity of light in vacuo, and the "translational kinetic energy" of a photon may be taken as its total energy:

$$\varepsilon = pc = \mathbf{p} \cdot \mathbf{c}.$$

Hence the mean pressure exerted by such photons is

$$\bar{P} = (1/3)\int n(\mathbf{p})(pc)\mathrm{d}^3\mathbf{p} = (1/3)\int n(\mathbf{p})\varepsilon(\mathbf{p})\mathrm{d}^3\mathbf{p} = (1/3)u, \tag{10.9}$$

where u is the energy density of the radiation field. This result agrees with a similar result which we obtained from somewhat different considerations in Sect. 2.5. Note that (10.9) is valid even for a non-isotropic radiation field, provided that $\bar{P}$ is interpreted as the *mean* pressure, *i.e.*, the normal force per unit area averaged over all possible orientations of the surface element. For the special case of an isotropic radiation field, the pressure P is identically equal to the mean pressure $\bar{P}$, so that the result $P = (1/3)u$ is a special case of (10.9)

Finally, we may apply (10.6) and (10.7) to the case of a system whose particles are identical mass points, each of rest mass m_0 moving with relativistic velocities. The total energy E (rest energy plus kinetic energy) of such a mass point is given by

$$E^2 = p^2c^2 + {m_0}^2c^4 = \left(\frac{m_0c^2}{\sqrt{1-v^2/c^2}}\right)^2, \tag{10.10}$$

where p is the (relativistic) momentum of the particle and v is its speed. Solving (10.10) for v and forming the quantity pv, we have

$$pv = pc[1+(m_0c/p)^2]^{(-1/2)}. \tag{10.11}$$

On the other hand, the kinetic energy ε per particle is

$$\varepsilon(p) = E - m_0c^2 = pc[\sqrt{1+(m_0c/p)^2} - m_0c/p]. \tag{10.12}$$

Eliminating pc between (10.11) and (10.12), we obtain

$$pv = \alpha(p)\varepsilon(p), \tag{10.13}$$

where

$$\alpha(p) \equiv \{1+(m_0c/p)^2 - (m_0c/p)\cdot\sqrt{1+(m_0c/p)^2}\}^{-1}. \tag{10.14}$$

Note that $\alpha(p)\to 2$ as $m_0c/p\to\infty$ (non-relativistic case) and $\alpha(p)\to 1$ as $m_0c/p\to 0$ (extreme relativistic case); hence $1\leq\alpha(p)\leq 2$. For the average pressure we then have

$$\bar{P} = (1/3)\int n(\mathbf{p})\alpha(p)\varepsilon(p)\mathrm{d}^3\mathbf{p} = \frac{\bar{\alpha}}{3}u_{\mathrm{kin}}, \tag{10.15}$$

where $\bar{\alpha}$, the average value of α, is defined by the second equality in (10.15). Because $1\leq\alpha(p)\leq 2$, it follows also that $1\leq\bar{\alpha}\leq 2$, where the upper and lower limits correspond, respectively, to the non-relativistic and extreme relativistic cases. Extremely relativistic electrons, for example, would thus obey the relation $\bar{P}\approx(1/3)u_{\mathrm{kin}}$ and would therefore behave in some respects as photons.

Equation (10.15) may also be applied in at least one important case to photons in a dispersive medium, *i.e.*, to "quasi-photons." It was shown in Sect. 2.10d that quasi-photons which obey a dispersion relation of the form (2.163) or (2.163′) could be regarded for some purposes as material particles with a non-zero rest mass. The rest mass of a quasi-photon is $m_0 = \hbar\omega_P/c^2$ ($2\pi\hbar$ = Planck's constant), where ω_P is the plasma frequency (*cf.* (2.157) and Sect. 17.20c); its momentum is $p = \hbar k = \mu_v\hbar\omega/c$, where μ_v is the real refractive index and $\omega = 2\pi\nu$ is the angular frequency of the quasi-photon; and its velocity is $v = v_g = \mu_v c$, where v_g is the group velocity. The "relativity parameter" is $m_0c/p = \sqrt{1-\mu_v^2}/\mu_v$. With these substitutions, all the equations in the preceding paragraph then apply also to such quasi-photons. For example, (10.14) becomes

$$\alpha(p) = \frac{\mu_v{}^2}{1-\sqrt{1-\mu_v{}^2}}, \tag{10.15′}$$

which shows that $\alpha(p)\to 2$ as $\mu_v\to 0$ and $\alpha(p)\to 1$ as $\mu_v\to 1$. Hence, in the present context quasi-photons with $\mu_v\approx 0$ behave as non-relativistic material particles, whereas quasi-photons with $\mu_v\approx 1$ (*i.e.*, ordinary photons) behave as extremely relativistic material particles (*i.e.*, particles with negligible rest mass).

Hence we have the general result (valid for any distribution function $n(\mathbf{p})$) that

$$(1/3)\leq\bar{P}/u_{\mathrm{kin}}\leq(2/3), \tag{10.16}$$

where the upper and lower limits correspond, respectively, to the non-relativistic and extreme relativistic cases.

10.3 Statistical Mechanics Approach

In statistical mechanics one makes the fundamental assumption that the entropy of a system is given by the general relation

$$S = k \ln \Delta\Gamma, \tag{10.17}$$

where k is Boltzmann's constant and $\Delta\Gamma$, the *statistical weight*, is the total number of distinct quantum states by which a given *macroscopic* state of the system can be realized. Since the number of such states can never be less than unity, it follows that S is always positive; moreover, the definition (10.17) gives the value $S = 0$ for a system in a pure state (*i.e.*, one having only one possible quantum state) and the zero of the entropy scale is thereby fixed unambiguously (the zero of the entropy scale is left undetermined in thermodynamics). Finally, (10.17) makes S *additive* for statistically independent (*i.e.*, non-interacting) systems, since the statistical weight of several statistically independent systems is equal to the product of their individual statistical weights. (See, for example, Landau and Lifshitz [La58, Chap. 1] for further discussion and clarification of some of these points.)

In general, $\Delta\Gamma$ will depend on how the individual members of a system are distributed over their various quantum states, *i.e.*, on the *distribution function* for the system. In *statistical* (*i.e.*, *thermodynamic*) *equilibrium*, the distribution function will be such that S is a maximum subject to whatever given constraints are imposed on the system (maximizing S is, in fact, one way of calculating the distribution function for a system in statistical equilibrium; see, for example, Landau and Lifshitz [La58, Sects. 40 and 54]). The general distribution function which obtains in statistical equilibrium is usually called the *Gibbs distribution function*. We shall not derive the expression for this function (see, for example, Fowler and Guggenheim [Fo39, Chap. VI]; Landau and Lifshitz [La58, Sect. 35]), but shall always assume its validity under conditions of statistical equilibrium.

Making use of the Gibbs function, it can be shown (*cf.* Landau and Lifshitz [La58, Sect. 35]) that (10.17) will correctly give the entropy for a system in statistical equilibrium if the quantity PV (P = pressure, V = volume) is given by the expression

$$PV = kT \ln Z, \tag{10.18}$$

where T is the temperature of the system (assumed uniform throughout) and where Z is called the *grand partition function* for the system. (It is implicitly assumed in (10.18) that the system is in thermal equilibrium (*cf.* Sect. 3.1) and in the absence of any external or internal force fields, so that

the pressure P is uniform throughout the volume V. If the system *is* in an external or internal force field and *not* in thermal equilibrium, then (10.18) can always be applied *locally* to a small region, provided that the region is not so small that equilibrium does not obtain therein.) In the case of a system composed of "particles" of only one kind, assumed to be either fermions or bosons (*cf.* Sect. 3.2) and to be non-interacting or only weakly interacting, Z is given by the following expression:

$$\ln Z = \sum_k G_k \ln \{1 \pm \exp[(\mu - \varepsilon_k)/kT]\}^{\pm 1}, \tag{10.19}$$

where μ is the chemical potential (see Sect. 9.12) for the kind of particle under consideration, ε_k is the energy of a single, representative "particle" in the quantum state k, and G_k is the statistical weight for the state k. The upper $(+)$ sign is to be used for *fermions*, the lower $(-)$ sign for *bosons*. We are here using G_k in a generalized sense, as in Sects. 3.2 and 3.3, to denote the statistical weight either of a "discrete" state or of a group of closely spaced states (as in a system with continuous states) all having energies in a narrow range about ε_k; in the case of a particle having both continuous and discrete states (for example, a molecule or atom with electronic and possibly vibrational and rotational energy states whose center of mass is moving freely in space) G_k would consist of the *product* of the "discrete" and "continuous" statistical weights. The summation sign in (10.19) of course stands for either a true summation over discrete states, for an integration over continuous states, or for both a summation and an integration if the particle possesses both discrete and continuous states.

By making use of some of the thermodynamic quantities defined in Chap. 9, explicit expressions for all the thermodynamic quantities can be derived from (10.18) and (10.19) (in which the independent variables are clearly V, T, and μ). For example, we may combine (9.32), (9.35b), and (9.50) to obtain the identity

$$PV = TS - E + \mu N, \tag{10.20}$$

where N is the total number of particles of the kind under consideration in the whole system. Forming the differential, we have

$$\mathrm{d}(PV) = T\mathrm{d}S + S\mathrm{d}T - \mathrm{d}E + \mu\mathrm{d}N + N\mathrm{d}\mu. \tag{10.21}$$

However, we also have the fundamental thermodynamic identity (9.42):

$$T\mathrm{d}S = \mathrm{d}E + P\mathrm{d}V - \mu\mathrm{d}N, \tag{10.22}$$

which we may use to eliminate dE. We obtain

$$d(PV) = SdT + PdV + Nd\mu. \tag{10.23}$$

From (10.23) we see that we have the identities

$$\left(\frac{\partial PV}{\partial T}\right)_{V,\mu} = S, \tag{10.24a}$$

$$\left(\frac{\partial PV}{\partial V}\right)_{T,\mu} = P, \tag{10.24b}$$

$$\left(\frac{\partial PV}{\partial \mu}\right)_{T,V} = N, \tag{10.24c}$$

where the left sides of (10.24) may be evaluated explicitly from (10.18) and (10.19). The internal energy E can then be obtained from S, PV, and N by use of (10.20).

If it is desired to regard N, V, and T as the independent variables, rather than μ, V, and T, one may combine (9.35b) and (9.50) to write for the free energy

$$F = -PV + \mu N = -kT \ln Z + \mu N, \tag{10.25}$$

where now μ is to be regarded as a function of N, V, and T and is obtained from (10.24c). The internal energy E is then given by (*cf.* (9.51))

$$E = -T^2[\partial(F/T)/\partial T]_{V,N}, \tag{10.26}$$

and the entropy S is obtained from PV, μ, and E by use of (10.20).

If the system consists of a mixture of several different types of particles all occupying the same volume V and all at the same temperature T, we have, from the additivity property of internal energy and entropy for non-interacting particles,

$$E = \sum_i E_i,\ S = \sum_i S_i,\ P = \sum_i P_i,\ F = \sum_i F_i, \tag{10.27}$$

where

$$S_i = (E_i + P_i V - \mu_i N_i)/T, \tag{10.28a}$$

$$F_i = E_i - TS_i, \tag{10.28b}$$

and S_i, N_i, etc., denote the entropy and number density, etc., of particles of type i. In this case $\ln Z$ (*cf.* (10.19)) also involves a summation over i:

$$\ln Z = \sum_i \sum_k G_{i,k} \ln \{1 \pm \exp[(\mu_i - \varepsilon_{i,k})/kT]\}^{\pm 1}. \tag{10.29}$$

10.4 Explicit Expressions for N, E, P, and S for Assemblies of Non-Interacting Bosons and Fermions in Statistical Equilibrium

We again consider, for simplicity, an assembly containing "particles" of only a single kind, either bosons or fermions. We then have, from (10.18) and (10.19),

$$PV = kT \sum_k G_k \ln\{1 \pm \exp[(\mu - \varepsilon_k)/kT]\}^{\pm 1}, \tag{10.30}$$

where the upper (+) sign applies to fermions, the lower (−) sign to bosons. From (10.24c) we have

$$N = \left(\frac{\partial PV}{\partial \mu}\right)_{T,V} = \sum_k \frac{G_k \exp[(\mu - \varepsilon_k)/kT]}{1 \pm \exp[(\mu - \varepsilon_k)/kT]} = \sum_k \frac{G_k}{\exp[(-\mu + \varepsilon_k)/kT] \pm 1}, \tag{10.31}$$

which gives μ implicitly as a function of N, V, and T and is the same as (3.9) in Sect. 3.2.

It is important to note that the value of μ depends on the zero of the energy scale. Suppose, for example, that we were to measure all energies relative to the energy ε_0 of the *ground* state of the particle. The energy ε_k' of the state k relative to the ground state is then

$$\varepsilon_k' = \varepsilon_k - \varepsilon_0 . \tag{10.32}$$

The number N of particles in the system clearly cannot depend on the choice of the energy zero-point; hence, if ε_k' is to be used in (10.31) for N, then μ must be replaced by

$$\mu' = \mu - \varepsilon_0 . \tag{10.33}$$

Next, we write down the expression for the free energy F:

$$F = -PV + \mu N = -kT \sum_\kappa G_k \ln\{1 \pm \exp[(\mu - \varepsilon_k)/kT]\}^{\pm 1} + \mu N. \tag{10.34}$$

We then have (regarding μ as a function of N, V, and T) for the internal energy E

$$E = -T^2[\partial(F/T)/\partial T]_{V,N} = \sum_k \frac{G_k \varepsilon_k}{\exp[(-\mu + \varepsilon_k)/kT] \pm 1}, \tag{10.35}$$

which is also intuitively correct and which is consistent with our work in Sect. 10.2. In terms of the energy ε_0 of the ground state of the particle of the kind under consideration, we have

$$E = \sum_k \frac{G_k \varepsilon_k' + G_k \varepsilon_0}{\exp[(-\mu' + \varepsilon_k')] \pm 1} = N\varepsilon_0 + E', \tag{10.36}$$

where E' is the internal energy of the system above the ground state internal energy. The energy $N\varepsilon_0$ is arbitrary and the value to be assigned to it depends on the system of interest. In the case of an ionizing gas, for example, ε_0 might be interpreted as the ionization potential of an atom (*cf.* Sect. 9.18 for a specific example).

For the pressure we have from (10.24b)

$$P = \left(\frac{\partial PV}{\partial V}\right)_{T,\mu} = kT \sum_k (\partial G_k/\partial V) \ln\{1 \pm \exp[(\mu - \varepsilon_k)/kT]\}^{\pm 1}, \quad (10.37)$$

since V appears only in the statistical weight G_k.

Equation (10.37) for P can be written in an alternative form. Let us suppose that our particles may have both discrete states and continuous states corresponding to translation of their centers of mass. We may then write

$$G_k = g_i \cdot \frac{4\pi p^2 \mathrm{d}pV}{h^3}, \quad (10.38)$$

where g_i denotes the statistical weight of the i^{th} discrete state and $4\pi p^2 \mathrm{d}pV/h^3$ (p = momentum of the particle) is the statistical weight for the continuous translational states (= number of "cells" each of "volume" h^3 in the "volume" $4\pi p^2 \mathrm{d}pV$ in phase space, *cf.* Sect. 3.1.b) of the particle. We may then write

$$\varepsilon_k = \varepsilon_i + \varepsilon(p,q), \quad (10.39)$$

where ε_i is the energy corresponding to the discrete state i and $\varepsilon(p,q)$ is the (kinetic plus potential) energy of the center of mass of the particle in some coordinate system ("q" stands for generalized position coordinates). Equation (10.37) then becomes

$$P = kT \sum_i g_i \int_0^\infty \frac{4\pi p^2 \mathrm{d}p}{h^3} \ln\{1 \pm \exp[(\mu - \varepsilon_i - \varepsilon(p,q))/kT]\}^{\pm 1}.$$

Integrating by parts, we have

$$P = kT \sum_i g_i \left\{ \frac{4\pi p^3}{3h^3} \ln\{1 \pm \exp[(\mu - \varepsilon_i - \varepsilon(p,q))/kT]\}^{\pm 1} \Big|_{p=0}^{\infty} \right.$$

$$\left. -(1/3) \int_0^\infty \frac{4\pi p^2}{h^3} \frac{p \exp[(\mu - \varepsilon_k)/kT]}{1 \pm \exp[(\mu - \varepsilon_k)/kT]} \left(\frac{\partial \varepsilon(p,q)}{\partial p}\right)\left(-\frac{\mathrm{d}p}{kT}\right) \right\}.$$

Because $\varepsilon(p,q)$ is always a monotonically increasing function of p, it follows that the integrated term in the above equation vanishes at both limits of integration. Moreover, $\partial\varepsilon(p,q)/\partial p = v$, where v is the speed of a particle of total energy $\varepsilon(p,q)$ and momentum p, from one of the Hamiltonian equations (see, for example, Goldstein [Go50, Chap. 7]). We may then write the above equation in the form

$$P = (1/3)\sum_i \int_0^\infty \frac{g_i 4\pi p^2 \mathrm{d}p}{h^3} \frac{pv}{\exp\left[(-\mu+\varepsilon_k)/kT\right]\pm 1} = (1/3)\int_0^\infty n(p)pv4\pi p^2 \mathrm{d}p, \tag{10.40}$$

where we have written

$$n(p) = \frac{1}{h^3}\sum_i \frac{g_i}{\exp\left[(-\mu+\varepsilon_i+\varepsilon(p,q))/kT\right]\pm 1} \tag{10.41}$$

for the number of particles per unit volume, in all (discrete) quantum states, having (translational) momentum p per unit volume of momentum space. Equation (10.40) agrees with (10.6), derived in Sect. 10.1, as it must for non-interacting particles. We note, finally, that (10.40) may also be written in the form

$$P = (1/3)n\langle pv\rangle = (1/3)n\langle \mathbf{p}\cdot\mathbf{v}\rangle = (1/3)n\langle \sum_{i=1}^{3} p_i v_i\rangle, \tag{10.42}$$

where n denotes the total number of particles per unit volume of the given kind and the brackets denote the average value of the quantity within, defined by comparison of (10.40) and (10.42).

Finally, an explicit equation for the entropy S can be obtained by substituting the above expressions for E, PV, and N into the basic identity (10.20):

$$S = (E+PV-\mu N)/T. \tag{10.43}$$

We note here an important property of the entropy as defined in this chapter. Whereas the internal energy E is arbitrary to within an additive constant, the value of S must be independent of the choice of this constant, because of the basic definition (10.17). To show that this is indeed the case, suppose we measure E relative to the "ground state energy" $N\varepsilon_0$, *i.e.*, we replace E by E' (*cf.* (10.36)). According to (10.33), we must then replace μ by μ'. In terms of the new scale of energy, we then have

$$S' = (E'+PV-\mu' N)/T. \tag{10.44}$$

Expressing E' and μ' in terms of E and μ, we see that $S' = S$, *i.e.*, that the value of S is independent of the zero point for the internal energy.

Perhaps the simplest example to cite for the entropy is black body radiation, for which $\mu = 0$ (*cf.* Sect. 9.12). We have (*cf.* Sect. 3.7c) $E = aT^4V$ (a = radiation constant) and $P = (1/3)aT^4$; (10.43) then gives simply

$$S = (4/3)aT^3V \tag{10.45}$$

for the entropy of black body radiation in an enclosure of volume V.

Other examples will be given in the next section, in Sect. 20.8, and in Chap. 24.

10.5 Non-Degenerate (Maxwell-Boltzmann) Systems

Non-degenerate (*or Maxwell-Boltzmann*) systems are those for which $-\mu/kT \gg 1$. From (10.31) we see that in this case the expression for the total number of particles of the kind under consideration becomes

$$N = e^{\mu/kT}\sum_k G_k \exp(-\varepsilon_k/kT), \tag{10.46}$$

from which the chemical potential μ may be calculated:

$$\mu = kT \ln\left[N/\sum_k G_k \exp(-\varepsilon_k/kT)\right]. \tag{10.47}$$

Note that in this limit which corresponds, as may be seen from (10.47), to small particle densities the symmetric or anti-symmetric character of the particles no longer plays any role in determining the properties of the system. Because of its importance and wide range of applicability in astrophysics, we devote a special section to this non-degenerate case and collect together some of the more widely used formulae.

Expanding (10.30) for this non-degenerate case, we obtain

$$PV = kTe^{\mu/kT}\sum_k G_k \exp(-\varepsilon_k/kT) = NkT, \tag{10.48}$$

from (10.46). We conclude, therefore, that *non-degenerate systems composed of non-interacting particles always obey the perfect gas law* (10.48). Note that this conclusion is valid whether or not the particles in the system are relativistic. Hence even an extremely relativistic gas will obey the perfect gas law (10.48), provided that the gas is non-degenerate.

The expression (10.35) for the internal energy E becomes

$$\begin{aligned} E &= e^{\mu/kT}\sum_k G_k\varepsilon_k \exp(-\varepsilon_k/kT) \\ &= N\sum_k G_k\varepsilon_k \exp(-\varepsilon_k/kT)/\sum_k G_k \exp(-\varepsilon_k/kT) \end{aligned} \tag{10.49}$$

from (10.46). Finally, the entropy S is given by (*cf.* (10.20))

$$S = E/T + kN - \mu N/T. \tag{10.50}$$

We consider now more explicit formulae for some of the above quantities. We suppose that our particles may have both discrete states with statistical weights g_i and continuous states with statistical weight $V \cdot 4\pi p^2 \mathrm{d}p/h^3$ (p = momentum of a particle, V = volume of system) (*cf.* Sect. 3.3), so that the total statistical weight of a "state" is

$$G_k = g_i V 4\pi p^2 \mathrm{d}p/h^3 . \tag{10.51}$$

For generality, let us suppose the particles to be in the presence of an external force field. Then we have for the total energy of a particle

$$\varepsilon_k = \varepsilon_i + \phi(\mathbf{r}) + \varepsilon(p), \tag{10.52}$$

where ε_i is the energy of the i^{th} discrete state, $\phi(\mathbf{r})$ is the potential energy of the particle in the force field, and $\varepsilon(p)$ is the translational kinetic energy of the particle in some coordinate system.

We first evaluate the "sum over states" in (10.46). We have

$$\sum_k G_k \exp(-\varepsilon_k/kT) = \sum_i g_i \exp(-\varepsilon_i/kT) \cdot V e^{-\phi(\mathbf{r})/kT} \int_0^\infty e^{-\varepsilon(p)/kT} 4\pi p^2 \mathrm{d}p/h^3 .* \tag{10.53}$$

We wish to measure the energies ε_i of the discrete states relative to the energy ε_0 of the *ground* state; denoting the energies so measured by ε_i', we have

$$\varepsilon_i' = \varepsilon_i - \varepsilon_0 . \tag{10.54}$$

We may then write (10.53) in the form

$$\sum_k G_k \exp(-\varepsilon_k/kT) = \exp\left[-\varepsilon_0/kT - \phi(\mathbf{r})/kT\right] \cdot V \cdot B(T)\mathscr{B}(T), \tag{10.55}$$

where

$$B(T) \equiv \sum g_i \exp(-\varepsilon_i'/kT) , \tag{10.56}$$

$$\mathscr{B}(T) \equiv \int_0^\infty e^{-\varepsilon(p)/kT} 4\pi p^2 \mathrm{d}p/h^3 ; \tag{10.57}$$

* The factor $Ve^{-\phi(\mathbf{r})/kT}$ should actually be replaced by the integral $\int_V e^{-\phi(\mathbf{r})/kT} \mathrm{d}V$; however, we may assume, for simplicity, that the volume V is so small that $\phi(\mathbf{r})$ is practically constant everywhere therein. In case this assumption is not permissible, we may replace $\phi(\mathbf{r})$ by an *average* value $\overline{\phi}$, defined by the relation $\overline{\phi} \equiv -kT \ln [V^{-1} \int_V e^{-\phi(\mathbf{r})/kT} \mathrm{d}V]$.

$B(T)$ and $\mathscr{B}(T)$ are the partition functions for, respectively, the discrete and the continuous states. Equation (10.47) for the chemical potential μ then becomes, setting $N/V = P/kT$ from the perfect gas law (10.48),

$$\mu = \varepsilon_0 + \phi(\mathbf{r}) + kT \ln\left[\frac{P}{kTB(T)\mathscr{B}(T)}\right]. \tag{10.58}$$

We may use (10.46) and (10.57) to obtain the *fraction* of all particles of the given kind having momenta whose magnitudes lie between p and $p+\mathrm{d}p$,

$$\frac{\mathrm{d}N(p)}{N} = \frac{e^{-\varepsilon(p)/kT}4\pi p^2 \mathrm{d}p}{h^3\mathscr{B}(T)}, \tag{10.59}$$

which is seen to be *independent of the presence of the external force field.*

For the internal energy E we use (10.54) and (10.52) in (10.49), again assuming V to be so small that $\phi(\mathbf{r})$ is practically constant therein (see the footnote in this section). We obtain

$$E = N\varepsilon_0 + N\phi(\mathbf{r}) + N\bar{\varepsilon} + N\Big[\sum_i g_i\varepsilon_i' \exp(-\varepsilon_i'/kT)\Big]/B(T), \tag{10.60}$$

where

$$\bar{\varepsilon}(T) \equiv \Big\{\int_0^\infty e^{-\varepsilon(p)/kT}\varepsilon(p)4\pi p^2 \mathrm{d}p/h^3\Big\}/\mathscr{B}(T) \tag{10.61}$$

is the average translational kinetic energy per particle and the last term in (10.60) is the excitation energy. If ε_1', *i.e.*, the excitation energy of the first excited level, is large compared with kT, this excitation energy is usually small compared with the other terms in (10.60) and is often neglected in practical applications.

Finally we have for the entropy S, using (10.50), (10.58), and (10.60), and neglecting the excitation energy,

$$S = kN\frac{\bar{\varepsilon}(T)}{kT} + kN + kN \ln\left[\frac{kTB(T)\mathscr{B}(T)}{P}\right], \tag{10.62}$$

which is also seen to be *unaffected by the presence of the external force field*; note also that, in accordance with a conclusion reached in the preceding section, the value of S is independent of the zero-point energy ε_0.

In general, $\varepsilon(p)$ is given by the special relativistic relation (10.12). In the non-relativistic ("*N.R.*," $\varepsilon(p) = p^2/2m$, m = particle mass) and extreme

relativistic ("*E.R.*," $\varepsilon(p) = pc$) limits, the functions $\mathscr{B}(T)$ and $\bar{\varepsilon}(T)$ can be easily evaluated. We have

$$\mathscr{B}(T) = (2\pi mkT)^{(3/2)}/h^3 \quad (N.R.) \tag{10.63a}$$

$$= 8\pi(kT/hc)^3 \quad (E.R.), \tag{10.63b}$$

$$\bar{\varepsilon}(T) = (3/2)kT \quad (N.R.) \tag{10.64a}$$

$$= 3kT \quad (E.R.). \tag{10.64b}$$

Equation (10.59) for the momentum distribution function becomes, in these two limiting cases,

$$\left.\begin{aligned} \frac{dN(p)}{N} &= \frac{1}{(2\pi mkT)^{(3/2)}} e^{-p^2/2mkT} \cdot 4\pi p^2 dp \quad (N.R.) \\ &= \frac{c^3}{8\pi(kT)^3} e^{-pc/kT} \cdot 4\pi p^2 dp \quad (E.R.). \end{aligned}\right\} \tag{10.65a, 10.65b}$$

Equation (10.58) for the chemical potential becomes

$$\left.\begin{aligned} \mu &= \varepsilon_0 + \phi(\mathbf{r}) + kT \ln\left[\frac{P}{T^{(5/2)}} \cdot \frac{h^3}{B(T)(2\pi m)^{(3/2)}k^{(5/2)}}\right] \quad (N.R.) \\ &= \varepsilon_0 + \phi(\mathbf{r}) + kT \ln\left[\frac{P}{T^4} \cdot \frac{c^3h^3}{8\pi B(T)k^4}\right] \quad (E.R.). \end{aligned}\right\} \tag{10.66a, 10.66b}$$

Neglecting the excitation energy, (10.60) for the internal energy becomes

$$E = N\varepsilon_0 + N\phi(\mathbf{r}) + (3/2)NkT \quad (N.R.) \tag{10.67a}$$

$$= N\varepsilon_0 + N\phi(\mathbf{r}) + 3NkT \quad (E.R.). \tag{10.67b}$$

Finally, we obtain for the entropy from (10.62) (again neglecting the excitation energy)

$$\left.\begin{aligned} S &= kN \ln\left[\frac{T^{(5/2)}}{P} \cdot \frac{B(T)(2\pi m)^{(3/2)}k^{(5/2)}e^{(5/2)}}{h^3}\right] \quad (N.R.) \\ &= kN \ln\left[\frac{T^4}{P} \cdot \frac{B(T)8\pi k^4e^4}{c^3h^3}\right] \quad (E.R.), \end{aligned}\right\} \tag{10.68a, 10.68b}$$

where e is the base of the natural logarithm.

It should be mentioned that if our particles are "fundamental" particles (such as protons or electrons), the partition function $B(T)$ is to be replaced simply by $g_0 = 2S+1$, where S is a whole or half integer (or zero) and $|\mathbf{S}| = \sqrt{S(S+1)}\hbar$ is the spin angular momentum of a particle. For example, for electrons, $S = (1/2)$ and $B(T) = 2$. Even in the case where our particles are atoms, the first term g_0 is often a good approximation to $B(T)$, particularly if the temperatures are low enough that ε_1 is several times kT.

Equation (10.68a) for the entropy will be applied to convection zones of stars in Sect. 20.6.

10.6 Principle of the Equipartition of Energy

The principle of the equipartition of energy is strictly a "classical" principle in that it applies *only* to non-degenerate (or Maxwell-Boltzmann) systems. The principle may be stated in a variety of ways, but we shall here follow Tolman's treatment (Tolman [To38, Sect. 35]).

We suppose that the particles in our system (all assumed of the same kind*) may have both discrete and continuous states, but we now allow forms of energy other than translational (such as rotational or vibrational) to be described by the continuous states. We therefore write, for the total energy of a particle

$$\varepsilon_k = \varepsilon_i + \varepsilon(q_1,\ldots,q_r,p_1,\ldots,p_r), \tag{10.69}$$

where ε_i is the energy corresponding to the discrete state i and $\varepsilon(q_1,\ldots,p_r)$ is the remaining energy in the continuous states; the q's and p's denote the generalized position and momentum coordinates which describe a particle (*cf.* Goldstein [Go50, Chap. 2]).

We now write for the total number of particles of the given kind in the system (*cf.* (10.46))

$$\begin{aligned} N &= e^{\mu/kT}\sum_k G_k \exp(-\varepsilon_k/kT) \\ &= e^{\mu/kT}B(T)h^{-r}\int\ldots\int \mathrm{d}q_1\ldots\mathrm{d}p_r \exp[-\varepsilon(q_1,\ldots,p_r)/kT], \end{aligned} \tag{10.70}$$

where $B(T)$ is the partition function for the discrete states (where the discrete states now are *not* measured relative to the ground state; *cf.* previous section) and the product of h^{-r} and the integral in (10.70) is the generalized partition function for the continuous states; the integration is to be carried out over

* This assumption is made merely for simplicity of exposition; it will be clear that the conclusions of this section are not restricted to systems composed only of like particles.

all relevant values of the q's and p's. Let us now choose any one of the q's or p's, say q_1, and perform the integration in that variable by parts. We have

$$N = e^{\mu/kT}B(T)h^{-r}\left\{\int\cdots\int\left[q_1 e^{-\varepsilon/kT}\right]_a^b \mathrm{d}q_2\cdots\mathrm{d}p_r\right.$$
$$\left.+\frac{1}{kT}\int\cdots\int\left[q_1\left(\frac{\partial\varepsilon}{\partial q_1}\right)\right]\exp\left[-\varepsilon(q_1,\ldots,p_r)/kT\right]\mathrm{d}q_1\cdots\mathrm{d}p_r\right\}. \quad (10.71)$$

We note now that in many cases the integrated quantity in (10.71) vanishes at both limits of integration. For example, this happens in the case of all the p's on which ε depends, since ε is always a monotonically increasing function of the magnitudes of the p's, and the limits of integration are $-\infty$ and $+\infty$ in this case. This integrated term may also vanish or become small at both limits of integration in the case of atoms bound together into a molecule, for example, provided that the potential energy increases sufficiently with increasing separation of the atoms from one another (consider, for example, a simple harmonic oscillator model of a diatomic molecule). If part of ε is contributed by the potential energy arising from an *external* force field, however, the integrated term may not vanish at one or the other of the limits of integration.

We henceforth consider only those q's, say q_i, and p's, say p_j, for which the integrated term vanishes at both limits. We now note that the remaining integral in (10.71) can be expressed in terms of the *average* value $\langle q_1\partial\varepsilon/\partial q_1\rangle$ of $q_1\partial\varepsilon/\partial q_1$, where the average value of some quantity f with respect to the q's and p's is defined by

$$\langle f\rangle \equiv \frac{e^{\mu/kT}B(T)h^{-r}}{N}\int\cdots\int f\exp\left[-\varepsilon(q_1,\ldots,p_r)/kT\right]\mathrm{d}q_1\cdots\mathrm{d}p_r. \quad (10.72)$$

(Note that this average is essentially just a generalization of the average defined in (10.40) and (10.42).) We then have, using (10.72) in (10.71),

$$\left\langle q_i\frac{\partial\varepsilon}{\partial q_i}\right\rangle = \left\langle p_j\frac{\partial\varepsilon}{\partial p_j}\right\rangle = kT \quad (10.73)$$

for each q_i and each p_j for which the integrated term in (10.71) vanishes at both limits of integration. Equation (10.73) is called by Tolman the "general principle of equipartition" and includes the usual statement of the principle (see sentence following (10.76) below) as a special case.

We consider two general applications of this principle. First, we may use it to show again that the perfect gas law is valid for all non-degenerate (Maxwell-Boltzmann) systems composed of non-interacting particles of any nature whatsoever. For this purpose we consider only the three degrees of

freedom* associated with the translational motion of the center of mass of the particle. Since by one of the Hamiltonian equations (*cf.* Goldstein [Go50, Chap. 7]) $\dot{q}_j = v_j = \partial\varepsilon/\partial p_j$ is the component of translational velocity associated with the translational momentum coordinate p_j, (10.73) may be written as $\langle p_j v_j \rangle = kT$, since (10.73) applies to the translational p's. Since there are three translational degrees of freedom, we have

$$\sum_{j=1}^{3} \langle p_j v_j \rangle = \langle \mathbf{p} \cdot \mathbf{v} \rangle = 3kT, \tag{10.74}$$

where in the first equality we have assumed p_j and v_j to be the components of the vectors $\mathbf{p}$ and $\mathbf{v}$ in a rectangular Cartesian coordinate system (but note that the second equality in (10.74) is independent of the coordinate system assumed). Comparing (10.74) with the general relation (*cf.* (10.42)) $P = (1/3)n\langle \mathbf{p} \cdot \mathbf{v} \rangle$, where n is the total number of particles of the given kind per unit volume, we see that

$$P = nkT, \tag{10.75}$$

i.e., that the perfect gas law applies, in general, to non-degenerate systems composed of non-interacting particles of any nature whatsoever (relativistic or not, for example). This conclusion was also reached from a somewhat different standpoint in the previous section.

As a second general application of (10.73), we suppose that the part of the total energy ε associated with q_1 or p_j is of the form $\varepsilon_i = aq_i^{m_i}$ or $\varepsilon_j = bp_j^{n_j}$, where a,b,m_i, and n_j are not functions of any of the q_i or of any of the p_j. We then have from (10.73)

$$\left.\begin{aligned} \langle q_i \partial\varepsilon_i/\partial q_i \rangle &= m_i \langle \varepsilon_i \rangle = kT \quad \text{or } \langle \varepsilon_i \rangle = (1/m_i)kT, \\ \langle p_j \partial\varepsilon_j/\partial p_j \rangle &= n_j \langle \varepsilon_j \rangle = kT \quad \text{or } \langle \varepsilon_j \rangle = (1/n_j)kT \end{aligned}\right\} \tag{10.76a, 10.76b}$$

for the average energy associated with each of the respective q's or p's. For example, the "classical" case is the one where $m_i = n_j = 2$, so that for this case *an amount of energy* $(1/2)kT$ *is associated, on the average, with each quadratic degree of freedom.* In the extreme relativistic case, however, we would have $n_j = 1$, so that in this case the average energy per kinetic degree of freedom would be kT and not $(1/2)kT$. If each particle has a total of f degrees of freedom, then, the total average energy per particle associated with these f degrees of freedom is

$$\langle \varepsilon \rangle = \sum_{k=1}^{f} \langle \varepsilon_k \rangle = kT \sum_{k=1}^{f} (m_k^{-1} + n_k^{-1}). \tag{10.77}$$

* We use the term "number of degrees of freedom" here to denote the total number of q's and p's for which the integrated term in (10.71) vanishes at both limits of integration, *i.e.*, for which (10.73) is applicable.

We note that $\langle\varepsilon\rangle$ may not necessarily be the *total* energy per particle; for example, in the case of particles in an external force field, the potential energy arising from this field would have to be added to $\langle\varepsilon\rangle$ in order to obtain the total energy.

10.7 Application of the Equipartition Principle to Some Simple Systems

10.7a Perfect Monatomic Gases with Only Three Degrees of Freedom per Particle

We consider here particles which may be treated as point masses, so that each particle possesses only three degrees of freedom, all translational. Then the average energy per particle is $\langle\varepsilon\rangle = (3/2)kT$ or $3kT$ in the non-relativistic (*N.R.*) and extreme relativistic (*E.R.*) limits, respectively. Then, considering one mole of the gas, we have that the total internal energy per mole is $E = N_0\langle\varepsilon\rangle$, where N_0 is Avogadro's number. Noting that $N_0 k = \mathscr{R}$, the gas constant per mole, we have

$$E = (3/2)\mathscr{R}T \qquad (N.R.) \qquad (10.78a)$$

$$= 3\mathscr{R}T \qquad (E.R.). \qquad (10.78b)$$

We may now compute c_V, the specific heat per mole at constant volume. We have

$$c_V \equiv (\partial E/\partial T)_V = (3/2)\mathscr{R} \qquad (N.R.) \qquad (10.79a)$$

$$= 3\mathscr{R} \qquad (E.R.), \qquad (10.79b)$$

from which we see that c_V is constant in these two limits. However, from thermodynamics there exists the relation (*cf.* (9.77)) $c_P = c_V + \mathscr{R}$ when $c_V =$ constant, so that, also,

$$c_P = (5/2)\mathscr{R} \qquad (N.R.) \qquad (10.80a)$$

$$= 4\mathscr{R} \qquad (E.R.). \qquad (10.80b)$$

Hence the ratio of specific heats, $\gamma \equiv c_P/c_V$, becomes in these two cases

$$\gamma = (5/3) \qquad (N.R.) \qquad (10.81a)$$

$$= (4/3) \qquad (E.R.). \qquad (10.81b)$$

Since the equation of state for the kinds of systems we are now considering is the perfect gas law, we have $\chi_\rho = (\partial \ln P/\partial \ln \rho)_T = 1$ and $\chi_T = (\partial \ln P/\partial \ln T)_\rho = 1$, so that $\gamma = \Gamma_1 = \Gamma_2 = \Gamma_3$, and we do not need to distinguish between the various gammas here and in the next subsection (see also Sect. 9.14b).

10.7b More General Perfect Gases with Constant Specific Heats

We start with the perfect gas law which applies to all non-degenerate systems of non-interacting particles:

$$P = nkT = \mathscr{R}T/V, \tag{10.82}$$

where V is the volume per mole. Using the relation $\mathscr{R} = c_P - c_V$, we may also write (10.82) in the form

$$P = (\gamma - 1)c_V T/V = (\gamma - 1)u, \tag{10.83}$$

where $\gamma = c_P/c_V$ and u is the total internal energy per unit volume (since c_V is assumed to be constant). (Note that (10.83) is a special case of (9.93) or (9.98′), obtained by setting $\Gamma_3 = \gamma$ and $\chi_T = 1$.)

We now assume the principle of the equipartition of energy to be valid and suppose that each particle (all of the same kind) has f degrees of freedom with each of which is associated the average energy kT/r, where $r = 2$ (*N.R.*) or 1 (*E.R.*). Then, using (10.82) and (10.83), we have

$$P = (\gamma - 1)n(f/r)kT = nkT,$$

whence

$$\gamma = 1 + r/f = 1 + 2/f \qquad (N.R.) \tag{10.84a}$$

$$= 1 + 1/f \qquad (E.R.). \tag{10.84b}$$

Consider now some applications of (10.84): (1) One, two, and three dimensional monatomic gases ($f = 1$, 2, or 3, respectively): $\gamma = 3$ (*N.R.*) or 2 (*E.R*), 2 (*N.R.*) or (3/2) (*E.R.*), and (5/3) (*N.R.*) or (4/3) (*E.R.*), respectively. (2) Diatomic molecules that are "rigid rotators": In this case two kinetic degrees of freedom are associated with the rotation of the molecule (assuming the rotational energy about the "figure axis" of the molecule to be negligible); hence we have $f = 3+2$, so that $\gamma = (7/5)$ in the *N.R.* case. (3) Diatomic molecules that are "rotation-oscillators": Here in the *N.R.* case we have $3+2+1$ kinetic degrees of freedom and one potential degree of freedom (assuming a simple harmonic oscillator model), *i.e.*, $f = 7$, whence $\gamma = (9/7)$. (4) For $f \to \infty$, $\gamma \to 1$ and $c_P \to c_V$ in both the *N.R.* and *E.R.* cases. Physically, this case would correspond to a situation in which *all* the energy added to a mass of the gas goes into "internal" forms of energy and *none* into kinetic forms of energy. This case is then effectively an *isothermal* case. A gas undergoing ionization behaves, qualitatively, as if it had a large number of "internal" degrees of freedom; this case then provides one way of understanding how the gammas in a partially ionized gas may have values near unity.

It should be noted that examples (2) and (3) above give the values of γ correctly for real gases only when the relevant "internal" degrees of freedom are excited.

Finally, we relate u_{kin}, the internal kinetic energy of *translation* per unit volume, to u, the *total* internal energy per unit volume, in terms of $\gamma-1$ and f. We had from (10.83) and (10.16)

$$P=(\gamma-1)u=(r/3)u_{\text{kin}},$$

where $r=2$ (*N.R.*) or 1 (*E.R.*), from which

$$\gamma-1=(r/3)(u_{\text{kin}}/u). \tag{10.85}$$

Thus, for non-degenerate systems $(\gamma-1)$ may be interpreted, approximately, as a measure of the ratio of *translational kinetic* internal energy to *total* internal energy. We also have $\gamma-1=r/f$, so that

$$u_{\text{kin}}/u=3/f\ (f\geqslant 3), \tag{10.86}$$

which applies to both the *N.R.* and the *E.R.* cases.

CHAPTER 11

Importance of Radiation Pressure in Stellar Interiors

In this chapter we shall establish an *upper limit* to the fraction of the total pressure that is due to radiation at a stellar center. We do this by following Chandrasekhar's development (the "β^* Theorem," *cf.* [Ch39, Chap. 3]).

We express the total pressure in a star as the sum of the gas pressure and the radiation pressure. We assume that the gas pressure is given by the perfect gas law and that the radiation pressure is that appropriate to black body radiation. We then have for the total pressure

$$\begin{aligned} P &= p_g + p_r \\ &= (\mathscr{R}/\mu)\rho T + (1/3)aT^4, \end{aligned} \tag{11.1}$$

where μ is the mean molecular weight of the stellar material (*cf.* Chap. 15).

We note that (11.1) may also be used when the gas is partially degenerate, if μ is replaced by $\mu\Lambda$, where the dimensionless factor Λ includes the effects of degeneracy. As will be shown in Chap. 24, degeneracy effects always *increase* the gas pressure over the perfect gas value, so that, always, $\Lambda \leq 1$; the equality sign applies to the case of non-degeneracy. To simplify the notation, we shall simply use μ, but shall interpret μ as $\mu\Lambda$ if we wish to consider the effects of degeneracy on the results we are about to obtain.

We showed in Chap. 6 that the expression for the radiation pressure in (11.1) (the black body value) should be extremely accurate throughout practically the entire stellar interior, provided that the refractive index is close to unity throughout most of the star. If the refractive index is not unity and is given by (2.163) (appropriate for conditions of nearly complete ionization), then p_r is less than the black body value $(aT^4/3)$(*cf.* Chap. 6). The qualitative effect of a non-unity refractive index can in this case be

formally taken into account by using an "effective" value of a, smaller than the actual value.

We shall work in terms of β, the ratio of gas pressure to total pressure:

$$\beta \equiv p_g/P = 1-p_r/P. \tag{11.2}$$

We wish to express P as an explicit function of ρ and β instead of p_r and p_g. We have $p_r = (1/3)aT^4 = [(1-\beta)/\beta]p_g = [(1-\beta)/\beta](\mathscr{R}/\mu)\rho T$, whence

$$T = \left(\frac{3\mathscr{R}}{a\mu}\cdot\frac{1-\beta}{\beta}\right)^{(1/3)}\rho^{(1/3)}. \tag{11.3}$$

Using (11.3) for T in the relation $P = p_g/\beta = (\mathscr{R}/\mu)\rho T/\beta$, we obtain

$$P = \left[\left(\frac{\mathscr{R}}{\mu}\right)^4\frac{3}{a}\,\frac{1-\beta}{\beta^4}\right]^{(1/3)}\rho^{(4/3)}, \tag{11.4}$$

which is the desired expression for P as a function of ρ and β. We have, in particular, for the *central* pressure,

$$P_c = \left[\left(\frac{\mathscr{R}}{\mu_c}\right)^4\frac{3}{a}\,\frac{1-\beta_c}{\beta_c{}^4}\right]^{(1/3)}\rho_c{}^{(4/3)}. \tag{11.5}$$

We showed, however (*cf.* Chap. 1), that the central pressure in a star in hydrostatic equilibrium was given by

$$P_c = \frac{G}{4\pi}\int_0^M \frac{M(r)\mathrm{d}M(r)}{r^4} = \frac{GM^2}{4\pi R^4}I_{1,4}, \tag{11.6}$$

where

$$I_{\sigma,\nu} \equiv \int_0^1 \left[\frac{M(r)}{M}\right]^\sigma\left(\frac{r}{R}\right)^{-\nu}\mathrm{d}\left[\frac{M(r)}{M}\right] \qquad [3(\sigma+1)>\nu]. \tag{11.7}$$

We also showed in Chap. 1, making only the assumption that $\bar{\rho}(r)$ does not increase outward, that an upper limit to the value of $I_{\sigma,\nu}$ exists:

$$I_{\sigma,\nu} \le \frac{3}{3(\sigma+1)-\nu}\left(\frac{\rho_c}{\bar{\rho}}\right)^{(\nu/3)} = \frac{3}{3(\sigma+1)-\nu}\left(\frac{4\pi\rho_c R^3}{3M}\right)^{(\nu/3)}, \tag{11.8}$$

where we have used the relation $\bar{\rho} = M/(4\pi R^3/3)$. In particular,

$$I_{1,4} \le \frac{3}{2}\left(\frac{4\pi\rho_c R^3}{3M}\right)^{(4/3)}, \tag{11.9}$$

which leads to the result

$$P_c \leq (\pi/6)^{(1/3)} G \rho_c^{(4/3)} M^{(2/3)}. \tag{11.10}$$

Comparing (11.10) with (11.5), we see that

$$\frac{1-\beta_c}{\beta_c^4} \leq \frac{\pi a}{18}\left(\frac{\mu_c}{\mathscr{R}}\right)^4 G^3 M^2. \tag{11.11}$$

Following Chandrasekhar, we define β^* such that

$$\frac{1-\beta^*}{\beta^{*4}} \equiv \frac{\pi a}{18}\left(\frac{\mu_c}{\mathscr{R}}\right)^4 G^3 M^2 \tag{11.12a}$$

$$= 0.03236 \cdot \mu_c^4 M^2 \tag{11.12b}$$

if M is in solar units. The condition (11.11) is then equivalent to the condition

$$(1-\beta_c)/\beta_c^4 \leq (1-\beta^*)/\beta^{*4}. \tag{11.13}$$

Now, $(1-\beta)/\beta^4$ is a monotonically decreasing function of β and a monotonically increasing function of $(1-\beta)$. Thus (11.13) may also be replaced by

$$(p_r/P)_c = 1-\beta_c \leq 1-\beta^*, \tag{11.14}$$

so that $1-\beta^*$ is the maximum possible value for the fraction of the total pressure that is due to radiation at a stellar center. It should be cautioned that the upper limit $1-\beta^*$ to the ratio p_r/P applies *only* to the stellar center. The value of p_r/P may possibly exceed $1-\beta^*$ at points in the star other than the center. However, because the ratio p_r/P is generally a slowly varying function of position in a star (*cf.* Chap. 23), its value is not likely to exceed $1-\beta^*$ by a very large factor (say less than 2), if at all, anywhere in the interior.

We see from (11.12b) that $1-\beta^*$, the upper limit to the value of $(p_r/P)_c$, increases with both μ_c and M but is independent of the radius R of the star. The dependence of $(1-\beta^*)/\beta^{*4}$ on μ_c and M given by (11.12b) can, in fact, be readily understood by reference to the equation of state in the form (11.4). If P and ρ are mean (or representative) values of pressure and density in a star, we have, to order of magnitude,

$$P \sim \frac{G}{4\pi}\frac{M^2}{R^4}, \quad \rho \sim \frac{M}{(4/3)\pi R^3}.$$

Substitution of these expressions into (11.4) immediately yields (11.12b), aside from the value of the constant factor.

Table 11.1 gives some values of the quantity $\mu_c^2 M$ as a function of $1-\beta^*$. We see from these values that for $M = 1$, $1-\beta^* \approx 0.0025$, 0.03, and 0.20 for,

respectively, $\mu_c = 1/2$, 1, and 2 (the case $\mu_c \approx 1/2$ applies approximately to the sun). For a star such as Capella ($M \approx 4$) we have $1-\beta^* \approx 0.03$ and 0.2 for, respectively, $\mu_c = 1/2$ and 1.

Since $\mu_c < 1$ for most stars, we see that radiation pressure at the stellar center can become appreciable only for stars of mass greater than two or three solar masses.

We note that the effect of degeneracy is to *lower* the value of $1-\beta^*$, *i.e.*, to *decrease* the upper limit to the value of $(p_r/P)_c$. This conclusion follows from (11.12b) by recalling that degeneracy has the same effect as reducing the value of μ_c (see the third paragraph of this chapter). Hence radiation pressure is likely to be less important in a partially degenerate star than in a non-degenerate star of similar characteristics.

Table 11.1

$M\mu^2{}_c$ AS A FUNCTION OF $1-\beta^*$

$1-\beta^*$	$M\mu_c^2$ $(M_\odot = 1)$
0.01	5.672×10^{-1}
0.02	8.186×10^{-1}
0.03	1.023×10^{0}
0.05	1.377×10^{0}
0.08	1.858×10^{0}
0.10	2.170×10^{0}
0.20	3.884×10^{0}
0.30	6.214×10^{0}
0.40	9.766×10^{0}
0.50	1.572×10^{1}
0.60	2.691×10^{1}
0.70	5.168×10^{2}
0.80	1.243×10^{1}
0.90	5.274×10^{2}
1.00	∞

We note, finally, that the effect of a non-unity refractive index in the case of nearly complete ionization is also to *lower* the value of $1-\beta^*$ (see the fourth paragraph of this chapter).

CHAPTER 12

Polytropic Changes

In this chapter we consider *polytropic* changes, which are a generalization of *adiabatic* changes, in a thermodynamic system. Specifically, we consider only *quasi-static* or *reversible* polytropic changes, so that the system is always only infinitesimally removed from a state of thermodynamic equilibrium. Hence the state of any general non-simple system may be completely described, thermodynamically, by specifying only two of the three state variables P, T, and V (or ρ) (see Sect. 9.13). We shall generalize the usual definition of a polytropic change (*cf.* Chandrasekhar [Ch39, Chap. 4]) and shall apply this generalized definition to a general system.

We define a *polytropic change* to be a quasi-static change carried out in such a way that the derivative

$$c \equiv \mathrm{d}Q/\mathrm{d}T \tag{12.1}$$

(the "specific heat") varies in a specified way throughout the change. For example, in an *adiabatic* change $\mathrm{d}Q = 0$ or $c = 0$; hence an adiabat is a polytrope of *zero* specific heat. In an *isothermal* change $\mathrm{d}T = 0$ or $c = \infty$; hence an isotherm is a polytrope of *infinite* specific heat. If $c = c_V$ or c_P, the polytrope is, respectively, an *isometric* (V = constant) or *isobaric* (P = constant) change.

For a general thermodynamic system in thermodynamic equilibrium the three state variables P, V (or ρ), and T are related by the equation of state of the material, so that only two of these variables are independent. If we now impose some *additional* condition on the system, such as requiring $\mathrm{d}Q/\mathrm{d}T$ to vary in some specified way during a quasi-static change, then we are imposing via the first law of thermodynamics a *second* relation between the state variables. Just as in the case of an adiabatic change, then, only *one* variable can be varied independently in a polytropic change. From the examples cited in the preceding paragraph we see that assuming different values of c amounts to assuming different variations of T with, say, V (or ρ)

during the quasi-static change. By allowing c to vary arbitrarily, it is clear that any desired dependence of T on V (or ρ), say, can be obtained.

We shall work in terms of specific quantities and shall consider unit mass of material. For the quasi-static change under consideration we define the three *polytropic exponents*:

$$\Gamma_1' \equiv \mathrm{d}\ln P/\mathrm{d}\ln\rho, \tag{12.2a}$$

$$\Gamma_2'/(\Gamma_2'-1) \equiv \mathrm{d}\ln P/\mathrm{d}\ln T, \tag{12.2b}$$

$$\Gamma_3'-1 \equiv \mathrm{d}\ln T/\mathrm{d}\ln\rho. \tag{12.2c}$$

It is clear from these definitions that we have the identity

$$\frac{\Gamma_2'}{\Gamma_2'-1} \equiv \frac{\Gamma_1'}{\Gamma_3'-1}, \tag{12.3}$$

so that, given any two of the polytropic exponents, the third can always be computed. From the equation of state $P = P(\rho,T)$, it is clear that we have the additional identity (*cf.* Sect. 9.14b)

$$\Gamma_1' \equiv \chi_\rho + (\Gamma_3'-1)\chi_T, \tag{12.4}$$

where (*cf.* (9.81), (9.82))

$$\chi_\rho \equiv (\partial\ln P/\partial\ln\rho)_T, \qquad \chi_T \equiv (\partial\ln P/\partial\ln T)_\rho. \tag{12.5}$$

Thus, given any one of the polytropic exponents, the other two may be evaluated from (12.3) and (12.4) if the values of χ_ρ and χ_T are known from the equation of state of the material. We shall now obtain expressions for Γ_1', Γ_2', and Γ_3' in terms of the adiabatic exponents $\Gamma_1, \Gamma_2, \Gamma_3$ (*cf.* Sect. 9.14a), c, c_V, and c_P.

We have the first law for an infinitesimal, quasi-static change in unit mass of material:

$$\mathrm{d}Q = \mathrm{d}E - (P/\rho^2)\mathrm{d}\rho. \tag{12.6}$$

Regarding $E = E(\rho,T)$ as a function of ρ and T and making use of (12.1), we have

$$\mathrm{d}Q = \left[\rho\left(\frac{\partial E}{\partial\rho}\right)_T - \frac{P}{\rho}\right]\frac{\mathrm{d}\rho}{\rho} + c_V T\frac{\mathrm{d}T}{T} = cT\frac{\mathrm{d}T}{T}, \tag{12.7}$$

where $c_V = (\partial E/\partial T)_\rho$ (*cf.* Sect. 9.13). We then have, using (12.2c),

$$\Gamma_3'-1 = \frac{P/\rho - \rho(\partial E/\partial\rho)_T}{(c_V-c)T}.$$

However, the right side of this equation gives the value of the adiabatic exponent $\Gamma_3 - 1$ when $c = 0$ (*cf.* (9.91)); hence we obtain

$$\Gamma_3' - 1 = \frac{\Gamma_3 - 1}{1 - (c/c_V)}. \tag{12.8}$$

For $c = 0$, we have $\Gamma_3' = \Gamma_3$ (adiabatic change); for $c = c_V$, we have $\Gamma_3' = \infty$ (ρ = constant); for $c = \infty$, we have $\Gamma_3' = 1$ (isothermal). For the special case where $\Gamma_3 = \gamma = c_P/c_V$ (a simple perfect gas, for example, *cf.* Sect. 9.15), we have

$$\Gamma_3' = \frac{c_P - c}{c_V - c}. \tag{12.9}$$

To relate Γ_2' to Γ_2, we regard ρ and E as functions of P and T. We then have from (12.6) and (12.1)

$$\mathrm{d}Q = \left[P\left(\frac{\partial E}{\partial P}\right)_T - \frac{P^2}{\rho^2}\left(\frac{\partial \rho}{\partial P}\right)_T\right]\frac{\mathrm{d}P}{P} + \left[T\left(\frac{\partial E}{\partial T}\right)_P - \frac{PT}{\rho^2}\left(\frac{\partial \rho}{\partial T}\right)_P\right]\frac{\mathrm{d}T}{T} = cT\frac{\mathrm{d}T}{T}, \tag{12.10}$$

where the factor multiplying $\mathrm{d}T/T$ is simply Tc_P (*cf.* (9.79)). We then obtain from (12.2b)

$$\frac{\Gamma_2'}{\Gamma_2' - 1} = \frac{(c_P - c)T}{(P^2/\rho^2)(\partial\rho/\partial P)_T - P(\partial E/\partial P)_T}.$$

The right side of this equation, however, is the expression for the adiabatic exponent $\Gamma_2/(\Gamma_2 - 1)$ when $c = 0$. We thus obtain

$$\frac{\Gamma_2'}{\Gamma_2' - 1} = \left(1 - \frac{c}{c_P}\right)\frac{\Gamma_2}{\Gamma_2 - 1}. \tag{12.11}$$

For $c = 0$, we have $\Gamma_2' = \Gamma_2$ (adiabatic); for $c = c_P$, we have $\Gamma_2' = 0$ (P = constant); for $c = \infty$, we have $\Gamma_2' = 1$ (isothermal). If $\Gamma_2 = \gamma = c_P/c_V$, then Γ_2' reduces to the value given by the right side of (12.9).

Finally, to relate Γ_1' to Γ_1, we make use of the identity (12.3). We obtain, making use of the corresponding identity (9.89) among the adiabatic exponents,

$$\Gamma_1' = \frac{\Gamma_1}{\gamma}\frac{c_P - c}{c_V - c}, \tag{12.12}$$

which shows that $\Gamma_1' = \Gamma_1$ for $c = 0$ (adiabatic). We have $\Gamma_1' = 0$ or ∞ according as $c = c_P$ (P = constant) or $c = c_V$ (ρ = constant); and $\Gamma_1' = \Gamma_1/\gamma$ if $c = \infty$ (isothermal). For the case $\Gamma_1 = \gamma = c_P/c_V$, Γ_1' is also given by the right side of (12.9). Clearly, $\Gamma_1' = 1$ (or Boyles law, $P = const.\ \rho$, obtains) if $\Gamma_1 = \gamma$ and $c = \infty$.

For the case where Γ_1', Γ_2', and Γ_3' are all constant, (12.2) can be integrated, and the integrated polytropic equations become

$$P = \text{const.}\, \rho^{\Gamma_1'}, \tag{12.13a}$$

$$P = \text{const.}\, T^{\Gamma_2'/(\Gamma_2'-1)}, \tag{12.13b}$$

$$T = \text{const.}\, \rho^{\Gamma_3'-1}. \tag{12.13c}$$

In general, however, Γ_1', Γ_2', and Γ_3' will be constant only if Γ_1, Γ_2, Γ_3, c, c_P, and c_V are also constant.

Aside from the question of the constancy of the three polytropic exponents, it is clear that they all, in general, have values differing from one another. It follows from the identities (12.3) and (12.4) that the polytropic exponents will all be equal to one another if $\chi_\rho = \chi_T = 1$, *i.e.*, if the equation of state is of the perfect gas law form $P = \textit{const.}\, \rho T$. This, however, is not a *necessary* condition for equality of all the polytropic exponents. It will be recalled (*cf.* Sect. 9.14b) that $\chi_\rho = \chi_T = 1$ is also a *sufficient* condition that $\Gamma_1 = \Gamma_2 = \Gamma_3$ and a *necessary* and *sufficient* condition that $\Gamma_1 = \Gamma_2 = \Gamma_3 = \gamma = c_P/c_V$.

We shall define the *polytropic index* n by the relations

$$1+n \equiv (\mathrm{d} \ln P/\mathrm{d} \ln T) \tag{12.14a}$$

$$= \Gamma_2'/(\Gamma_2'-1), \tag{12.14b}$$

so that n is constant only if Γ_2' is constant. In some contexts (for example, Sect. 23.1; Chandrasekhar [Ch39, Chap. 4]) the polytropic index n is defined by the relation

$$1+\frac{1}{n} \equiv \frac{\mathrm{d} \ln P}{\mathrm{d} \ln \rho}. \tag{12.14c}$$

These two definitions are equivalent if the equation of state is of the form $P = \textit{const.}\, \rho T$.

In applications to stars the derivative $(\mathrm{d} \ln P/\mathrm{d} \ln T)$ may be considered as being evaluated from the runs of P and T in an actual stellar model.* In this case the value of this derivative will, in general, be a function of the radial distance r from the stellar center. When this derivative is interpreted in this manner, the corresponding value of n in (12.14a) is called the *effective polytropic index*, $n_e(r)$, and is defined by the relation

$$1+n_e(r) \equiv (\mathrm{d} \ln P/\mathrm{d} \ln T)_r, \tag{12.15a}$$

$$= \left[\Gamma_2'/(\Gamma_2'-1)\right]_r, \tag{12.15b}$$

* See the comments made in connection with (9.9′) and (9.9″).

where the subscript r means that the derivative is (in general) a function of r.

It is possible to express the other two derivatives in (12.2) in terms of n_e, χ_ρ, and χ_T by use of (12.15b) and the identities (12.3) and (12.4). We obtain

$$(\Gamma_3' - 1)_r = \left(\frac{\mathrm{d} \ln T}{\mathrm{d} \ln \rho}\right)_r = \frac{\chi_\rho}{1 + n_e - \chi_T} \tag{12.16a}$$

$$= \frac{1}{n_e} \text{ if } \chi_\rho = \chi_T = 1 \tag{12.16b}$$

and

$$(\Gamma_1')_r = \left(\frac{\mathrm{d} \ln P}{\mathrm{d} \ln \rho}\right)_r = \chi_\rho \left(\frac{1 + n_e}{1 + n_e - \chi_T}\right) \tag{12.17a}$$

$$= \frac{1 + n_e}{n_e} \text{ if } \chi_\rho = \chi_T = 1. \tag{12.17b}$$

Hence all three derivatives in (12.2) can be described in terms of n_e alone if (but not *only* if) $\chi_\rho = \chi_T = 1$.

CHAPTER 13

Stability of the Radiative Gradient

We found that if radiative equilibrium obtains, *i.e.*, if the energy is transported through the layers of a star by radiation alone, the temperature gradient is given to good approximation in the stellar interior in terms of the local values of opacity κ (Rosseland mean mass absorption coefficient, see Chap. 8), density ρ, "interior luminosity" $L(r)$ (net outward rate of flow of energy through a sphere of radius r), and r by the relation

$$\frac{\mathrm{d}}{\mathrm{d}r}((1/3)aT^4) = -\frac{\kappa\rho}{c}\frac{L(r)}{4\pi r^2}, \tag{13.1a}$$

which may also be written as

$$\frac{\mathrm{d}T}{\mathrm{d}r} = -\frac{3}{4ac}\frac{\kappa\rho}{T^3}\frac{L(r)}{4\pi r^2}. \tag{13.1b}*$$

We now wish to decide whether the stellar material would be, dynamically, in stable or unstable equilibrium under such a gradient as that given by (13.1). In other words, with such a gradient, would the matter be stable or unstable to small local perturbations?

Suppose the *actual* temperature gradient at the point under consideration is that appropriate to radiative transfer, or, in other words, is given by (13.1). Suppose now that an element of mass δm at the radial distance r suddenly undergoes an arbitrarily small increase in temperature $\Delta T(r) > 0$, where

$$\Delta T(r) \equiv T_{\delta m}(r) - T(r), \tag{13.2}$$

$T_{\delta m}(r)$ being the temperature of the element and $T(r)$ being the temperature of the unperturbed surrounding material at the point r. The pressure

* With a slight generalization of the definition of κ, (13.1a) and (13.1b) apply also when some (or all) of the energy is being transported by *conduction*; see Sect. 16.7.

within the element will increase to a value slightly greater than the surrounding external pressure, and the volume of the element will increase quickly until the internal and external pressures are equal. Thus the density $\rho_{\delta m}(r)$ of the element will have decreased below the density $\rho(r)$ of the unperturbed surroundings, so that the excess density

$$\Delta\rho(r) \equiv \rho_{\delta m}(r) - \rho(r) \tag{13.2'}$$

of the element will now be negative. The element will accordingly begin to move outward under the influence of the buoyant forces of the surrounding material. We assume that the pressure within the element acting upon the surrounding material is always equal to the external pressure of the surrounding material acting on the element; *i.e.*, we assume that pressure equilibrium obtains at all times:

$$P_{\delta m}(r) = P(r),$$

whence

$$\left(\frac{dP}{dr}\right)_{\delta m} = \frac{dP}{dr}, \tag{13.3}$$

where $P(r)$ and dP/dr are the pressure and pressure gradient of the surrounding material. In other words, we are assuming that the time required for pressure equilibrium to become established in the element is small compared with the times of interest. (This assumption is discussed further in Sect. 14.3.)

The values of $T_{\delta m}(r)$ and $\rho_{\delta m}(r)$ of the element as it moves outward will depend, in general, on $P(r)$ (since the pressure forces may do work on the element) and on the way in which the element exchanges heat with its surroundings. In the special case of *adiabatic* motion (no net gains or losses of heat by the element) $T_{\delta m}(r)$ and $\rho_{\delta m}(r)$ would be determined only by the value of $P(r)$ at each point through the adiabatic relations between T and P and between ρ and P. However, we consider here the general (not necessarily adiabatic) case.

It is clear, now, that for a condition of *stable* equilibrium to exist the density $\rho_{\delta m}(r)$ of the element must eventually become equal to the unperturbed density $\rho(r)$ of the surroundings at some point further out; for, when these two densities are equal, the upward buoyant force is exactly balanced by the downward gravitational force, and the net force on the element is zero. In other words, the $\rho_{\delta m}(r)$ curve shown in Fig. 13.1 must cross the unperturbed $\rho(r)$ curve at some value of r greater than that at which the temperature increase occurred. Since we have chosen ΔT, and hence $\Delta\rho$, to be arbitrarily

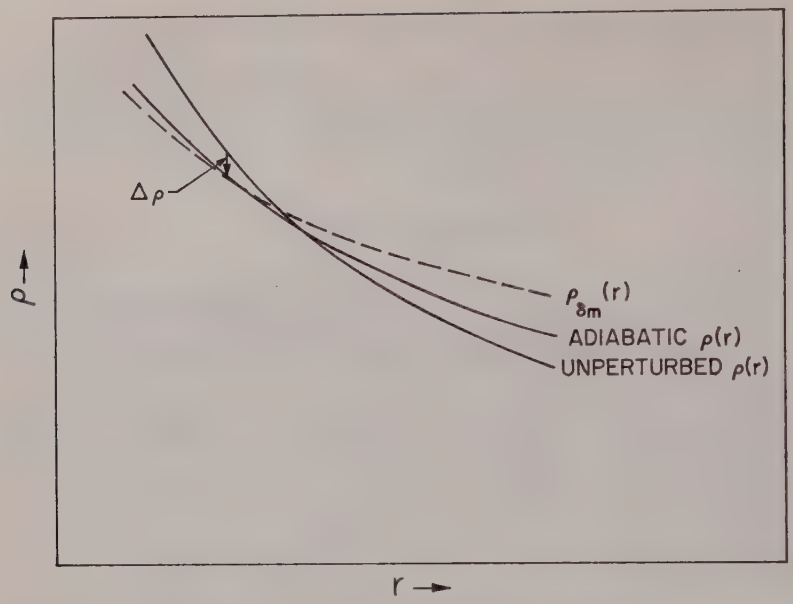


Fig. 13.1 A case of *stability* against convection. $\Delta\rho$ is to be regarded as an infinitesimal perturbation in density.

small, and since $\Delta\rho$ is negative in the present example where ΔT is positive, then, clearly, a necessary and sufficient general condition for stability of the radiative gradient is that

$$\mathrm{d}(\Delta\rho)/\mathrm{d}r > 0 \tag{13.4}$$

or

$$(\mathrm{d}\rho/\mathrm{d}r)_{\delta m} > (\mathrm{d}\rho/\mathrm{d}r)_{\text{unpert}}, \tag{13.5}$$

where the subscript "unpert" refers to the unperturbed surroundings.

The forms (13.4) and (13.5) of the condition for stability against convection may appear somewhat unconventional, as this condition is usually expressed (see Sect. 13.1) in terms of ΔT rather than of $\Delta\rho$. However, (13.4) and (13.5) are perfectly general, whereas the usual expressions ((13.6) and (13.7) below) are valid only in the case of uniform chemical composition and under certain assumed conditions (see Sect. 13.1).

Rather than proceding immediately from the general equations (13.4) and (13.5), we consider, first, in Sect. 13.1 the somewhat more restricted case of uniform chemical composition. This is the case of interest in most applications; moreover, most of the conventional terminology and notation regarding convective stability is based on this case. The more general case of a non-uniform (but continuously varying in space) chemical composition

is considered in Sect. 13.3. (The case of a *discontinuously* varying composition is considered in Sect. 23.6a.) A crude order-of-magnitude estimate of the degree of "superadiabaticity" of the temperature gradient in a convective zone in the deep stellar interior is presented in Sect. 13.2, and a general discussion of convective stability in stars is given in Sect. 13.4.

(Recently, a rigorous study of the conditions for instability against convection, based on (essentially) a detailed linear stability analysis, has been published by Lebovitz [Le65]. This analysis yields the same criterion for convective instability (sometimes called the "Schwarzschild" criterion) as does the conventional, more intuitive, treatment presented in Sect. 13.1. The validity of the Schwarzschild criterion in the case of general relativistic fluid dynamics has been established by Chandrasekhar [Ch65] and by Thorne [Th66a].)

13.1 Case of Uniform Chemical Composition

In this section we assume that the chemical composition is constant in space. Hence, in the case of complete (or zero) ionization or dissociation the mean molecular weight μ (*cf.* Chap. 15) is constant in space and the "material" pressure P may be regarded, through the pressure equation of state, as (in

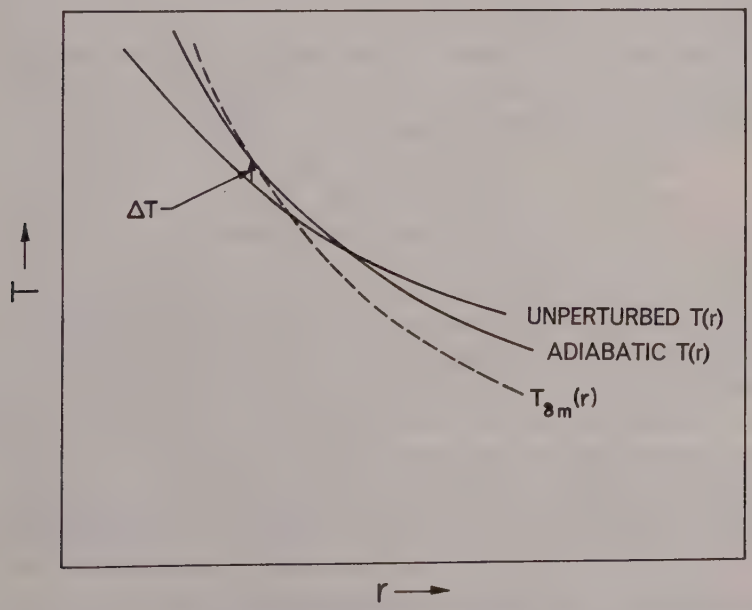


Fig. 13.2 A case of *stability* against convection. ΔT is to be regarded as an infinitesimal perturbation in temperature.

general) a function only of density ρ and temperature T (*i.e.*, only changes in ρ and T contribute to changes in P). In the case of partial ionization or dissociation we assume instantaneous chemical equilibrium (see Sect. 9.12), so that, again, P may be regarded, in general, as a function only of ρ and T.

Consider now the rising element of mass δm which was conceptually followed in the introduction to this chapter. The assumption of continuous pressure equilibrium with its surroundings, together with the assumptions stated in the preceding paragraph, require that, when the element attains the same density as its surroundings, its temperature is also the same as that of its surroundings. Hence, in this case of uniform chemical composition we may base the discussion of the criterion for stability against convection on the temperature excess $\Delta T(r)$, rather than on the density excess $\Delta\rho(r)$, and we shall do so in the remainder of this section.

Reference to Fig. 13.2 and to the discussion in the introduction to this chapter shows that in this case the condition for stability of the radiative gradient is that

$$\mathrm{d}(\Delta T)/\mathrm{d}r < 0 \tag{13.6}$$

or

$$(\mathrm{d}T/\mathrm{d}r)_{\delta m} < (\mathrm{d}T/\mathrm{d}r)_{\text{unpert}}. \tag{13.7}$$

Since $\mathrm{d}T/\mathrm{d}r$ is always negative in the stellar interior (except possibly in regions where neutrino energy losses are important, *cf.* Sects. 17.20, 26.4g, and 26.4h; and possibly in degenerate stellar cores, *cf.* Eggleton [Eg66]) and since T is always positive, we may write the necessary and sufficient condition for stability of the radiative gradient (13.7) in the present case in the form

$$\left|\frac{1}{T}\left(\frac{\mathrm{d}T}{\mathrm{d}r}\right)_{\delta m}\right| > \left|\frac{1}{T}\left(\frac{\mathrm{d}T}{\mathrm{d}r}\right)_{\text{unpert}}\right|. \tag{13.8}$$

According to (13.8), then, for stability of the radiative gradient the $T_{\delta m}(r)$ curve must be *steeper* than the unperturbed $T(r)$ curve of the surroundings.

The condition (13.8) is quite general (within the assumptions stated in the first paragraph of this section) but not very useful for practical computations of stellar models: in order to evaluate $(\mathrm{d}T/\mathrm{d}r)_{\delta m}$, consideration of heat exchange mechanisms with the surroundings of the element would be required and, in general, an iterative procedure of constructing a stellar model might be necessary in this general case. In the important special case where *no energy sources are effective* in the element,* however, (13.8) can be replaced by a simpler condition. If there are no effective energy sources in

* For further clarification and discussion of this point, see Chap. 14.

the rising element, then the element can only *lose* heat to its surroundings, either by radiation or by conduction.* Hence $T_{\delta m}(r)$ will fall more sharply with increasing r than would be the case if the element were moving *adiabatically* (*i.e.*, exchanging no heat, in the net, with its surroundings); thus the $T_{\delta m}(r)$ curve will in this case be *steeper* than the adiabatic $T(r)$ curve:

$$\left|(\mathrm{d}T/\mathrm{d}r)_{\delta m}\right| \geq \left|(\mathrm{d}T/\mathrm{d}r)_{\mathrm{ad}}\right| . \tag{13.9}$$

It is clear, then, that whenever (13.9) is valid, the necessary and sufficient condition (13.8) can be replaced by the simpler sufficient condition for stability of the radiative gradient:

$$\left|\frac{1}{T}\left(\frac{\mathrm{d}T}{\mathrm{d}r}\right)_{\mathrm{ad}}\right| > \left|\frac{1}{T}\left(\frac{\mathrm{d}T}{\mathrm{d}r}\right)_{\mathrm{unpert}}\right| , \tag{13.10}$$

since satisfaction of (13.10) will guarantee satisfaction of (13.8), provided that (13.9) is valid.

In case (13.9) is not satisfied, as may be the case if effective energy sources (such as nuclear sources) are present in the element, then of course (13.10) is not strictly the correct condition for stability and the more general condition (13.8) should be used. This case, however, presents no difficulty in practice since, as will be shown in Sect. 13.2, $(\mathrm{d}T/\mathrm{d}r)_{\delta m}$ is likely to be equal to $(\mathrm{d}T/\mathrm{d}r)_{\mathrm{ad}}$ to high accuracy even in those parts of a star (generally, the deeper regions) where nuclear energy sources *are* effective. Consequently, the condition (13.10) may be used, in practical calculations, as the condition for stability even in this case, although (13.8) is really the rigorously correct condition under the assumed conditions. The condition (13.10) is universally used in calculations of chemically homogeneous stellar models.

If (13.8) is satisfied, the inertia of the upward-moving element will cause it to overshoot its equilibrium position (defined by the intersection of the $T_{\delta m}(r)$ and the $T(r)$ curves). Subsequently, however, a restoring force on the element will develop because $T_{\delta m}(r)$ will then be less than $T(r)$, whence $\rho_{\delta m}(r) > \rho(r)$, and the gravitational force on the element will exceed the buoyant force. The motion of the element will thus eventually be reversed, and there is clearly no tendency for convective motions to develop, *i.e.*, the material is stable against convection.

We now assume that the radiative gradient is stable, *i.e.*, that (13.8) is satisfied at the point under consideration. For simplicity, we shall assume that the simpler condition (13.10) can be used as the stability criterion. To

* As is pointed out in Chap. 14, the heat loss here refers only to the "horizontal" loss, and has nothing to do with the heat gains or losses associated with *thermal equilibrium* (*cf.* Chap. 5).

describe the *unperturbed* temperature gradient at the point of interest, we may make use of the effective polytropic index which was defined in Chap. 12 by the relation

$$(\mathrm{d}\ln P/\mathrm{d}\ln T)_r - 1 = n_e(r),$$

which may be written in the form

$$\left(\frac{1}{T}\frac{\mathrm{d}T}{\mathrm{d}r}\right)_{\text{unpert}} = \frac{1}{n_e+1}\left(\frac{1}{P}\frac{\mathrm{d}P}{\mathrm{d}r}\right). \tag{13.11}$$

Equation (13.11) expresses the actual, unperturbed temperature gradient at the point of interest in terms of the pressure gradient and the effective polytropic index at that point.

For the *adiabatic* relation between P and T at the point of interest, we have (*cf.* (9.88)) $(\Gamma_2-1)/\Gamma_2 \equiv (\mathrm{d}\ln T/\mathrm{d}\ln P)_{\text{ad}}$, which may also be written in the form

$$\left(\frac{1}{T}\frac{\mathrm{d}T}{\mathrm{d}r}\right)_{\text{ad}} = \frac{\Gamma_2-1}{\Gamma_2}\left(\frac{1}{P}\frac{\mathrm{d}P}{\mathrm{d}r}\right). \tag{13.12}$$

Using (13.11) and (13.12) in (13.10), we obtain

$$n_e > \frac{1}{\Gamma_2-1} \tag{13.13}$$

as the condition expressing stability of the radiative gradient under the assumed conditions when the simple condition (13.10) is appropriate. (When the more general condition (13.8) must be used, the adiabatic exponent Γ_2 in (13.13) may be replaced by the polytropic exponent Γ_2' which relates P and T during the motion of the moving element. However, in this book we shall always assume that the simpler condition (13.10) is an adequate criterion for stability of the radiative gradient under the assumed conditions.)

For example, if the equation of state is of the perfect gas law form, $P = const.\ \rho T$, we have $\Gamma_2 = \gamma = c_P/c_V$ (*cf.* Sect. 9.14b), and (13.13) becomes $n_e > 1/(\gamma-1)$. For a non-relativistic, perfect monatomic gas $\gamma = 5/3$ (*cf.* Sect. 10.7a), whence $n_e > 1.5$ for stability of the radiative gradient. As another example, consider the case where the gas pressure is negligible compared with the radiation pressure. Then we have $\Gamma_2 = 4/3$ (*cf.* Sect. 9.16), so that $n_e > 3$ for stability against convection. In an actual star n_e usually decreases inward, at least at points below the regions of hydrogen and helium ionization but not too near the stellar center, *cf.* Part II of this book. Consequently, it follows that radiation pressure may be expected to increase the value of r below which instability against convection exists over the value which would

obtain in the absence of radiation pressure (this expectation is confirmed in the models of Deinzer and Salpeter [De64] and Meggitt [Me65]). Thus the effect of radiation pressure is to reduce the steepness of the adiabatic temperature gradient at a given point and thus to favor instability against convection.

To summarize: At every point in a chemically homogeneous star at which the radiative gradient is stable, we must have, combining (13.10) and (13.12),

$$\left(\frac{\mathrm{d}\ln T}{\mathrm{d}\ln P}\right)_r < \frac{\Gamma_2 - 1}{\Gamma_2}. \tag{13.14}$$

We now consider the case in which the radiative gradient is *unstable*, *i.e.*, (13.8) is not satisfied. In this case the radiative temperature gradient that would prevail under perfectly static conditions (*i.e.*, with all existing convective motions artifically suppressed) is *steeper* than the gradient $(\mathrm{d}T/\mathrm{d}r)_{\delta m}$ which a rising element would follow. This clearly represents a case of unstable

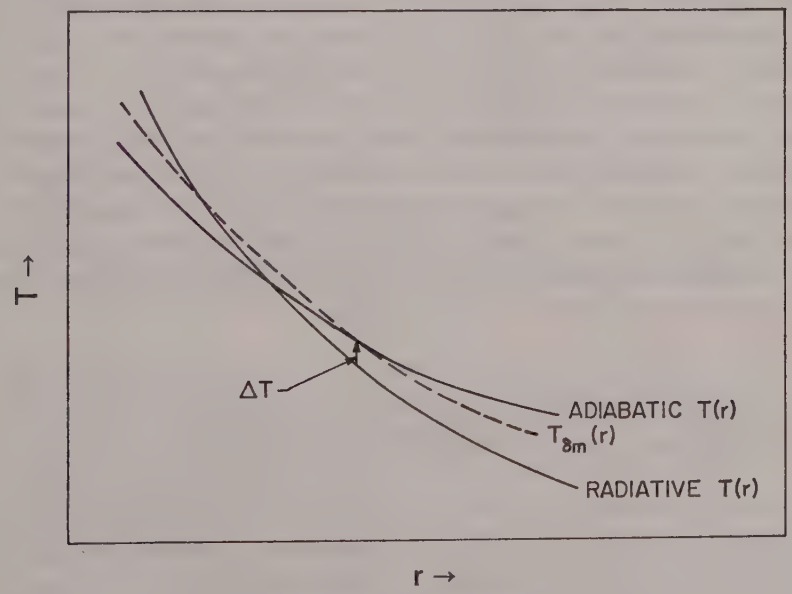


Fig. 13.3 A case of *instability* against convection. ΔT is to be regarded as an infinitesimal perturbation in temperature.

equilibrium; for, if the temperature of a small element of matter suddenly increased by an infinitesimal amount, the element would rise, following the $T_{\delta m}(r)$ curve in Fig. 13.3, and would continue to rise until it reached a point, considerably further out, at which the $T_{\delta m}(r)$ curve and the radiative $T(r)$

curve crossed again, or else until the element had dissolved through turbulent mixing. Similarly, a slight decrease in the temperature of the element would cause it to descend toward the center of the star. If no effective energy sources are present in the element, then $(dT/dr)_{\delta m}$ will clearly be intermediate in value between (dT/dr), the "static" gradient, and $(dT/dr)_{ad}$, the adiabatic gradient. In this case, then, the actual average gradient in the convection zone will also be steeper than $(dT/dr)_{ad}$ under conditions of instability against convection.

Under these conditions, then, upward and downward convection currents would be set up which would carry large amounts of energy from the interior of the star, thus effecting an overall decrease in the steepness of the $T(r)$ curve which would otherwise obtain in a perfectly static condition. Eventually a "steady" situation would be realized, in which both radiation and convection compete as transport mechanisms, each carrying a part of the total energy flux. The actual $T(r)$ curve within the convective region would then have a slope intermediate between that of the radiative $T(r)$ curve and that of the adiabatic $T(r)$ curve. That is, the actual $T(r)$ curve would be *superadiabatic* (*steeper* than the adiabatic $T(r)$ curve) but *less steep* than the radiative $T(r)$ curve. It is clear that the actual $T(r)$ curve in the convecting region must be superadiabatic if there are no effective energy sources in the region; if the actual gradient were exactly adiabatic under these conditions, then there would be no energy transport by convection, and hence no "driving force" for the convection. The extent to which the superadiabatic gradient would differ from the adiabatic would depend on the ratio of the *excess* heat energy that each unit mass of the gas would have to carry away from the interior to the total internal thermal energy of the unit mass.

13.2 Estimate of the Degree of Superadiabaticity in the Deep Interior

In the deep interior of a star the superadiabatic gradient is always only negligibly steeper than the adiabatic one. This question of the magnitude of the superadiabaticity will be discussed much more fully in Chap. 14; in this section, however, we shall give a rough order-of-magnitude estimate of the extent to which the superadiabatic gradient exceeds the adiabatic in the deep interior.

Consider a spherical shell of thickness Λ within a convective region in the deep interior of a star (see Fig. 13.4). We assume that there is no nuclear energy production occurring in the regions of interest. We take Λ to be the "mixing length" (*cf.* Chap. 14); *i.e.*, the "mean free path" of an average

turbulence element. Also, let t be the "mean life" of an average turbulence element, or the time required for the element to convect through the distance Λ. It is clear, then, that a time of the order of t must elapse before an amount of energy $L(r)t$ incident on the lower boundary of the shell in time t can emerge from the top of the shell at radius $r+\Lambda$, where $L(r)$ is the net rate at which energy is carried by convection outward through a sphere of radius r (we are here neglecting the energy transport by radiation, since the present

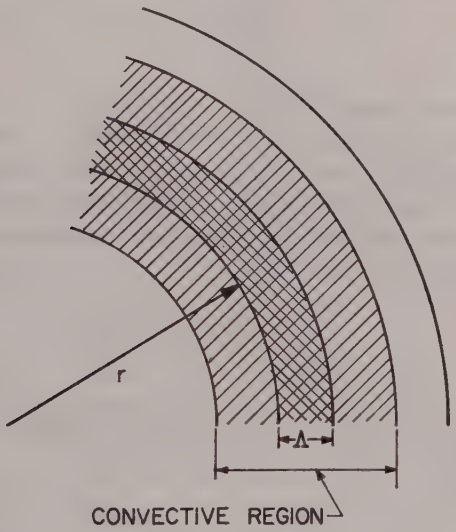


Fig. 13.4 Illustration for estimating the superadiabaticity of the temperature gradient.

argument is only an order-of-magnitude one). (Note that, if we consider luminosities averaged over times comparable to or longer than t, we must have $L(r) = L(r+\Lambda)$ if there is negligible gravitational or nuclear energy released within the shell of thickness Λ in the time t.) The matter of mass ΔM within the shell effectively "stores" this energy $L(r)t$ incident in time t on the bottom of the shell, and then releases it at the top of the shell. Thus, in time t each unit mass of material in the shell will absorb the amount of energy $L(r)t/\Delta M$ and will consequently "heat up" by the amount $\Delta T = L(r)t/\Delta M/c_V$, where we take $c_V \simeq (3/2)\mathscr{R}/\mu$ as the specific heat per unit mass at constant volume. If the average temperature of the material during the time t is T, then the excess of the superadiabatic gradient over the adiabatic will be of order

$$\frac{\Delta|(dT/dr)|}{|(dT/dr)_{ad}|} \sim \frac{\Delta T}{T} \sim \frac{L(r)t/\Delta M}{c_V T}. \tag{13.15}$$

For typical conditions in the deep interior of a star of approximately solar type we may take $L(r) \sim 10^{33}$ erg/sec, $t \sim 10^6$ sec (see Chap. 14*), $\Delta M \sim$ $\sim (1/10) M_\odot \sim 10^{32}$ gm, and $T \sim 10^7$°K. Thus $L(r)t/\Delta M \sim 10^7$ erg/gm is the amount of energy absorbed per gram during the time t, and $c_V T = (3/2)(\mathcal{R}/\mu)T \sim 10^8 \times 10^7 \sim 10^{15}$ erg/gm is the average internal energy per gram of the stellar material. Hence the fractional excess of the actual temperature gradient in the convective region over the adiabatic is

$$\frac{\Delta|dT/dr|}{|(dT/dr)_{ad}|} \sim 10^{-8},$$

which is a very small excess indeed.

Thus, in a convective region in the deep interior of a star we expect that the actual, superadiabatic temperature gradient can be approximated to a very high degree of accuracy by the *adiabatic* temperature gradient. Hence, *within* a convective region in the deep interior the temperature gradient should be given very accurately by the relation

$$\frac{1}{T}\frac{dT}{dr} = \frac{\Gamma_2 - 1}{\Gamma_2}\frac{1}{P}\frac{dP}{dr}. \tag{13.16}$$

Outside such a region, where the material is in radiative equilibrium, the temperature gradient is given by (13.1):

$$\frac{dT}{dr} = -\frac{3}{4ac}\frac{\kappa\rho}{T^3}\frac{L(r)}{4\pi r^2}. \tag{13.17}$$

It should be mentioned that in the outer stellar layers, where the internal energy per unit mass is small due to the relatively low temperatures, the actual temperature gradient in a convective region may depart appreciably from the adiabatic gradient. Hence, in the outer layers a slightly superadiabatic gradient may not provide an adequate representation of the actual temperature gradient (see Chap. 14).

13.3 Case of Non-Uniform Chemical Composition

We consider now the more general case where the chemical composition varies (continuously) with radial distance. The "material" pressure P must now be considered a function not only of ρ and T, but also of mean molecular

* At the end of Sect. 14.6 an order-of-magnitude derivation of the value of $\Delta T/T$, as well as values of other convective quantities, in the deep stellar interior will be presented, which does not require a priori knowledge of the value of t.

weight μ. In this case it is necessary to go back to the general forms (13.4) and (13.5) of the condition for stability against convection. We use $\ln P$ instead of r as the independent radial variable, where P is the total pressure (which always increases inward in a star in hydrostatic equilibrium). Equation (13.4) becomes

$$\frac{\mathrm{d}(\Delta\rho)}{\mathrm{d}\ln P}<0 \tag{13.18}$$

and (13.5) becomes

$$\left(\frac{\mathrm{d}\ln\rho}{\mathrm{d}\ln P}\right)_{\delta m}<\left(\frac{\mathrm{d}\ln\rho}{\mathrm{d}\ln P}\right)_{\text{unpert}}. \tag{13.19}$$

Before expressing (13.19) in terms of temperature gradients, let us digress for a moment to consider from a physical standpoint the fate of an element of mass δm whose temperature $T_{\delta m}(P)$ abruptly increases over the temperature $T(P)$ of its surroundings by a small amount and whose density $\rho_{\delta m}(P)$ decreases (still assuming pressure equilibrium of the element with its surroundings) below the density $\rho(P)$ of its surroundings by a small amount. Just as in Sect. 13.1, the element will move outward because of the presence of the unbalanced buoyant forces. As the element moves outward, we may assume that the mean molecular weight in the element remains constant and equal to the value characteristic of the location in the star where the element originated.* Because of the assumed spatial variation of mean molecular weight, a difference between the mean molecular weight $\mu_{\delta m}$ of the element and the mean molecular weight $\mu(P)$ of its immediate surroundings develops. In the case of stability against convection, the element will eventually reach a position where $\rho_{\delta m}(P) = \rho(P)$ and the buoyancy and the gravitational forces balance each other. After a slight overshoot and possibly some oscillations about this position, the element will soon (at least for the moment) come to rest. However, because $\mu_{\delta m}\neq\mu(P)$ at this "equilibrium" position and because the "material" pressure P is, through the pressure equation of state, a function of ρ, T, and μ, it follows that $T_{\delta m}(P)\neq T(P)$, *i.e.*, although the element experiences zero net force and is (temporarily) at rest, it has a different temperature from that of its immediate surroundings.

The subsequent fate of the element depends on the value of the time, say t_d, during which it can retain its identity against diffusive mixing with its

* We ignore here and until the end of this section any possible changes in μ resulting from ionization and/or dissociation within the element; see the last paragraph in this section.

surroundings, as compared with the value of the time, say t_h, required for significant heat exchanges with its surroundings. If $t_d \ll t_h$, the element will mix with its surroundings and lose its identity. If, on the other hand, $t_d \gg t_h$, the element will exchange heat with its surroundings and tend to acquire their temperature, with a consequent change in density from that of its surroundings. This change in density will lead to further motions of the element, either inward or outward, but on a much longer time scale than that associated with the original motion of the element. If $\mu(P)$ decreases outward (as is normally the case with evolving stars, see Sects. 23.6 and 26.4) and if the equation of state is the perfect gas law (possibly modified to include black body radiation pressure), then when the element has just come to rest, we will have $T_{\delta m}(P) > T(P)$, and the element will cool down, contract, and so move inward. It is clear that the element, if it did not first dissolve through diffusive mixing with its surroundings, would eventually have to return to the level from which it originated. Hence, in the present case of μ decreasing outward, satisfaction of the condition (13.19) will insure stability not only against ordinary convection, but also against slow, "convective"-like motions having a much longer time scale than for ordinary convective motions. If $\mu(P)$ *increases* outward, then satisfaction of the condition (13.19) will, again, insure stability against ordinary convection. However, it is easy to see that the material will in this case be *unstable* against slow, "convective"-like mixing on a much longer time scale than for ordinary convection; we may refer to such slow, "convective"-like motions as "quasi-convection." This case of $\mu(P)$ increasing outward, however, does not ordinarily arise in stars during the course of slow, quasi-static evolutionary processes (disregarding effects of ionization and/or dissociation; see the end of this section).

At any rate, (13.19) is certainly the correct criterion for stability against convection as understood in the ordinary sense. We now wish to express this criterion in terms of temperature gradients.

In order to express $(\mathrm{d} \ln \rho / \mathrm{d} \ln P)_{\delta m}$ in terms of a temperature gradient, we recall that we have assumed that the mean molecular weight μ of the element does not change during its motion.* Hence, as we follow the element, the pressure change within the element is made up only of changes in density ρ and temperature T, not in μ. We write the pressure equation of state in the general form $P = P(\rho, T, \mu)$ and take the logarithmic differential, keeping μ fixed. We obtain

$$\mathrm{d} \ln P = \chi_\rho \mathrm{d} \ln \rho + \chi_T d \ln T, \tag{13.20}$$

* See the earlier footnote in this section.

where (*cf.* (9.81) and (9.82))

$$\chi_\rho \equiv \left(\frac{\partial \ln P}{\partial \ln \rho}\right)_{T,\mu}, \tag{13.21}$$

$$\chi_T \equiv \left(\frac{\partial \ln P}{\partial \ln T}\right)_{\rho,\mu}. \tag{13.22}$$

We then obtain from (13.20)

$$\left(\frac{\mathrm{d} \ln \rho}{\mathrm{d} \ln P}\right)_{\delta m} = \frac{1}{\chi_\rho} - \frac{\chi_T}{\chi_\rho}\left(\frac{\mathrm{d} \ln T}{\mathrm{d} \ln P}\right)_{\delta m}, \tag{13.23}$$

where we assume that, in view of the arbitrarily small initial temperature perturbation in the element, χ_ρ and χ_T have the same values in the element as in the unperturbed surroundings at each level.

In the case of $(\mathrm{d} \ln \rho / \mathrm{d} \ln P)_{\text{unpert}}$, we must take into account the change in μ that is experienced as one moves about in the unperturbed surroundings (because of the non-uniform composition). In this case we must have, instead of (13.20),

$$\mathrm{d} \ln P = \chi_\rho \mathrm{d} \ln \rho + \chi_T \mathrm{d} \ln T + \chi_\mu \mathrm{d} \ln \mu, \tag{13.24}$$

where

$$\chi_\mu \equiv \left(\frac{\partial \ln P}{\partial \ln \mu}\right)_{\rho,T}, \tag{13.25}$$

so that

$$\left(\frac{\mathrm{d} \ln \rho}{\mathrm{d} \ln P}\right)_{\text{unpert}} = \frac{1}{\chi_\rho} - \frac{\chi_T}{\chi_\rho}\left(\frac{\mathrm{d} \ln T}{\mathrm{d} \ln P}\right)_{\text{unpert}} - \frac{\chi_\mu}{\chi_\rho}\left(\frac{\mathrm{d} \ln \mu}{\mathrm{d} \ln P}\right). \tag{13.26}$$

Using (13.23) and (13.26) in (13.19), and replacing the subscripts "unpert" by "rad" (for "radiative"), we obtain as the (necessary and sufficient) condition for stability against convection in regions of continuously varying (in space) composition

$$\left(\frac{\mathrm{d} \ln T}{\mathrm{d} \ln P}\right)_{\text{rad}} < \left(\frac{\mathrm{d} \ln T}{\mathrm{d} \ln P}\right)_{\delta m} - \frac{\chi_\mu \, \mathrm{d} \ln \mu}{\chi_T \, \mathrm{d} \ln P}, \tag{13.27}$$

which reduces to the usual condition (see Sect. 13.1) when μ is constant in space.

If there are no (nuclear) energy sources in the regions of interest, then (13.27) may be replaced by a simpler and more useful (but less general) con-

dition. In this case we have (see Sect.13.1) $(\mathrm{d}\ln T/\mathrm{d}\ln P)_{\delta m} \geq (\mathrm{d}\ln T/\mathrm{d}\ln P)_{\mathrm{ad}}$, so that (13.27) becomes

$$\left(\frac{\mathrm{d}\ln T}{\mathrm{d}\ln P}\right)_{\mathrm{rad}} < \left(\frac{\mathrm{d}\ln T}{\mathrm{d}\ln P}\right)_{\mathrm{ad}} - \frac{\chi_\mu \,\mathrm{d}\ln\mu}{\chi_T \,\mathrm{d}\ln P} \tag{13.28a}$$

or, in terms of $(\Gamma_2-1)/\Gamma_2 \equiv (\mathrm{d}\ln T/\mathrm{d}\ln P)_{\mathrm{ad}}$ (*cf.* Sect. 9.14),

$$\left(\frac{\mathrm{d}\ln T}{\mathrm{d}\ln P}\right)_{\mathrm{rad}} < \frac{\Gamma_2-1}{\Gamma_2} - \frac{\chi_\mu \,\mathrm{d}\ln\mu}{\chi_T \,\mathrm{d}\ln P}, \tag{13.28b}$$

which differs from (13.14) only in the presence of the last term. Condition (13.28b) is sufficient (but not necessary) in the same sense as is condition (13.10).

Consider, for example, an equation of state appropriate to a mixture of an ideal gas and black body radiation:

$$P = \frac{\mathscr{R}}{\mu}\rho T + \frac{1}{3}aT^4, \tag{13.29}$$

where all symbols have their usual meaning (see Chap. 1). We readily obtain from (13.29) the results $\chi_\mu = -\beta$ and $\chi_T = 4-3\beta$, where β is the ratio of gas to total (gas plus radiation) pressure. Hence (13.28b) becomes in this case

$$\left(\frac{\mathrm{d}\ln T}{\mathrm{d}\ln P}\right)_{\mathrm{rad}} < \frac{\Gamma_2-1}{\Gamma_2} + \frac{\beta}{4-3\beta}\frac{\mathrm{d}\ln\mu}{\mathrm{d}\ln P}, \tag{13.30}$$

an equation which was derived by Sakashita, Ono, and Hayashi [Sa59] and which has been used in a number of investigations of stellar evolution (for example, Stothers [St66c, 66d] and Hofmeister, Kippenhahn, and Weigert [Ho64]). According to this equation, a mean molecular weight μ which *increases* (continuously) inward (the usual situation in evolving stars, *cf.* Sects. 23.6 and 26.4) tends to *stabilize* the corresponding regions against convection, since $\mathrm{d}\ln\mu/\mathrm{d}\ln P > 0$ in this case.

Consider now the case where the chemical composition is constant in space but where μ varies with position as a result of ionization and/or dissociation. At least in a static star chemical equilibrium (*cf.* Sects. 9.12 and 13.1) may be assumed to obtain in the unperturbed surroundings, and μ for the surroundings is then a function of ρ and T. The effects of the variable μ are then absorbed into the χ_ρ and χ_T in (13.24). If instantaneous chemical equilibrium is assumed to obtain in the perturbed element, then μ for the element is also a function of ρ and T, and effects of the variable μ do not have to be considered explicitly. Hence, in this case of instantaneous chemical

equilibrium the usual condition (13.10) or (13.28b) (without the last term) for stability against convection obtains, even in regions where spatial variations in μ are brought about by ionization and/or dissociation.

If, on the other hand, instantaneous chemical equilibrium is *not* assumed to obtain in the perturbed element, then a term similar to the last term in (13.24) must be added to the right side of (13.20) for the element. Also, the values of χ_ρ and χ_T for the element could no longer be assumed to be the same as for the unperturbed surroundings at the given level, even if the initial temperature perturbation of the element were infinitely small. It is clear that a much more complicated expression than (13.28b) for stability against convection would result for this last case, which is not ordinarily of much interest anyway.

13.4 General Discussion of Stability Against Convection

In view of the above considerations, we see that, in general, the onset of convection at some point in a star is governed by the steepness of the radiative temperature gradient relative to the adiabatic gradient at that point, corrected, if necessary, for the presence of a non-uniform chemical composition. In the deep interior of a star, where Γ_2 may be essentially constant or slowly varying (and greater than 4/3), the steepness of the adiabatic gradient (with respect to pressure) is essentially constant, and therefore the onset of convection will be determined essentially by the steepness of the local *radiative* gradient (with respect to pressure). (We are here and in the next few paragraphs assuming that the correction term for the effects of a non-uniform composition is negligible or zero.) The radiative gradient with respect to pressure may be obtained by combining the equation of radiative transfer,

$$-\frac{1}{T}\frac{dT}{dr} = \frac{3}{4ac}\frac{\kappa\rho}{T^4}\frac{L(r)}{4\pi r^2},$$

and the equation of hydrostatic equilibrium,

$$-\frac{1}{P}\frac{dP}{dr} = G\frac{M(r)}{r^2}\frac{\rho}{P},$$

to obtain

$$\left(\frac{d\ln T}{d\ln P}\right)_{\text{rad}} = \frac{3}{16\pi acG}\cdot\frac{\kappa P}{T^4}\cdot\frac{L(r)}{M(r)}. \qquad (13.30')$$

Since the ratio P/T^4 is generally (but not always) a slowly varying function of position in a star in radiative equilibrium (this statement will be justified in

Sect. 23.2), it is seen that the onset of convection in the deep interior is determined primarily by the values of κ and $L(r)/M(r)$. A large value of κ means that a steeper temperature gradient is required for a given flux of energy to be transported by radiation than a small value. In the deep interior of an actual star κ generally decreases toward the center; this effect then serves to *hinder* the onset of convection. The factor $L(r)/M(r)$, however, may tend to *favor* the onset of convection as r decreases. If the energy sources are strongly concentrated toward the center, then $L(r)$ will remain equal to L until r becomes very small. But the interior mass $M(r)$ will steadily decrease with decreasing r (as r^3 at points sufficiently close to the center). Thus, as one descends toward the center, the ratio $L(r)/M(r)$ will probably become sufficiently large at some point to initiative convection, and a convective core will result. Hence stars with highly concentrated energy sources will almost certainly have convective cores. If the energy sources are more-or-less uniformly distributed, however, then $L(r)$ may decrease with decreasing r almost as rapidly as $M(r)$, so that the ratio $L(r)/M(r)$ may increase only very slowly (or perhaps remain constant) with decreasing r. The radiative gradient may therefore never become steep enough to initiate convection. Stars with not-so-highly concentrated energy sources therefore may or may not have convective cores.

In the outer layers of a star, now, where $L(r) \simeq L$ and $M(r) \simeq M$, the factor $L(r)/M(r)$ no longer plays any direct role in determining the onset of convection. In these regions, however, the steepness of the *adiabatic* temperature gradient is not constant, but is very sensitive to the state of ionization (or dissociation) of the dominant constituents (such as hydrogen and helium) of the stellar envelope. In a region of partial ionization of hydrogen or helium Γ_2 will drop to a value near unity, so that the adiabatic gradient will become relatively small, as may be seen from the relation

$$\frac{\Gamma_2 - 1}{\Gamma_2} = \left(\frac{\mathrm{d} \ln T}{\mathrm{d} \ln P}\right)_{\mathrm{ad}}$$

(see Sect. 9.18). If the adiabatic gradient becomes flatter than the radiative gradient (as will almost certainly be the case for hydrogen ionization, and probably also for one or both stages of helium ionization), then a convective zone will be initiated, in which convection *may or may not* carry an appreciable fraction of the total flux (see Chap. 14).

The role played by κ in the outer layers is the same as in the deep interior; *i.e.*, an inward *increase* in the value of κ will favor *convective* transfer, and an inward *decrease* in the value of κ will favor *radiative* transfer. Moreover, a small value of Γ_2 will also tend to favor convective transfer. It might be added that both of these factors that favor convective transfer are strongly

operative in the region of hydrogen ionization. Therefore, probably all stars that contain a large abundance of hydrogen (which includes practically all stars), and which are not so hot that hydrogen is completely ionized even at the photosphere, have hydrogen convection zones near their surfaces. The importance of such a hydrogen convection zone in influencing the *structure* of the star, however, depends on the fraction of the flux carried by convection and on the depth of the convection zone. These effects will be discussed in later chapters.

As was pointed out in Sect. 13.3, a non-uniform chemical composition for which the mean molecular weight increases inward continuously (*cf.* Sects. 23.6 and 26.4) tends to stabilize the corresponding regions against convection.

Another useful and enlightening viewpoint is obtained by relating the question of convective stability to the question of whether the local specific entropy S (entropy per unit mass, say) of the unperturbed material is increasing or decreasing inward in the star. This viewpoint is useful at least in the case of uniform chemical composition and in case no irreversible processes (such as nuclear reactions, *cf.* Sects. 3.1, 9.12, and the paragraph following (17.75″); or viscous heating) are occurring at the point under consideration. We shall see that in these cases the specific entropy in a star in hydrostatic equilibrium always *decreases inward* in convectively *stable* regions.

To show this, we regard total pressure P and temperature T as the independent thermodynamic variables, and we consider an infinitesimal *reversible* process, in which S changes by the amount $\mathrm{d}S$. Using the relation (valid for reversible processes, *cf.* Chap. 9) $T\mathrm{d}S = \mathrm{d}E + P\mathrm{d}(1/\rho)$ (E = internal energy per unit mass), assuming that E is a function only of ρ and T, and using some of the identities established in Sects. 9.11 and 9.14, we obtain

$$\mathrm{d}S = c_P(\nabla - \nabla_{\mathrm{ad}})\,\mathrm{d}\ln P, \tag{13.31}$$

where c_P is the specific heat per unit mass at constant pressure, and where we have used the abbreviations (*cf.* Chap. 14)

$$\nabla \equiv (\mathrm{d}\ln T/\mathrm{d}\ln P), \tag{13.32}$$

$$\nabla_{\mathrm{ad}} \equiv (\Gamma_2 - 1)/\Gamma_2 = (\mathrm{d}\ln T/\mathrm{d}\ln P)_{\mathrm{ad}}. \tag{13.33}$$

In the present application of (13.31) ∇ may be taken as the actual temperature gradient (with respect to pressure) of the unperturbed material at the point of interest, ∇_{ad} as the corresponding adiabatic gradient (*cf.* (13.12)), and $\mathrm{d}\ln P$ as an increment in $\ln P$ of the unperturbed material corresponding to an inward or outward displacement of the point of observation (see remarks made in connection with (9.9′) and (9.9″)). Since

P always increases inward in a star in hydrostatic equilibrium, it follows from (13.31) that S in such a star always *decreases* inward in convectively *stable* regions ($\nabla < \nabla_{ad}$, *cf.* (13.10)) under the assumed conditions, Q.E.D. In convectively *unstable* regions ($\nabla > \nabla_{ad}$), S would clearly *increase* inward in a star in hydrostatic equilibrium if convective motions were artificially suppressed. Hence, in the present case of uniform composition where no irreversible processes are occurring and for a star in hydrostatic equilibrium, the local behavior of the specific entropy S (assuming convective motions to be suppressed) can also be used as a criterion for convective stability: The material is convectively *stable or unstable* according as S (in the absence of convective motions) *decreases* or *increases* inward, respectively.

Note that S is *constant* with depth in the absence of viscous heating in a convective zone in adiabatic equilibrium ($\nabla = \nabla_{ad}$). As we have pointed out earlier in this chapter, the actual gradient in a real convection zone must be slightly *superadiabatic* (∇ only slightly greater than ∇_{ad}), if no nuclear reactions are occurring, so that under these conditions S *increases inward* slowly (perhaps negligibly so, *cf.* Chap.20) in an actual convection zone.

CHAPTER 14

Mixing Length Theory of Convection

Here we shall describe the so-called "mixing length" theory of convection, and shall follow in part the work of Vitense [Vi53] and Böhm-Vitense [Bö58]. Our detailed derivations, however, will not necessarily be the same as those presented in those works. The mixing length theory replaces, conceptually, the complicated situation in an actual convection zone, consisting of convective elements (or eddies) of assorted sizes, shapes, velocities, and lifetimes, by a group of "average" convective elements, all of which have the same physical properties at a given radial distance r from the stellar center. Each convective element is assumed to travel, on the average, through a distance Λ, the *mixing length* (or *mean free path*), before mixing with the surrounding matter and thereby losing its identity. Also, the convective elements are all assumed to be of the same characteristic dimension in all directions, but the precise shape is not specified. This characteristic dimension of the convective elements is assumed to be equal to Λ, the mixing length, in general a function of r. In addition, the convective elements are all assumed to have, at a given value of r, the same speed, $\bar{v}$, which is supposed to approximate the average speed of all the actual convective elements at this radial distance. Thus the mean life of a convective element, or the average time it retains its individuality, is given by $t = \Lambda/\bar{v}$.

It will be clear even at this stage that the mixing length theory represents an extreme simplification of the actual physical process of convection. One does not therefore expect quantitative results derived on the basis of this theory to have high accuracy or reliability. One of the principal sources of uncertainty in the theory is the value to be used for the mixing length itself. Arguments could also be raised against some of the factors of two, etc., which (as we shall see) are introduced in the course of development of the theory. In view of the basic crudity of the theory, the exact values adopted for such factors are hardly significant. The theory does, however, yield a *qualitatively*

reasonable picture of convective heat transfer. Although various attempts are being made at present to develop a more refined theory of convection (see, for example, Spiegel and Veronis [Sp60]; Ledoux, Schwarzschild, and Spiegel [Le61]; and Spiegel [Sp63, 64, 65], a really satisfactory theory does not yet exist. It is therefore simply for want of a better theory that astrophysicists commonly use the mixing length theory in the construction of stellar models in spite of its drawbacks.* Moreover, stellar models constructed on the basis of the mixing length theory do not show any obvious contradictions with observations. Let it be emphasized again, however, that users of this theory should be well aware of the possible quantitive unreliability of results obtained by its use.

We consider a region in a star in which the matter is convectively unstable, in the sense implied in Chap. 13. We assume complete pressure equilibrium; this means that all elements of matter at a given distance r from the center of the star have precisely the same value of the pressure. (This assumption will be discussed further in Sect. 14.3.) Thus the pressure exerted by a convective element on the surrounding matter at a given point is exactly equal to the pressure which the surrounding matter exerts on the convective element at that point. This equality is assumed to be maintained continuously as the element moves through its surroundings.

Consider now matter at a given level which is a distance r from the center of the star. The matter at such a level in a convection zone will not have a uniform temperature; some elements will be hotter than others. We can, however, define an *average* temperature, $T(r)$, at each such level, which is just the average temperature of all the matter which is (instantaneously) at that level. Matter whose temperature is equal to this average temperature will, on the average, be essentially stationary. Because we have assumed pressure equilibrium, matter whose temperature *exceeds* this average temperature will be less dense than the stationary matter and will therefore tend to rise. Similarly, matter which is cooler than the average of the surrounding matter will be more dense than the stationary matter and will therefore tend to sink. The cool, sinking elements will eventually be warmed by their surroundings, thereby absorbing heat from the lower lying regions. The warm, rising elements, on the other hand, will eventually lose their excess heat to the regions further out. The total effect is thus a net upward transfer of heat energy.

We shall also assume hydrostatic equilibrium at each level. This implies that the star is in a steady state, at least over times large compared to the

* Recently, a "non-mixing length" theory of convection has been proposed by Faulkner, Griffiths, and Hoyle (Fa65). It is not yet clear whether or not this theory is superior to conventional mixing length theories.

"free-fall" time, *cf.* Chaps. 0, 1, and 27. The mass of rising matter at each level is thus equal to the mass of sinking matter at that level, at each instant of time.

We shall further neglect turbulent pressure, which arises from the transfer of momentum by the moving masses. An argument will be presented in Sect. 14.6 which shows that as long as the convection is not supersonic, the turbulent pressure is not likely to exceed about one-half the sum of the gas pressure and radiation pressure. Although the turbulent pressure may not always be negligible, its neglect is therefore not expected to lead to errors much larger than a factor of the order of two in the pressure stratification anywhere in a convection zone having no supersonic convective velocities. On the other hand, the mixing length theory itself has many uncertainties in it, each of which might easily introduce a significant error in the results. If the convective velocities should be supersonic (in which case the turbulent pressure might well be important), the basic assumptions of the mixing length theory are violated (*cf.* Sects. 14.3 and 14.9) and the theory in its present form is then not really applicable anyway. We feel, accordingly, that the errors arising from neglect of the turbulent pressure may well be smaller than the "resolving power" of the mixing length theory itself in its present form.

Calculations with the mixing length theory have, however, been carried out with the turbulent pressure included, by Henyey, Vardya, and Bodenheimer [He65], in the manner suggested by Böhm-Vitense [Bö58]. The interested reader is referred to these papers. In this first paper the effect on the structure of convective envelope models of adjusting the values of the various parameters appearing in the mixing length theory is examined.

In Sect. 14.1 we shall introduce and define the four gradients originally defined by Böhm-Vitense [Vi53]. In Sects. 14.2–14.4 expressions will be derived for the convective flux, the average speed of convecting elements, and the total (radiative plus convective) flux. The efficiency of convection (related to the exchange of heat by a convecting element with its surroundings) will be discussed in Sect. 14.5. In Sect. 14.6 we shall derive expressions for and numerical estimates of upper limits to values of various quantities related to the mixing length theory. Finally, in Sects. 14.7–14.9 the solution of the equations of the mixing length theory for several situations of interest for stellar interiors will be discussed.

14.1 The Four Gradients

Consider an element of matter which begins to rise at the level whose distance from the center of the star is r. We shall neglect the initial increment of temperature which started the element convecting upward at r. Thus we

assume that $T'(r) = T(r)$, where $T'(r)$ and $T(r)$ are, respectively, the temperature of the rising element and the average temperature of the surrounding matter, both evaluated at the radial distance r. In a region of convective instability the temperature, $T'(r)$, of the rising element will follow a curve

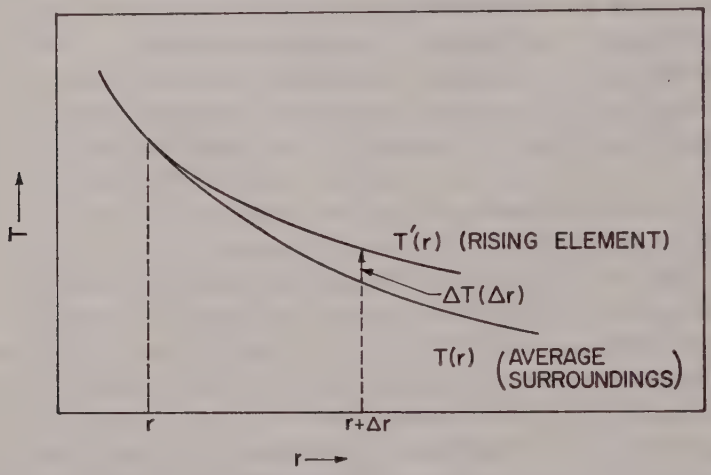


Fig. 14.1 Temperature of a rising convective element relative to that of its surroundings.

which is less steep than the curve, $T(r)$, which represents the average temperature of all the matter at each level (see Fig. 14.1). When the element has risen through a distance Δr its excess temperature over the average temperature of the surrounding matter at this point will be given (neglecting terms of higher order in Δr than the first*) by

$$\Delta T(\Delta r) = T'(r+\Delta r) - T(r+\Delta r) = \Delta r[(\mathrm{d}T'/\mathrm{d}r) - (\mathrm{d}T/\mathrm{d}r)]$$

or

$$\Delta T(\Delta r) = \Delta r \Delta \nabla T, \tag{14.1}$$

where, in terms of Schwarzschild's notation [Sc58],

$$\Delta \nabla T \equiv [(-\mathrm{d}T/\mathrm{d}r) - (-\mathrm{d}T'/\mathrm{d}r)]. \tag{14.2}$$

* Note that this "linearization" is justified only as long as T does not change drastically within the distance Δr, which will later be replaced by Λ, the mixing length. (More specifically, all terms in the Taylor expansion for $\Delta T(\Delta r)$ must be small compared to the first.) Drastic spatial changes in T do in fact occur in the region of hydrogen ionization in some stars (*cf.* Sect. 20.4 and Faulkner, Griffiths, and Hoyle [Fa65]).

Assuming that $T' \simeq T$ for all radial distances between r and $r+\Delta r$,* we can also write

$$\Delta T(\Delta r) = \Delta r T[(-\mathrm{d}\ln T/\mathrm{d}r)-(-\mathrm{d}\ln T'/\mathrm{d}r)]. \tag{14.3}$$

We now define the *total pressure scale height*, λ_P, at radial distance r:

$$-\frac{\mathrm{d}\ln P}{\mathrm{d}r} \equiv \frac{1}{\lambda_P} = \frac{\rho g}{P}, \tag{14.4}$$

where the second equality in (14.4) follows from the equation of hydrostatic equilibrium. Here P, ρ, and g denote, respectively, the total (gas plus radiation) pressure, density averaged over the entire surface of radius r, and gravitational acceleration ($= GM_r/r^2$, M_r = interior mass) at the level under consideration. The scale height, λ_P, is a measure of the distance over which the pressure changes by an appreciable fraction of itself. If λ_P were constant with r, then we would have $P \propto \exp(-\Delta r/\lambda_P)$, so that λ_P would be the "e-folding distance", or the distance over which the pressure would change by the factor $e^{\pm 1}$. Since we have assumed pressure equilibrium at each level, we can also write (14.3) in the form

$$\Delta T(\Delta r) = -\frac{\mathrm{d}\ln P}{\mathrm{d}r}\Delta r T\left[\left(\frac{\mathrm{d}\ln T}{\mathrm{d}\ln P}\right)-\left(\frac{\mathrm{d}\ln T'}{\mathrm{d}\ln P}\right)\right]$$

or

$$\Delta T(\Delta r) = \Delta r(T/\lambda_P)(\nabla-\nabla'), \tag{14.5}$$

where

$$\nabla \equiv \frac{\mathrm{d}\ln T}{\mathrm{d}\ln P}\left(=\frac{1}{n_e+1}\right), \tag{14.6}$$

$$\nabla' \equiv \frac{\mathrm{d}\ln T'}{\mathrm{d}\ln P}, \tag{14.7}$$

n_e being the effective polytropic index, defined in Chap. 12. Here ∇ represents the average temperature gradient (with respect to pressure) of all the matter at a given level and ∇' represents the temperature gradient (again with respect to pressure) of a rising (or falling) element of matter. In general, the value of ∇' depends, among other things, on the rate at which the moving element is exchanging heat with its surroundings. In Sects. 14.7 and 14.8 we derive approximate formulae which can be used to estimate the value of ∇'; we shall assume for the moment, however, that the value of ∇' is known. In many cases (especially in the deep interior), a good approximation to ∇' is

$$\nabla' = \nabla_{\mathrm{ad}}, \tag{14.8}$$

* See footnote p. 284.

where

$$\nabla_{ad} \equiv \left(\frac{d \ln T}{d \ln P}\right)_{ad} = \frac{\Gamma_2 - 1}{\Gamma_2} \tag{14.9}$$

(see Sect. 9.14a). Thus ∇_{ad} is the temperature gradient (with respect to pressure) of an element of matter that moves *adiabatically*.

We have now defined three of the four gradients. The fourth comes about as follows. Even in a convection zone some of the energy will be carried by radiation; this will be true whenever a temperature gradient exists. Again letting T denote the average temperature of all the matter at a given level, we will have (assuming the "diffusion" approximation, see Sect. 6.4, to be valid)

$$d\left(\frac{1}{3}aT^4\right)/dr = -\frac{1}{c}\kappa\rho F_r(r),$$

where $F_r(r) \equiv F_r$ is the flux carried by radiation at radial distance r (and averaged over all the matter at radial distance r).* Thus

$$F_r = -\frac{4ac}{3}\frac{T^3}{\kappa\rho}\frac{dT}{dr}, \tag{14.10}$$

which can also be written in the form

$$F_r = \left(-\frac{d \ln P}{dr}\right)\frac{4ac}{3}\frac{T^4}{\kappa\rho}\left(\frac{d \ln T}{d \ln P}\right)$$

or

$$F_r = \frac{4ac}{3}\frac{T^4}{\kappa\rho}\frac{1}{\lambda_P}\nabla = \frac{4ac}{3}\frac{T^4 g}{\kappa P}\nabla. \tag{14.11}$$

We now define a *fictitious* (but computable) radiative gradient by the relation

$$F = F_c + F_r \equiv \frac{4ac}{3}\frac{T^4}{\kappa\rho}\frac{1}{\lambda_P}\nabla_r, \tag{14.12}$$

where F is the total flux and F_c and F_r are the convective and radiative fluxes, respectively. Thus ∇_r is the gradient which would exist at a given point if all the energy were transported by radiation at that point (*i.e.*, if all convective motions were suppressed). All quantities in (14.12) are assumed to be averages over all matter at radial distance r.

* When we say "radiative" flux we shall actually mean "radiative plus conductive" flux. As we shall see in Sect. 16.7, thermal conduction can be formally taken into account in the "diffusion" formula (14.10) by a suitable re-definition of κ.

In a convection zone in which no (nuclear) energy is being produced, we will have the following inequalities among the four gradients:

$$\nabla_r > \nabla > \nabla' > \nabla_{ad} \tag{14.13}$$

(see Fig. 14.2). It may be noted that in a region in which energy is being released by nuclear sources, we could have $\nabla' < \nabla_{ad}$; this effect would tend to

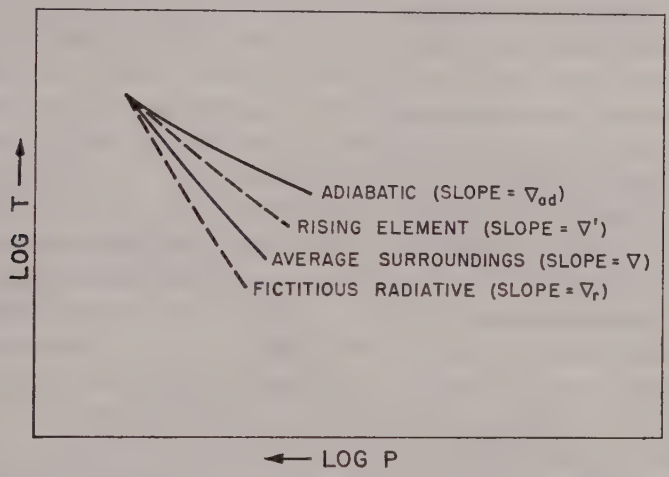


Fig. 14.2 The four gradients in a convective region which contains no (nuclear) energy sources (*cf.* (14.13)).

enhance convective instability (*cf.* Chap. 13). Nevertheless, the first two inequalities in (14.13) are valid independently of whether or not nuclear energy sources are present.

14.2 Convective Flux

We suppose that a rising element retains its identity while it moves through a distance Δr, after which it mixes with its surroundings and thereby gives up its excess heat energy. Thus the heat transferred per unit area per unit time by upward moving elements which move through the distance Δr before dissolving is

$$F_c(\Delta r) = \frac{1}{2}\rho v c_P \Delta r[(-dT/dr)-(-dT'/dr)] = \frac{1}{2}\rho v c_P \Delta r \Delta\nabla T, \tag{14.14}$$

where ρ and v are the density and speed, respectively, of the rising elements, averaged over the distance Δr, and c_P is the average value (over the distance

Δr) of the specific heat per unit mass at constant *pressure*. We must use c_P, rather than c_V, here since, in accordance with our assumption of pressure equilibrium, the heat exchange with the surrounding matter occurs at constant pressure at each level (the pressure is different at different levels, but the heat exchange is supposed always to take place with matter which is at the same instantaneous level as that of the rising element). The factor (1/2) in (14.14) comes from the fact that at each level approximately one-half of the matter is rising and one-half is descending. A similar expression would have resulted had we considered only *downward* moving elements (cooler than the average surroundings), since such elements will eventually absorb heat, become warmer than their surroundings, and will then appear as rising elements. The *total* convective flux is then given by (14.14), but with the factor (1/2) replaced by unity. Note that $F_c(\Delta r)$ is proportional to Δr.

We must now average $F_c(\Delta r)$ over all possible values of Δr. We effect this averaging by replacing Δr in (14.14) by the product of a numerical factor and Λ, the mixing length, *i.e.*, the mean free path of a convecting element. We take the value of this numerical factor such that we obtain, ultimately, numerical agreement with Böhm-Vitense's [Bö58] results. It is clear that the other factors (ρ, v, c_P, and $\Delta\nabla T$) on the right side of (14.14) must, strictly speaking, be considered as averaged over the vertical distance Λ as well as over the entire surface of radius r. The *average* (over the distance Λ) convective flux, which we associate with the actual convective flux, is then given by the relation

$$F_c = \frac{1}{2}\rho\bar{v}c_P\Lambda\Delta\nabla T, \tag{14.16}$$

where $\bar{v}$ is the speed of upward and downward moving elements, averaged over the distance Λ and over the entire surface defined by the radial distance r under consideration.* Since

$$(\nabla - \nabla') = \frac{\lambda_P}{T}\Delta\nabla T = \frac{P}{\rho g T}\Delta\nabla T, \tag{14.17}$$

we can also write for the average convective flux

$$F_c = \frac{1}{2}\rho\bar{v}c_P T\frac{\Lambda}{\lambda_P}(\nabla - \nabla'). \tag{14.18}$$

* The reader should be cautioned against taking this "derivation" and the picture on which it is based too literally. Indeed, the final formula (14.16) could have been written down immediately on the basis of dimensional arguments, with the factor (1/2) replaced by some arbitrary numerical factor of order unity.

14.3 Average Speed of Convecting Elements

Before (14.16) and (14.18) can be used, in practice, an expression must be obtained for the speed, $\bar{v}$, which an average convecting element will have at radial distance r. We obtain the expression by means of the following dynamical considerations.

Suppose that $\Delta\rho$ represents the excess density of an element of matter over the average of its surroundings at a given point. The net force acting on this element can then be obtained from the equation of motion for the element:

$$\ddot{r} = -g - \frac{1}{\rho}\frac{\partial P}{\partial r},$$

where g is the local gravitational acceleration. We now write $\rho = \rho_0(1+\Delta\rho/\rho_0)$, where the subscript "0" denotes values at the point r in the "equilibrium" state (*i.e.*, no net forces present) of the convecting element, *i.e.*, $\ddot{r}_0 = -g_0 - (\partial P/\partial r)_0/\rho_0 = 0$ when $\rho = \rho_0$. Expanding ρ to the first order in $\Delta\rho/\rho_0$, we have

$$\begin{aligned}\ddot{r} &= -g_0 - \frac{1}{\rho_0}\left(\frac{\partial P}{\partial r}\right)_0\left[1 - \frac{\Delta\rho}{\rho_0}\right] \\ &= -g_0 - \frac{1}{\rho_0}\left(\frac{\partial P}{\partial r}\right)_0 + \frac{\Delta\rho}{\rho_0}\cdot\frac{1}{\rho_0}\left(\frac{\partial P}{\partial r}\right)_0 \\ &= -g_0\Delta\rho/\rho_0,\end{aligned}$$

making use of the equation of motion in the "equilibrium" state of the convecting element. Dropping subscripts, we then have that the net force (the buoyant force minus the gravitational force) per unit volume acting on the element is

$$f_B = -g\Delta\rho. \tag{14.19}$$

We suppose that the element starts moving with zero velocity at r and is accelerated upward through a distance Δr, where it then mixes with its surroundings. We assume that the excess density is virtually zero at r and increases linearly with Δr until at $r+\Delta r$ it has the value $\Delta\rho(\Delta r)$. The net force per unit volume at $r+\Delta r$ then has the value

$$f_B(\Delta r) = -g\Delta\rho(\Delta r),$$

where we neglect the variation of g over the distance Δr. The work done on unit volume in moving it through the distance Δr is then

$$W(\Delta r) = \int_0^{\Delta r} f_B((\Delta r)')\mathrm{d}(\Delta r)' = -g\int_0^{\Delta r}\Delta\rho((\Delta r)')\mathrm{d}(\Delta r)' = -\frac{1}{2}g\Delta\rho(\Delta r)\Delta r, \tag{14.20}$$

since $\Delta\rho$ increases linearly with $(\Delta r)'$. Note that $W(\Delta r)$ increases *quadratically* with Δr. We must now average $W(\Delta r)$ over all possible values of Δr. Just as was done in the steps immediately preceding (14.16), we effect this averaging by replacing $W(\Delta r)$ by the product of a numerical factor and $W(\Lambda)$, the value of $W(\Delta r)$ obtained by replacing Δr by Λ, the mixing length. We set this numerical factor equal to $(1/4)$, so as to obtain numerical agreement with Böhm-Vitense's [Bö58] work. (A value less than unity for this numerical factor may be plausible because of the quadratic dependence of $W(\Delta r)$ on Δr.) We obtain* for the average of $W(\Delta r)$ over all values of Δr

$$\overline{W}(\Lambda) = \frac{1}{4} W(\Lambda) = -\frac{1}{8} g \Delta\rho(\Lambda)\Lambda . \tag{14.21}$$

If there were no dissipative forces (friction), no heat loss, no transfer of kinetic energy to the surrounding matter (via "pushing aside" of gas masses which are in the path of the moving element), etc., then this average work $\overline{W}(\Lambda)$ done on unit volume of matter would all be transformed into kinetic energy of the moving element itself (neglecting the difference in gravitational potential energy between r and $r+\Lambda$). The average (over a distance Λ) kinetic energy imparted to unit volume of the element would then be $\overline{1/2\rho v^2} = \overline{W}(\Lambda)$. We suppose, however, that only about one-half of this work $\overline{W}(\Lambda)$ appears as kinetic energy of the moving elements; as we shall see, the final result is not very sensitive to the value we adopt for this fraction. (We are justified in assuming that an appreciable fraction of this work done on the element appears as kinetic energy, essentially because of the large sizes of the convective elements. Their large volumes imply that the buoyant forces will be much greater than the retarding viscous forces, and therefore that these latter forces will not be of great importance. Alternatively, we may say that the large scale of the convective elements implies a very large value of the *Reynolds number*, defined as

$$Re \equiv v D \rho / \eta , \tag{14.22}$$

where v is a typical velocity of mass motion, D is a typical linear dimension associated with the mass motions, and η is the coefficient of viscosity. If $D = \Lambda$, then the large values of Λ (typically of the order of or greater than a few hundred kilometers, in stars) normally cause Re to be large compared to unity, which implies that the convection is *turbulent*. Most of the retarding force acting on the moving element then arises from its having to "push

* The reader should again be warned against taking this "derivation" too literally; see footnote in Sect. 14.2.

aside" other (stationary or slower-moving) masses, thereby imparting some of its kinetic energy to them.) Thus we have

$$\overline{\frac{1}{2}\rho v^2} \simeq \frac{1}{2}\rho \bar{v}^2 = \frac{1}{2}\overline{W}(\Lambda) = -\frac{1}{16} g \Delta\rho(\Lambda)\Lambda ,$$

whence

$$\bar{v}^2 = -\frac{1}{8} g(\Delta\rho(\Lambda)/\rho)\Lambda , \tag{14.23}$$

where $\bar{v}$ is the average speed of convecting elements at the level where the mixing length is Λ. This $\bar{v}$ may be identified, to adequate accuracy, with the $\bar{v}$ appearing in (14.16) for the convective flux.

We now relate $\Delta\rho$ and ΔT by making use of the equation of state of the stellar material. This equation may be written in the general form $\rho = \rho(\mu,P,T)$, where μ is the mean molecular weight. Thus, keeping P constant at a given level, in keeping with our assumption of pressure equilibrium, we have

$$\Delta \ln \rho = -Q \Delta \ln T ,$$

where

$$-Q \equiv \left(\frac{\partial \ln \rho}{\partial \ln \mu}\right)_{P,T} \left(\frac{\partial \ln \mu}{\partial \ln T}\right)_P + \left(\frac{\partial \ln \rho}{\partial \ln T}\right)_{\mu,P} . \tag{14.24}$$

For a perfect gas equation of state, $P=(\mathscr{R}/\mu)\rho T$, we have $(\partial \ln \rho/\partial \ln \mu)_{P,T} = 1$, $(\partial \ln \rho/\partial \ln T)_{\mu,P} = -1$, whence

$$Q = 1-(\partial \ln \mu/\partial \ln T)_P . \tag{14.24a}$$

It is easy to show that in the case of a mixture of a perfect gas and black body radiation, we have

$$Q = \frac{4-3\beta}{\beta} - \left(\frac{\partial \ln \mu}{\partial \ln T}\right)_P , \tag{14.24b}$$

where β is the ratio of gas pressure to total pressure. Thus we have, in general,

$$\Delta\rho(\Lambda)/\rho = -Q\Delta T(\Lambda)/T , \tag{14.25}$$

where $Q \simeq 1$ and $Q=1$ exactly if μ is constant and if the perfect gas equation of state is obeyed.* Also, $\Delta\rho(\Lambda)$ and $\Delta T(\Lambda)$ are the excess density and temperature at $r+\Lambda$. Thus we now have

$$\bar{v}^2 = \frac{1}{8} gQ(\Delta T(\Lambda)/T)\Lambda . \tag{14.26}$$

* Note, however, that Q becomes very large in cases where $\beta \to 0$, *i.e.*, where radiation pressure is large compared to gas pressure. Eq. (27.28) is another general formula for Q.

Taking $\Delta T = 0$ at r, we have

$$\Delta T(\Lambda) = \Lambda \Delta \nabla T = \Lambda (T/\lambda_P)(\nabla - \nabla'),$$

whence

$$\bar{v}^2 = \frac{1}{8} g Q (\Lambda^2/T) \Delta \nabla T \tag{14.27}$$

or

$$\bar{v} = (2\sqrt{2})^{-1} g^{1/2} Q^{1/2} (\Lambda / T^{1/2}) (\Delta \nabla T)^{1/2} \tag{14.28a}$$

$$= (2\sqrt{2})^{-1} g Q^{1/2} (\rho/P)^{1/2} \Lambda (\nabla - \nabla')^{1/2}. \tag{14.28b}$$

We may also express $\bar{v}$ in terms of

$$v_S = \sqrt{(\partial P/\partial \rho)_{\text{ad}}} = \sqrt{\Gamma_1 P/\rho}, \tag{14.29}$$

the Laplacian (or adiabatic) sound velocity*, and λ_P, the total pressure scale height, given for a star in hydrostatic equilibrium by (14.4). We obtain

$$\frac{\bar{v}}{v_S} = \frac{(\Lambda/\lambda_P) Q^{1/2}}{2\sqrt{2}\Gamma_1^{1/2}} \left(\frac{\lambda_P}{T}\right)^{1/2} (\Delta \nabla T)^{1/2} \tag{14.30a}$$

$$= \frac{(\Lambda/\lambda_P) Q^{1/2}}{2\sqrt{2}\Gamma_1^{1/2}} (\nabla - \nabla')^{1/2}, \tag{14.30b}$$

where we note that the factor multiplying $(\nabla - \nabla')^{(1/2)}$ is of order unity if $\Lambda \sim \lambda_P$ (see Sect. 14.6).

* Equation (14.29) for v_S is valid only in the *non-relativistic* limit. If v_S should turn out to be near c (velocity of light in vacuo) in value, or if the energy density (exclusive of the rest-mass energy density ρc^2) should be comparable to or larger than ρc^2, then, according to Landau and Lifshitz [La 59, p. 502], ρ in the middle term of (14.29) should be replaced by u/c^2, where u is the total energy density, including the rest-mass energy density ρc^2; the equation will then be relativistically correct. Indeed, it can be shown that the general (relativistically correct) expression for v_S is the following:

$$v_S = c \sqrt{\Gamma_1 \left(1 + \frac{u}{P}\right)^{-1}}.$$

In the extreme relativistic case we have $u \to 3P$ and $\Gamma_1 \to (4/3)$ (see Sect. 26.5a), so that $v_S \to c/\sqrt{3}$. (Note that this case also applies to black body radiation with no matter present; see Sect. 9.16.) In the non-relativistic case we have $u \to \rho c^2 \gg P$, so that $v_S \to \sqrt{\Gamma_1 P/\rho}$, which is (14.29).

It should be mentioned that the present analysis would not be physically valid if $\bar{v}$ should ever exceed the local (Laplacian) sound velocity v_S. If this should happen, the assumption of pressure equilibrium between a convecting element and its surroundings would not be a very realistic condition. The reason for this is that the time t_P required for pressure equilibrium to become established in the element is of the order of Λ/v_S, whereas the mean eddy lifetime t is of the order of $\Lambda/\bar{v}$. Hence $t_P/t \sim \bar{v}/v_S$, whence t_P/t is smaller than unity only if $\bar{v}/v_S < 1$. Moreover, dissipation of energy through shock formation might well prevent $\bar{v}$ from exceeding v_S by an appreciable amount in a real star. We shall accordingly here (at the risk of committing an error!) take v_S to be an upper limit to the physically possible values of $\bar{v}$. In other words, if $\bar{v}$ should ever in the course of a calculation exceed v_S, our prescription would be to replace $\bar{v}$ in the equations by v_S. This case will be discussed in Sect. 14.9. Except in Sect. 14.9, we shall always assume that $\bar{v} < v_S$.

It should also be emphasized that all quantities in the final formulae for $\bar{v}$, F_c, etc., are actually to be regarded as *averages* over a distance Λ. For computational convenience, however, *local*, rather than *average*, values are normally used in practical calculations. However, Hofmeister and Weigert [Ho64c] have carried out calculations using a non-local mixing length. Such calculations must be carried out iteratively, since the mixing length (here chosen to be equal to the *density* scale height) at each point is evaluated from the actual run of the density in the vicinity of the point in question in a preliminary model. They conclude that the overall structure of the convection zone, obtained using their methods, does not differ significantly from that obtained using the local pressure scale height, provided that the ratio (mixing length/pressure scale height) is suitably chosen (1.5 for a solar-type convection zone, about 2.1 for a red giant-type convection zone). There are differences in the computed density distribution, however, in the very outermost layers of a red giant-type convection zone between the two methods of calculation.

Equations (14.16) and (14.18) for the convective flux may thus be written, using (14.28) and (14.30) for $\bar{v}$, in the forms

$$F_c = (4\sqrt{2})^{-1} g^{1/2} Q^{1/2} (\rho/T^{1/2}) c_P \Lambda^2 (\Delta\nabla T)^{3/2} \tag{14.31a}$$

$$= (Q^{1/2}/(4\sqrt{2}\Gamma_1^{1/2}))(\Lambda/\lambda_P)^2 \rho v_S c_P T (\lambda_P/T)^{3/2} (\Delta\nabla T)^{3/2} \tag{14.31b}$$

$$= (Q^{1/2}/(4\sqrt{2}\Gamma_1^{1/2}))(\Lambda/\lambda_P)^2 \rho v_S c_P T (\nabla - \nabla')^{3/2} \tag{14.31c}$$

$$= (4\sqrt{2})^{-1} c_P Q^{1/2} \rho^{5/2} g^2 T P^{-3/2} \Lambda^2 (\nabla - \nabla')^{3/2}. \tag{14.31d}$$

14.4 The Net Flux

Using (14.11) and (14.31c), we obtain for the total flux

$$F(r) = \frac{L(r)}{4\pi r^2} = F_r + F_c = \frac{4ac}{3}\frac{T^4 g}{\kappa P}\nabla + \left(\frac{Q^{1/2}}{4\sqrt{2}\Gamma_1^{1/2}}\right)\left(\frac{\Lambda}{\lambda_P}\right)^2 \rho v_S c_P T(\nabla - \nabla')^{3/2}. \tag{14.32}$$

Expressing $F(r)$ in terms of the fictitious radiative gradient ∇_r, defined by (14.12), and dividing through by the factor $4acT^4g/3\kappa P$, we obtain

$$\nabla_r = \nabla + \frac{(Q^{1/2}/4\sqrt{2}\Gamma_1^{1/2})(\Lambda/\lambda_P)^2 \rho v_S c_P T}{4acT^4 g/3\kappa P}(\nabla - \nabla')^{3/2} \tag{14.33a}$$

$$= \nabla + \frac{3c_P \kappa Q^{1/2}\rho^{5/2} g\Lambda^2}{16\sqrt{2}acP^{1/2}T^3}(\nabla - \nabla')^{3/2}. \tag{14.33b}$$

We note that the factor multiplying ∇ in (14.32) may be regarded as a "radiative conductivity" (with dimensions of energy flux). Similarly, the factor multiplying $(\nabla - \nabla')^{3/2}$ in (14.32) may be regarded as a "convective conductivity" (with the same units). Consequently, the factor multiplying $(\nabla - \nabla')^{3/2}$ in (14.33), the ratio of the "convective" to the "radiative" conductivities, is clearly some measure of the ability of the material to transport energy by convection.

Equation (14.33) relates ∇, the actual gradient, to ∇_r, the fictitious radiative gradient, and ∇', the gradient of a rising or falling convective element, in terms of local values of the physical variables c_P, κ, Q, Λ, ρ, etc. We assume the local values of all these physical variables to be known. We are also temporarily assuming the local value of ∇' to be known. For example, we might have to sufficient accuracy $\nabla' = \nabla_{ad} = (\Gamma_2 - 1)/\Gamma_2$, where Γ_2 is a function of ρ, T, and chemical composition. If the value of ∇ is regarded as known at a given point, then (14.33) immediately gives the corresponding value of ∇_r, from which the total flux $L(r)/4\pi r^2$ may be evaluated at that point by use of (14.12). Also, the corresponding values of $\bar{v}$, F_c, and F_r are given by (14.28), (14.31), and (14.11). If, on the other hand, the value of ∇_r (*i.e.*, of $L(r)/4\pi r^2$) is regarded as known at a given point, then (14.33) must be solved for the corresponding value of ∇. The quantity $\nabla - \nabla'$ can then be found, and values of the remaining quantities of interest can be calculated, just as above.

If $\nabla' = \nabla_{ad}$ is not an adequate approximation, then one must obtain another expression relating the four gradients from a postulated mechanism of heat exchange between the moving element and the surrounding matter.

This other expression, along with (14.33), then yields two relations involving the four gradients. One of the gradients, ∇_{ad}, can always be evaluated at a point if values of the physical variables (ρ, T, etc.) are known there.* Consequently, given the value of *any one* of the remaining three gradients, corresponding values of the other two may be calculated from these relations. Normally the gradient whose value is assumed known is either ∇_r or ∇, depending on the nature of the problem. The other gradient of the pair (∇_r, ∇) is then regarded as a function of the known one. We now develop this other expression (actually, *two* other expressions since we shall introduce another unknown, the *efficiency*).

14.5 Efficiency of Convection

A rising turbulence element, because it is hotter than its surroundings, will lose heat to its surroundings by radiation during its lifetime. On the other hand, if the element possesses (nuclear) energy sources, it may also gain heat relative to its surroundings during its lifetime from the action of these sources. It is important to note that these gains and losses of heat by the element are not considered as contributing *directly* to the net outward flux of heat, nor as *directly* affecting in any way the condition of thermal equilibrium (see Chap. 5) averaged over a spherical shell. These gains and losses of heat may be regarded as "horizontal", since they are taken relative to the instantaneous average surroundings of an element at a given level. Since there are, on the average, as many cool elements as hot ones over a spherical shell, there can be no direct contribution to the net "vertical" heat transport by these gains and losses of heat. They do, however, contribute *indirectly* to the net outward heat flux, through their effect on the efficiency (see later in this section) of convection. We shall in this section obtain approximate expressions for these "horizontal" gains and losses of heat and shall define, following Böhm-Vitense [Bö58], the convective efficiency.

(The present picture of convective heat transfer can be likened, roughly, to the process of transporting water from one reservoir to a higher one by means of a closed, continuous conveyor belt to which are attached buckets for scooping up the water from the lower reservoir. The "horizontal" losses, for example, would correspond in this analogy to "leaky" buckets, which would obviously cut down the efficiency of transport of water.)

* This statement is true only if the turbulent pressure has been neglected. If $P_{tot} = P + P_{turb}$ (gas + radiation + turbulent pressure), then the gradient of an element which moves adiabatically is $(\mathrm{d} \ln T/\mathrm{d} \ln P_{tot})_{ad} = [(\Gamma_2 - 1)/\Gamma_2]\,(\mathrm{d} \ln P/\mathrm{d} \ln P_{tot})$, where the relation between P and P_{tot} involves the quantity $(\bar{v}/v_S)$ (see Sect. 14.7), which is part of the *solution* of the problem.

Consider first the "horizontal" losses by radiation. We shall compute these by assuming the turbulence elements to be optically thick; this is likely to be accurate in most cases because of the large sizes of the turbulence elements (say, of the order of the pressure scale height) and the relatively great opacity of stellar material.*

For optically thick turbulence elements we may use an expression of the form of (14.10) to compute the energy flowing outward across unit area of the surface of the turbulence element in unit time. This is, approximately,

$$\frac{4ac}{3}\frac{T^3}{\kappa\rho}\frac{\Delta T}{(\Lambda/2)} \quad \left(\kappa\rho\frac{\Lambda}{2}\gg 1\right), \tag{14.34}$$

where ΔT is the temperature difference between the center of the turbulence element and its surface (or the surrounding matter), averaged over its lifetime, and $\Lambda/2$ is approximately the distance over which this temperature difference occurs. (Note that, because $a = 4\sigma/c$, (14.34) is equal to $(4\pi/3)\Delta B(T)/\Delta\tau$, where $B(T) = (\sigma/\pi)T^4$ and $\Delta\tau \simeq \kappa\rho(\Lambda/2)$.) Multiplying (14.34) by the surface area, $\mathscr{A}$, of the turbulence element, and the lifetime, $\Lambda/\bar{v}$, gives the net energy radiated to its surroundings by the turbulence element during its lifetime:

$$\frac{4ac}{3}\frac{T^3}{\kappa\rho}\frac{\Delta T}{(\Lambda/2)}\frac{\Lambda}{\bar{v}}\cdot\mathscr{A}. \tag{14.35a}$$

If energy is being generated within the turbulence element at the rate ε per unit mass, then the excess energy generated within the turbulence element relative to its average surroundings during its lifetime is

$$(\Delta\varepsilon)\rho V\Lambda/\bar{v}, \tag{14.35b}$$

where $\Delta\varepsilon$ is the excess rate of energy production per unit mass due to the temperature and density excesses ΔT and $\Delta\rho$ of the element over the temperature and density of its average surroundings. Defining

$$\lambda \equiv (\partial \ln \varepsilon/\partial \ln \rho)_T, \nu \equiv (\partial \ln \varepsilon/\partial \ln T)_\rho \tag{14.35c}$$

* Calculations of Henyey, Vardya, and Bodenheimer [He65] have shown that the convective elements are not always optically thick. However, elements which are not optically thick exist only at and immediately above the photosphere (defined by $\tau=(2/3)$, see Sect. 20.1), where convection is always inefficient. In such regions of inefficient convection the results of the mixing length theory are most sensitive to the basic assumptions and to the value of the mixing length Λ (see. Sect. 14.7), and are accordingly least reliable. We feel that the basic crudity of the mixing length theory in its present form does not warrant our distinguishing in this book between optically thin and optically thick convective elements in these outermost regions.

as the density and temperature exponents in the energy generation law and using (14.25) to express $\Delta\rho$ in terms of ΔT, we may also write (14.35b) in the form

$$(\nu - Q\lambda)(\Delta T/T)\varepsilon\rho V\Lambda/\bar{v}\,. \tag{14.35d}$$

Note that this term could be negative (and so add to the effect of the radiative losses given by (14.35a)) if neutrino energy losses (see Sect. 17.20) should predominate over nuclear energy production.

Now, the "excess heat content" of the rising element over its surroundings, just before it dissolves and gives up this excess heat, is given by

$$c_P\rho\Delta T_{\max}V \tag{14.36}$$

where V is the volume of the turbulence element and $\Delta T_{\max}$ is the temperature excess of the element over its surroundings at the end of its path. We shall set $\Delta T_{\max} = 2\Delta T$, following Böhm-Vitense [Bö58], where ΔT is defined in the paragraph before last.

We now define, following [Bö58], the *convective efficiency*, Γ, by the following expression:

$$\Gamma = \frac{\text{"}\textit{Excess heat content}\text{"}\textit{ just before dissolving}\,.}{\textit{Energy radiated during lifetime}} \tag{14.37}$$

Note that this definition takes into account only energy *losses* from the element. Using (14.35a) and (14.36), we obtain

$$\Gamma = \frac{3}{4ac}\frac{c_P\kappa\rho^2\bar{v}}{T^3}\frac{V}{\mathscr{A}}. \tag{14.38}$$

For a sphere of diameter Λ, we have $V/\mathscr{A} = \Lambda/6$; a cube of side Λ and a cylinder of diameter and height Λ also both have this same value for $V/\mathscr{A}$. In order to obtain numerical agreement with the results of Böhm-Vitense [Bö58], whose derivation differs slightly from ours, we shall take $V/\mathscr{A} = (2/9)\,\Lambda$ instead of $\Lambda/6$. Equation (14.38) then becomes

$$\Gamma = \frac{c_P}{6ac}\frac{\kappa\rho^2\bar{v}\Lambda}{T^3}. \tag{14.39}$$

Using (14.28b) for $\bar{v}$, we have

$$\Gamma = \frac{c_P}{12\sqrt{2}ac}\frac{\kappa g Q^{1/2}\rho^{5/2}\Lambda^2}{P^{1/2}T^3}(\nabla - \nabla')^{1/2}. \tag{14.40}$$

We note that the factor multiplying $(\nabla-\nabla')^{(1/2)}$ in (14.40) differs by only a numerical factor from the factor multiplying $(\nabla-\nabla')^{(3/2)}$ in (14.33b). According to the discussion immediately following (14.33), then, this factor in (14.40) is essentially the ratio of the "convective" to the "radiative" conductivity. We then obtain

$$\nabla_r = \nabla+\frac{9}{4}\Gamma(\nabla-\nabla'). \tag{14.41}$$

We now require a second expression for Γ. In Chap. 12 (see (12.11)) we derived a relation between the polytropic exponent Γ_2' and the adiabatic exponent Γ_2. Noting that $\nabla' = (\Gamma_2'-1)/\Gamma_2'$, we may write this relation in the form

$$\nabla' = \nabla_{ad}\frac{1}{1-(c/c_P)}, \tag{14.41a}$$

where

$$c \equiv dQ/dT' \tag{14.41b}$$

is a generalized specific heat per unit mass, which may be arbitrarily specified during a polytropic change, and T' is taken here as the temperature of a turbulence element during its motion. We write $(dQ/dt) = (dQ/dT')(dT'/dt) = \Delta\varepsilon-\Delta(\nabla\cdot\mathbf{F}/\rho)$, where $\Delta\varepsilon$ and $\Delta(\nabla\cdot\mathbf{F}/\rho)$ are, respectively, the excess rate of energy generation (*cf.* (14.35b)) and the excess rate of energy loss by radiation (*cf.* (14.35a)), both per unit mass, of the element relative to its surroundings. We then have from (14.41b)

$$\frac{c}{c_P} = \frac{\Delta\varepsilon-\Delta(\nabla\cdot\mathbf{F}/\rho)}{c_P(dT'/dt)} = \frac{-\Delta(\nabla\cdot\mathbf{F}/\rho)[1-(\Delta\varepsilon)/\Delta(\nabla\cdot\mathbf{F}/\rho)]}{c_P[(dT'/dt)-(dT/dt)]+c_P(dT/dt)},$$

where (dT/dt) is the rate of change (as seen from the moving turbulence element and resulting from its motion) of the average temperature of all the surrounding matter. We note (see (14.37)) that the quantity $\Delta(\nabla\cdot\mathbf{F}/\rho)/[c_P(dT'/dt)-c_P(dT/dt)]$ may be identified with $(1/\Gamma)$, where Γ is the convective efficiency, and that (see (14.4) and (14.6)) $(dT/dt) = -(T/\lambda_P)(dr/dt)\nabla = -(T/\lambda_P)\bar{v}\nabla$, where we have set (dr/dt) equal to $\bar{v}$, the mean convective velocity. We then have

$$\frac{c}{c_P} = \frac{-(1-\eta)/\Gamma}{1-[\nabla/(\nabla-\nabla')]}, \tag{14.41c}$$

where

$$\eta \equiv \frac{\Delta\varepsilon}{\Delta(\nabla\cdot\mathbf{F}/\rho)}. \tag{14.41d}$$

Multiplying numerator and denominator of (14.41d) by $\rho V\Lambda/\bar{v}$, we see that η is just the ratio of (14.35b) to (14.35a), *i.e.*, the ratio of the excess energy generated within the element to the excess energy radiated by the element, both during its lifetime. Using (14.35d), (14.35a), and taking $V/\mathscr{A} = \Lambda/6$ *in this derivation alone*, we obtain

$$\eta = \frac{1}{16ac}\frac{(v-\lambda Q)\varepsilon\kappa\rho^2\Lambda^2}{T^4}. \tag{14.41e}$$

Using (14.41c) for c/c_P in (14.41a), we obtain, finally, after rearranging,

$$\frac{\Gamma}{1-\eta} = \frac{\nabla-\nabla'}{\nabla'-\nabla_{ad}}, \tag{14.42}$$

which is the desired expression.

It is clear that the factor $(1-\eta)^{-1}$ on the left side of (14.42) gives the effect of energy sources on convection. This factor is unity if $\varepsilon = 0$ (in which case (14.42) reduces to the equation given by Böhm-Vitense [Bö58]), greater than unity (or negative if $\eta > 1$) if $\eta > 0$ (as would normally be the case for thermonuclear energy sources), and less than unity (but positive) if $\eta < 0$ (as would normally be the case if neutrino energy losses exceeded nuclear energy production, see Sect. 17.20).

Equation (14.42) may be interpreted as follows. The "heat excess" of the turbulence element is proportional to $(\nabla-\nabla')$, which is essentially the excess temperature of the element over the average temperature of its surroundings. If the element exchanged *no* heat with its surroundings during its motion, then the element would be moving *adiabatically* and its "heat excess" would be proportional (with the same factor) to $(\nabla-\nabla_{ad})$ (see Fig. 14.2, obtained from the results of Sect. 14.1). The excess energy lost by radiation, less the excess energy generated, during the lifetime of the element is then proportional to the difference between these two quantities, or is proportional to $(\nabla-\nabla_{ad})-(\nabla-\nabla') = (\nabla'-\nabla_{ad})$. Equation (14.42) is then (recalling the meaning of η) obviously consistent with the word definition (14.37) of Γ.

Equations (14.40), (14.41), and (14.42) are three equations involving the four gradients, the convective efficiency Γ, and the physical variables c_P, ε, κ, Λ, ρ, T, etc. We assume, as usual, that values of ∇_{ad} and of all the physical variables are available at some point in a convection zone (see first footnote in this section). Of the four quantities Γ, ∇_r, ∇, and ∇', clearly only one is independent: values of the remaining three are determined at a given point by the simultaneous solution of (14.40)–(14.42). Solutions of these equations will be presented in Sects. 14.7, 14.8, and 14.9 for the case $\eta = 0$ (no energy sources), the only case of interest for most purposes.

We note from (14.42) and (14.41), respectively, that $\nabla' \to \nabla_{ad}$ and $\nabla \to \nabla'$ (*i.e.*, both ∇ and ∇' approach ∇_{ad}) for $\Gamma \to \infty$ (great convective efficiency); and that ∇ and ∇' both approach ∇_r for $\Gamma \to 0$ (small convective efficiency). To find the fraction F_c/F of the total flux which is being carried by convection in these two limiting cases, we note from (14.11) and (14.12) that $F_r/F = \nabla/\nabla_r$, whence

$$\frac{F_c}{F} = \frac{F - F_r}{F} = \frac{\nabla_r - \nabla}{\nabla_r}. \tag{14.43}$$

Clearly, (14.43) is valid whether there are energy sources or not. For $\Gamma \to 0$, $\nabla \to \nabla_r$ and $F_c/F \to 0$. *Hence a small convective efficiency means that, although convection may be occurring, it carries only a small fraction of the total flux and causes only minor deviations of the actual gradient* ∇ *from the fictitious radiative gradient* ∇_r. This clearly means that the convection will have only a minor effect on the structure of that portion of the envelope where Γ is small. For $\Gamma \to \infty$, we have $\nabla \to \nabla_{ad}$, whence we have from (14.43)

$$\frac{F_c}{F} \simeq \frac{\nabla_r - \nabla_{ad}}{\nabla_r} \quad (\Gamma \to \infty). \tag{14.44}$$

Hence $F_c/F \to 1$ if $\nabla_r \gg \nabla_{ad}$ or $F_c/F \to 0$ if $\nabla_r \to \nabla_{ad}$. This latter case might apply, for example, to a region in a convective zone in the deep interior of a star very close to the boundary separating the convective zone from a region in radiative equilibrium (where all convective transfer must cease, aside from possible "overshoot", which we do not consider here*). Thus we see that *high convective efficiency does not, in itself, necessarily imply that a large percentage of the total flux is transported by convection.*

Before concluding this section, we note that one may also define a *gas pressure* scale height by the relation

$$\frac{1}{\lambda_g} \equiv -\frac{\mathrm{d} \ln P_g}{\mathrm{d}r}, \tag{14.45}$$

where P_g is the *gas* pressure. We may obtain an expression for λ_g in the following way. From the equation of hydrostatic equilibrium, $\mathrm{d}(P_g + P_r)/\mathrm{d}r = -\rho g$ (P_g = radiation pressure, $g = GM_r/r^2$ = gravitational acceleration), we have

$$\frac{\mathrm{d}P_g}{\mathrm{d}r} = -\rho g - \frac{\mathrm{d}P_r}{\mathrm{d}r} = -\rho g_e, \tag{14.46}$$

* Two recent discussions of "overshoot" at boundaries of convective regions are by Saslaw and Schwarzschild [Sa 65b] and by Roxburgh [Ro65a]. In both papers it is concluded that mixing by overshoot is not likely to be very important under typical stellar interior conditions.

where

$$g_e \equiv g + \frac{1}{\rho}\frac{dP_r}{dr} \tag{14.47}$$

is an *effective* gravitational acceleration for the gas pressure. Hence, from (14.45) and (14.46) we see that

$$\lambda_g = P_g/\rho g_e \tag{14.48}$$

(see the second equality in (14.4)). Assuming the "diffusion approximation" (see Sect. 6.4) to be valid, we may write

$$\frac{1}{\rho}\frac{dP_r}{dr} = -\frac{\kappa}{c}F_r(r) = -\frac{\kappa}{c}\frac{F_r(r)}{F(r)}F(r)\,,$$

where $F_r(r)$ and $F(r)$ denote, respectively, the radiative flux and the total flux, both at radial distance r. We then have

$$g_e = g - \frac{\kappa}{c}\frac{F_r(r)}{F(r)}F(r)$$

or, noting from (14.11) and (14.12) that $F_r/F = \nabla/\nabla_r$,

$$g_e = g - \frac{\kappa}{c}\frac{\nabla}{\nabla_r}F(r)\,. \tag{14.49}$$

This equation shows that $g_e \leq g$ and, as is pointed out by Böhm-Vitense [Bö58], g_e can even become negative in regions where the opacity κ is sufficiently large. In such regions the *gas* pressure could actually *decrease* inward (although the *total* pressure always increases inward in hydrostatic equilibrium). By using (14.12) for $F(r)$ and (14.4) for λ_P, we may also write (14.49) in the form

$$g_e = g[1-4(1-\beta)\nabla]\,, \tag{14.50}$$

where $1-\beta$ is the ratio of radiation pressure to total pressure.

These considerations regarding λ_g are introduced to caution the reader that a careful distinction between λ_P and λ_g may be necessary in the use of the mixing length theory. It is an open question as to which scale height has the greater physical significance in regard to the mixing length theory. We shall in this chapter always work in terms of λ_P rather than of λ_g. In this way we avoid (at least explicitly) difficulties arising from negative effective gravitational accelerations: the gravitational acceleration which appears in all the formulae is then simply $g = GM_r/r^2$, which is always positive and easily computable. Moreover, if we were using λ_g instead of λ_P, then the value of g_e at a point would not be known until the value of ∇ at the point is known. Hence our

discussion a few paragraphs back regarding the knowns and unknowns of the problem would have to be altered.

14.6 Upper Limits to Values of Various Quantities

We first estimate an approximate upper limit to the expected values of the turbulent pressure in a convection zone. Treating the convective elements as Newtonian mass points, we may take for the turbulent pressure (*cf.* Sect. 10.2)

$$P_{\text{turb}} = \frac{2}{3}\left(\frac{1}{2}\rho\bar{v}^2\right) = \frac{1}{3}\rho v_S^2(\bar{v}/v_S)^2 , \tag{14.51}$$

where v_S is the Laplacian (adiabatic) sound speed. Using (14.29) for v_S, we have

$$P_{\text{turb}}/P = \frac{1}{3}\Gamma_1(\bar{v}/v_S)^2 , \tag{14.52}$$

where P is the sum of the gas and radiation pressures. If we require that the convection always be subsonic (otherwise the physical basis of the mixing length theory would be invalid), then we have

$$P_{\text{turb}}/P \lesssim \frac{1}{3}\Gamma_1 . \tag{14.53}$$

Since Γ_1 is not likely to be much larger than (5/3), then

$$P_{\text{turb}}/P \lesssim \frac{5}{9} ; \tag{14.54}$$

i.e., the turbulent pressure is not likely to exceed about one-half the sum of the gas and radiation pressures if the convective velocities are subsonic. Numerical estimates of the value of P_{turb}/P under representative stellar conditions will be presented later in this section.

Let us now write

$$H_c = 4\pi r^2 F_c \tag{14.55}$$

and solve (14.31d) and (14.31a) for $(\nabla - \nabla')$ and $\Delta\nabla T$:

$$\nabla - \nabla' = \left[\frac{2}{\pi^2}\cdot\frac{H_c^2 P}{r^4 c_P^2 \rho^5 g^4 T^2 \Lambda^4 Q}\right]^{1/3} , \tag{14.56}$$

$$\Delta\nabla T = \left[\frac{2}{\pi^2}\cdot\frac{H_c^2 T}{r^4 g \rho^2 c_P^2 \Lambda^4 Q}\right]^{1/3} . \tag{14.57}$$

Using these equations in (14.30a) and (14.30b), we obtain

$$\frac{\bar{v}}{v_S} = \frac{1}{2(2\pi)^{1/3}} \left[\frac{g H_c \Lambda Q \rho^{1/2}}{(\Gamma_1 P)^{3/2} r^2 c_P T} \right]^{1/3}. \tag{14.58}$$

Also, from (14.40) we have

$$\Gamma = \frac{1}{12(2\pi)^{1/3} ac} \left[\frac{c_P^2 Q H_c \kappa^3 g \Lambda^4 \rho^5}{r^2 T^{10}} \right]^{1/3}. \tag{14.59}$$

Now let

$$P = (\mathscr{R}/\mu)\rho T/\beta,$$

$$g = GM_r/r^2, \tag{14.60}$$

where β is the ratio of gas to total pressure. Then we have

$$\nabla - \nabla' = \left(\frac{2\sqrt{2}}{5\pi}\right)^{2/3} \left(\frac{\mathscr{R}}{G^4}\right)^{1/3} \left(\frac{H_c^2 r^4}{\mu \Lambda^4}\right)^{1/3} \left(\frac{T}{\rho^2 M_r^4}\right)^{1/3} \frac{1}{Q^{1/3}\beta} \left(\frac{5\mathscr{R}}{2\mu c_P}\right)^{2/3}, \tag{14.61}$$

$$\Delta\nabla T = \left(\frac{2\sqrt{2}}{5\pi}\right)^{2/3} \frac{1}{(G\mathscr{R}^2)^{1/3}} \left(\frac{\mu^2 H_c^2}{r^2 \Lambda^4}\right)^{1/3} \left(\frac{T}{M_r \rho^2}\right)^{1/3} \frac{1}{Q^{1/3}} \left(\frac{5\mathscr{R}}{2\mu c_P}\right)^{2/3}. \tag{14.62}$$

If hydrogen and/or helium are present in large abundances, their ionization may increase the value of $(2\mu c_P/5\mathscr{R})$ by factors from, say, 5 to 20 and Q by probably less than a factor of 2. If all quantities are expressed in c.g.s. units, we have

$$\Delta\nabla T = 4.2 \times 10^{-4} \left(\frac{\mu^2 H_c^2}{r^2 \Lambda^4}\right)^{1/3} \left(\frac{T}{M_r \rho^2}\right)^{1/3} \frac{1}{Q^{1/3}} \left(\frac{5\mathscr{R}}{2\mu c_P}\right)^{2/3} \text{°K/cm}, \tag{14.63}$$

Also we have

$$\frac{\bar{v}}{v_S} = \frac{1}{(40\pi)^{1/3}} \frac{1}{\Gamma_1^{1/2}} \left(\frac{\mu}{\mathscr{R}}\right)^{5/6} G^{1/3} \left(\frac{H_c \Lambda}{r^4}\right)^{1/3} \left(\frac{M_r}{\rho}\right)^{1/3} \frac{1}{T^{5/6}} \beta^{1/2} Q^{1/3} \left(\frac{5\mathscr{R}}{2\mu c_P}\right)^{1/3} =$$

$$= 1.5 \times 10^{-10} \mu^{5/6} \left(\frac{H_c \Lambda M_r}{r^4 \rho}\right)^{1/3} \frac{1}{T^{5/6}} \beta^{1/2} Q^{1/3} \left(\frac{5\mathscr{R}}{2\mu c_P}\right)^{1/3} \left(\frac{5}{3\Gamma_1}\right)^{1/2}. \tag{14.64}$$

Finally, from (14.59) we have

$$\Gamma = \left(\frac{25}{\pi}\right)^{1/3} \frac{1}{96} \frac{(\mathscr{R}^2 G)^{1/3}}{\sigma} \kappa \left(\frac{H_c \Lambda^4}{\mu^2 r^4}\right)^{1/3} \left(\frac{M_r \rho^5}{T^{10}}\right)^{1/3} \left(\frac{2\mu c_P}{5\mathscr{R}}\right)^{2/3} Q^{1/3} =$$

$$= 2.8 \times 10^5 \kappa \left(\frac{H_c \Lambda^4}{\mu^2 r^4}\right)^{(1/3)} \left(\frac{M_r \rho^5}{T^{10}}\right)^{1/3} \left(\frac{2\mu c_P}{5\mathscr{R}}\right)^{2/3} Q^{1/3}. \tag{14.65}$$

Upper limits to the values of $\Delta\nabla T$, $\bar{v}/v_S$, and Γ are given by setting $H_c = L$ in the above expressions.* For the *deep interior* of a star of approximately solar type we take $\beta = 1$, $Q = 1$, $c_P = (5/2)\,(\mathscr{R}/\mu)$, $\mu = (1/2)$, $r = (1/2)R = 3\times10^{10}$ cm, $\Lambda = (1/10)R = 7\times10^{9}$ cm†, $T = 10^{7}\,^{\circ}$K, $\rho = 100$ gm/cm^3, $M_r = (1/2)M = 1\times10^{33}$ gm, $\kappa = 100$ cm^2/gm, and $L = 4\times10^{33}$ ergs/sec. Then we have from (14.63)

$$\Delta\nabla T \lesssim 5\times10^{-12}\,^{\circ}\text{K/cm}. \tag{14.66}$$

But $|dT/dr| \sim 10^{-4}\,^{\circ}$K/cm, whence

$$\Delta\nabla T/|dT/dr| \lesssim 5\times10^{-8}. \tag{14.67}$$

This is then approximately equal to the fractional excess of the superadiabatic gradient over the adiabatic in a convective core, for example. Note that this estimate agrees, to order of magnitude, with the estimate made in Chap. 13. Taking $\Lambda \simeq (1/10)R_\odot \approx 7\times10^{9}$ cm, we obtain

$$\Delta T = \Lambda\Delta\nabla T \lesssim 0.03\,^{\circ}\text{K}$$

for the temperature excess of an element over that of its surroundings just before it mixes and loses its identity. This temperature excess is to be compared to the value $10^{7}\,^{\circ}$K for the temperature of the surrounding matter.

We note, also, that an error in our estimate of the value of Λ by even several orders of magnitude would still render $\Delta\nabla T/|dT/dr| \ll 1$, so that representing the actual gradient in a convective core by a slightly superadiabatic gradient will normally yield very high accuracy. For this reason it is generally not necessary to use the mixing length theory of convection when dealing with convective cores. We will treat convective cores in a later chapter.

From (14.64) we have for a star like the sun

$$\bar{v} \lesssim 400\text{ cm/sec}.$$

But $v_S = 530$ km/sec under these conditions, whence $\bar{v}/v_S \lesssim 0.84\times10^{-5}$, *i.e.*, the convective velocity is very small compared to the sound velocity under

* It will be shown in later sections that H_c can approximately equal L only under certain conditions. Hence the upper limits that we obtain here may in some case be considerable overestimates of actual values of the various quantities. Another kind of upper limit is discussed in Sect. 14.9.

† It can easily be shown from the virial theorem (cf. Sect. 17.2) that the average value of the pressure scale height λ_P in a star is of the same order of magnitude as R, the stellar radius. In the deep interior of a star (not too close either to the center or to the surface), then, $\Lambda \sim \lambda_P$ (see discussion following (14.68)) $\sim R$.

these conditions. Taking $\Lambda \simeq (1/10)R_\odot \simeq 7 \times 10^9$ cm, we have for the lifetime of a turbulence element in a convective core

$$t = \Lambda/\bar{v} \gtrsim 10^7 \text{ sec} \sim 100 \text{ days}.$$

This time is much shorter than the time scale of nuclear evolution, which may be some 10^9–10^{10} years for stars of approximately solar type. Hence convective cores may be considered well mixed and of uniform chemical composition over times greater than some fraction of a year (for a star not too different from the sun).

From (14.65) we have for the convective efficiency

$$\Gamma \lesssim 4 \times 10^8$$

for the interior of a star like the sun. Note that this value is consistent (to order of magnitude) with our estimate in Chap. 13 of the superadiabaticity of the temperature gradient in the deep interior.

(At the end of this section order-of-magnitude estimates of the values of convective quantities in the deep interior will be made on the basis of more physical arguments.)

For the outer layers of a star it is customary to assume that

$$\Lambda = \alpha\lambda, \tag{14.68}$$

where λ is some scale height (usually pressure or density) and α is a number whose value is of order unity. The particular scale height that λ is to represent is very much open to question. It would appear on the surface that use of the density scale height might lead to difficulty, since this scale height becomes infinite or negative in the case of a density inversion (see Sect. 20.4). It has been argued, on the other hand, that density inversions cannot occur if the density scale height is used. It has also been suggested (see Faulkner, Griffiths, and Hoyle [Fa65]) that λ should be the *temperature* scale height in some applications. In view of the inconclusive state of opinions on the matter, we shall here arbitrarily adopt the *total* (gas plus radiation) *pressure* scale height λ_P (see (14.4)). Values generally used for α are in the range 1–4, say; but a value between 1 and 2 seems to give best results.

The principal arguments for assuming that $\alpha \sim 1$ are as follows. Convecting elements which are much smaller than a scale height in size will lose considerable energy by radiation during their motion, partly because of their small size and partly because of the short distance (of the order of Λ) through which they move before dissolving. These small elements will therefore not be very efficient carriers of convective flux; most of the convective flux will presumably be carried by the larger elements. On the other hand, a turbulence element is not likely to retain its identity when it moves into regions

which differ drastically in their physical conditions from those that obtained where the element originated. For example, the volume of a turbulence element will increase by approximately a factor of $2(\approx\exp(1/\Gamma_1))$ for every scale height through which it moves. It is therefore argued that turbulence elements cannot retain their identities if they move through a distance of much more than a few scale heights. These two arguments together then imply that $\alpha\sim 1$.

For our numerical illustration we shall again consider a star like the sun and shall adopt the values $\beta = 1$, $Q = 1$, $c_P = (5/2)(\mathcal{R}/\mu)$, $\mu = (1/2)$, $r\simeq R = R_\odot = 7\times 10^{10}$ cm, $\Lambda = 200$ km, $T = 1\times 10^4$ °K (corresponding to the outer regions of the hydrogen convection zone), $\rho = 10^{-6}$ gm/cm^3, $M_r = M_\odot = 2\times 10^{33}$ gm, $\kappa = 100$ cm^2/gm,* and $L_r = L_\odot = 4\times 10^{33}$ erg/sec. Then we have

$$\Delta\nabla T\lesssim 9\times 10^{-5}\,°\mathrm{K/cm}.$$

Taking $\Lambda = 200$ km, we obtain

$$\Delta T = \Lambda\Delta\nabla T\lesssim 1800\,°\mathrm{K}$$

for the temperature excess of a turbulence element just before it dissolves and mixes with the surrounding matter.

For the mean convective velocity we have

$$\bar{v}\lesssim 1.5\ \mathrm{km/sec}.$$

But

$$v_S\sim 17\ \mathrm{km/sec},$$

whence

$$\frac{\bar{v}}{v_S}\lesssim 0.088;$$

the convective velocity is several times smaller than the sound velocity under the given conditions. The lifetime of a turbulence element is then given by

$$t = \Lambda/\bar{v}\gtrsim 100\text{–}200\ \mathrm{sec}.$$

Note that this figure is comparable to the observed time scale for changes in the solar granulation.

Finally, for the convective efficiency we have

$$\Gamma\lesssim 140,$$

* Actually, in these regions κ may be larger than this by one or two orders of magnitude. However, according to (14.56)–(14.59), κ enters in our present estimates only into the expression for Γ, whose value is already probably considerably overestimated (see next paragraph).

which may be a considerable overestimate since the number here is an *upper limit* to the value of Γ. Actually, in the layers *immediately* below the photosphere, Γ will most likely have a value much less than 1, possibly of order 10^{-4}. At any rate, convection is far more inefficient at transferring heat in the outer layers than in the deep interior of a star. Basically, this is because of the extremely low densitites that prevail in these regions.

We can now estimate the importance of turbulent pressure, which has been neglected in these examples. Considering the moving eddies as Newtonian mass points, we had for the ratio of the turbulent pressure P_{turb} to the sum of the gas and radiation pressures P

$$P_{\text{turb}}/P = \frac{1}{3}\Gamma_1(\bar{v}/v_S)^2 = \frac{5}{9}(\bar{v}/v_S)^2 \tag{14.69}$$

if $\Gamma_1 = (5/3)$.

In the deep interior of a star we take as representative the value $\bar{v}/v_S \lesssim 10^{-5}$ derived above. We obtain

$$P_{\text{turb}}/P \lesssim 10^{-10}.$$

Thus turbulent pressure can certainly be neglected here.

In the outer stellar layers we take as representative the value $\bar{v}/v_S \lesssim 0.1$ derived above for a solar-type star; this gives the result

$$P_{\text{turb}}/P \lesssim 10^{-2}.$$

In the outer layers of the convection zones of red giant stars, however, $\bar{v}/v_S$ can approach unity, in which case $P_{\text{turb}}/P \sim 0.5$. The turbulent pressure may therefore not be negligible in these cases.

We may summarize some of the numerical results of the present section by giving rough orders of magnitude for the values to be expected for some of the relevant quantities in the deep interior and in the outer regions of a typical star. These are shown in Table 14.1.

Table 14.1

REPRESENTATIVE VALUES OF CONVECTIVE QUANTITIES IN THE STELLAR INTERIOR

Quantity	*Deep Interior*	*Outer Layers*
$\Delta\nabla T$	10^{-12} to 10^{-10} °K/cm	10^{-5} to 10^{-4} °K/cm
$\Delta T = \Lambda\Delta\nabla T$	0.01 to 1 °K	100 to 1000 °K
$\bar{v}/v_s$	10^{-6} to 10^{-4}	0.1 to 1
$\bar{v}$	1 to 100 m/sec	0.5 to 15 km/sec
$t = \Lambda/v$	1 to 100 days	1 to 100 minutes
Γ	10^{6} to 10^{9}	10^{-4} to 10^{2}

It may be pointed out that the primary factors which produce the differences shown in Table 14.1 between the values of the physical quantities in the deep interior and in the outer layers are the density ρ and the mixing length Λ; the temperature also plays a (smaller) role. The density drops perhaps by some 8 powers of ten and the mixing length by perhaps 2 to 3 powers of ten, in going from the central regions to the surface regions of a star, while the temperature drops by about 3 powers of ten.

We conclude this section by noting that the general orders of magnitude of the values of the convective quantities given in Table 14.1 for the deep interior of a star can also be derived very simply from the following physical arguments. In these arguments numbers lying in the approximate range 10^{-2} to 10^{+2} are regarded as "of order unity".

Consider first the average convective velocity $\bar{v}$. This was evaluated by equating (to order of magnitude) the kinetic energy per unit volume of a turbulence element to the work done per unit volume by the buoyant force in moving the element through a distance of one mixing length Λ. Noting that, normally, $\Delta\rho/\rho \sim -\Delta T/T$, where $\Delta\rho$ and ΔT are, respectively, the excess density and temperature of the element over its surroundings at the end of its path, we have

$$\frac{1}{2}\rho\bar{v}^2 \sim \rho g \Lambda(\Delta T/T).$$

Writing $g \sim GM/R^2$ and setting $\Lambda \sim R$ (see the second footnote in this section), we obtain

$$\bar{v} \sim (GM/R)^{1/2}(\Delta T/T)^{1/2}, \tag{14.69a}$$

where T is now regarded as a mean, or representative, temperature of the matter in the stellar interior. On the other hand, the mean value $\bar{v}_S$ of the Laplacian sound velocity (*cf.* (14.29)) for the whole star can be evaluated from the virial theorem (see Sect. 27.1) and is given (essentially independently of the equation of state), to order of magnitude, by

$$\bar{v}_S \sim (GM/R)^{1/2}. \tag{14.69b}$$

Comparison of (14.69a) and (14.69b) then yields the result

$$\frac{\bar{v}}{v_S} \sim \left(\frac{\Delta T}{T}\right)^{1/2}. \tag{14.69c}$$

Next, the mean eddy lifetime t is given by

$$t = \Lambda/\bar{v} \sim R/\bar{v} \sim \left(\frac{R^3}{GM}\right)^{1/2}\left(\frac{\Delta T}{T}\right)^{-1/2}$$

(using (14.69a)), or by

$$t \sim t_{\rm ff}(\Delta T/T)^{-1/2}, \tag{14.69d}$$

where $t_{\rm ff} \sim (R^3/GM)^{1/2}$ is the "free-fall" time (*cf.* Chaps. 0, 1, and 27).

Equations (14.69c) and (14.69d) can also be understood in the following intuitive terms. If $\Delta T/T$ were unity, then a turbulence element would be accelerated with the full gravitational acceleration, and hence, after having been accelerated through a distance $\Lambda \sim R$, would be moving with practically the full "free-fall" velocity, which (*cf.* Sect. 27.1) is of the same order of magnitude as the mean sound velocity $\bar{v}_S$. The mean convective velocity $\bar{v}$ should then be smaller than $\bar{v}_S$ by the approximate factor $(\Delta T/T)^{1/2}$, as is shown by (14.69c), where $(\Delta T/T)$ is the fraction of the full gravitational acceleration that a turbulence element "feels"; and the mean eddy lifetime t should be larger than $t_{\rm ff}$ by the approximate factor $(\Delta T/T)^{-1/2}$, as is shown by (14.69d).

Consider next the convective efficiency Γ, defined in (14.37). Taking a typical turbulence element in the deep interior to have a size and mass comparable in order of magnitude to those of the star, it is clear (see next paragraph) that the "excess heat content" of the turbulence element will be of order $c_V TM(\Delta T/T)$ (noting that, normally, $c_P \sim c_V$, where c_P and c_V are the specific heats per unit mass at constant pressure and at constant volume). To obtain the "excess heat radiated by the element to its surroundings during its lifetime" t, we note that, because the surface area of the element is of the same order of magnitude as that of the star, the "luminosity" of the element to its surroundings differs (to rough order of magnitude) from the "radiative luminosity" $L_{\rm rad}$ $(= 4\pi F_r$, *cf.* (14.10)) of the whole star (assumed of the same order of magnitude as the total stellar luminosity L) only because the temperature gradients in the element and in the star are different: in the element the temperature gradient is of order $\Delta T/\Lambda \sim \Delta T/R$, whereas that in the star is of order T/R. Hence the "luminosity" of the element is of order $L(\Delta T/T)$, and the excess heat radiated by the element to its surroundings during its lifetime t is accordingly $\sim Lt(\Delta T/T) \sim Lt_{\rm K}(\Delta T/T)^2 \sim c_V TM(\Delta T/T)^2$, where we have used a result established in the following paragraph ($t_{\rm K}$ = "Kelvin" time). Making use of the above expressions for the "excess heat content" and the "excess heat radiated by the element to its surroundings during its lifetime" in the definition of Γ (*cf.* (14.37)), we see that in a convection zone in the deep stellar interior Γ is given to rough order of magnitude by

$$\Gamma \sim (\Delta T/T)^{-1}. \tag{14.69d'}$$

To find the value of $(\Delta T/T)$, which enters into all of our order-of-magnitude expressions for $\bar{v}$, t, and Γ in the deep interior, we note that t is

the time during which a spherical shell of mass ΔM and thickness Λ in a convection zone in the deep interior must "store" the amount of energy Lt (L = luminosity of the star) incident on its lower boundary, since t is the time required for this energy to be transported by convection to the next shell further out (we are assuming that most of the energy is carried by convection and that L_r ("interior luminosity") $\sim L$; *cf.* the simplified model of a convection zone presented in Chap. 13). The increase in temperature of the shell resulting from the absorption of this energy Lt may be identified (in our order-of-magnitude considerations) with the ΔT appearing in the above equations. We accordingly have

$$Lt \approx c_V T \Delta M (\Delta T/T).$$

Taking $\Delta M \sim M$, we note that the quantity $c_V T \Delta M$ is then of the order of the total thermal energy of the star, which in turn is (by the virial theorem, *cf.* Sect. 17.2) of the order of the total gravitational energy $\sim GM^2/R$ (*cf.* Sect. 17.1) of the star. The above equation then leads to the result

$$t \sim t_K (\Delta T/T), \tag{14.69e}$$

where $t_K \sim GM^2/LR$ is the "Kelvin time" (*cf.* Chaps. 0, 1, and 27 and Sect. 17.4).

Comparison of (14.69d) and (14.69e) yields the desired result:

$$\frac{\Delta T}{T} \sim \left(\frac{t_{ff}}{t_K}\right)^{2/3}, \tag{14.69f}$$

where (*cf.* (0.8″)) $t_{ff}/t_K \sim LR^{5/2}/(G^{3/2}M^{5/2}) \sim 10^{-12}$ for a star similar to the sun. Hence $\Delta T/T \sim 10^{-8}$ for such a star, which is in agreement with our other estimates of the value of this quantity. Its smallness implies that the temperature gradient in a convection zone in the deep interior of a star is only negligibly superadiabatic, since the fractional excess of the temperature gradient over the adiabatic is of the same order of magnitude as $\Delta T/T$.

We now have from (14.69c), (14.69d), (14.69d′), and (14.69e)

$$\frac{\bar{v}}{v_S} \sim \left(\frac{t_{ff}}{t_K}\right)^{1/3} \sim 10^{-4}, \tag{14.69g}$$

$$t \sim t_{ff}\left(\frac{t_{ff}}{t_K}\right)^{-1/3} \sim 10^4 t_{ff}, \tag{14.69h}$$

$$t \sim t_K\left(\frac{t_{ff}}{t_K}\right)^{2/3} \sim 10^{-8} t_K, \tag{14.69i}$$

$$\Gamma \sim \left(\frac{t_{ff}}{t_K}\right)^{-2/3} \sim 10^8 \tag{14.69j}$$

(using the value $t_{ff}/t_K \sim 10^{-12}$ for a solar-type star). Taking $\bar{v}_S \sim 10^7$ cm/sec, $T \sim 10^7$°K, $t_{ff} \sim 10^3$ sec, and $t_K \sim 10^{15}$ sec (appropriate order-of-magnitude values for a solar-type star), we obtain $\bar{v} \sim 10^3$ cm/sec, $\Delta T \sim 10^{-1}$°K, and $t \sim 10^7$ sec. These values (and that for Γ given in (14.69j)) agree to order of magnitude with the values given in Table 14.1 for the deep interior of an approximately solar-type star.

14.7 Solution of the Equations When the Total Flux is Specified

In this section we shall assume that the total flux $L(r)/4\pi r^2 = F_r + F_c$ is specified at a given point and that values of ∇_{ad} and of the physical variables c_P, κ, Λ, ρ, T, etc. are all known at this point. Hence the value of ∇_r is known from (14.12) at the given point. We also neglect turbulent pressure and assume no energy sources to be present ($\varepsilon = 0$). Energy sources are usually not effective in regions of a star where it is necessary to use the mixing-length theory. We consider here the calculation of the corresponding actual gradient ∇ and other convective properties of interest at the given point in the convective zone. This case in which $L(r)/4\pi r^2$ is given would be appropriate, for example, to the calculation of the structure and properties of a convective envelope containing no nuclear energy sources in a star in thermal equilibrium (see Chap. 20). In such a convective envelope the "interior luminosity" $L(r) = L_r(r) + L_c(r)$ (radiative plus convective interior luminosities) would be constant throughout the convective envelope and equal to the luminosity L of the star.* It is clear that it must be possible to evaluate the derivative $\nabla = (\mathrm{d} \ln T/\mathrm{d} \ln P)$ at each point in a convective envelope (for given, constant $L(r)$) in order to be able to integrate the equations of stellar structure and thus to construct a model of a stellar convection zone. (Construction of stellar models is discussed in Chap. 21.) This case may also be appropriate in model calculations of the Henyey type (see Sect. 21.7, Henyey *et al.* [He59, He64, He65], and Divine [Di65]).

In terms of a dimensionless quantity

$$A \equiv \frac{Q^{1/2} c_P \kappa g \rho^{5/2} \Lambda^2}{12\sqrt{2} a c P^{1/2} T^3} \tag{14.70}$$

which is essentially the ratio of the "convective" to the "radiative" con-

* However, $L(r)$ need not be constant for applicability of the method to be described here. For example, the convective envelope may have $\varepsilon = 0$ and not be in thermal equilibrium (*cf.* Chap. 5), *i.e.*, it may be slowly expanding or contracting via absorption or release of gravitational energy (*cf.* Sect. 17.6).

ductivities, *cf.* Sect. 14.4, the three basic equations (14.40), (14.41), and (14.42) (with $\eta = 0$) are

$$\Gamma = A(\nabla - \nabla')^{1/2}, \tag{14.71}$$

$$\nabla_r - \nabla = a_0 A(\nabla - \nabla')^{3/2}, \tag{14.72}$$

$$\Gamma = (\nabla - \nabla')/(\nabla' - \nabla_{ad}), \tag{14.73}$$

where we have replaced the numerical factor 9/4 in (14.41) by the general constant a_0. We do this because different versions of the mixing length theory in use employ different values of this constant. It will be recalled that in this section ∇_r is regarded as a known quantity and the three unknowns are ∇, ∇', and Γ (values of ∇_{ad} and A are always assumed known (see the first footnote in Sect. 14.5)). We shall see that the equations (14.71)–(14.73) can be written as a single cubic equation in one unknown.

We first combine (14.71) and (14.72):

$$\nabla_r - \nabla = a_0 \Gamma(\nabla - \nabla') \tag{14.74}$$

and solve this equation for ∇:

$$\nabla = \frac{\nabla_r + a_0 \Gamma \nabla'}{1 + a_0 \Gamma}. \tag{14.75}$$

Using (14.75), we form the quantity $\nabla_r - \nabla$:

$$\nabla_r - \nabla = \frac{a_0 \Gamma(\nabla_r - \nabla')}{1 + a_0 \Gamma}. \tag{14.76}$$

We now eliminate ∇ between (14.73) and (14.75) and solve for ∇'. Using this expression for ∇', we form the quantity $\nabla_r - \nabla'$ and obtain

$$\nabla_r - \nabla' = \frac{\Gamma(1 + a_0 \Gamma)(\nabla_r - \nabla_{ad})}{1 + \Gamma(1 + a_0 \Gamma)}. \tag{14.77}$$

Substituting (14.77) into (14.76) and dividing the resulting equation by $\nabla_r - \nabla_{ad}$, we obtain

$$\frac{\nabla_r - \nabla}{\nabla_r - \nabla_{ad}} \equiv \zeta = \frac{a_0 \Gamma^2}{1 + \Gamma(1 + a_0 \Gamma)}. \tag{14.78}$$

We see that the quantity ζ, defined by the first equality in (14.78), is a function only of the convective efficiency Γ, and therefore may itself be regarded as a measure of convective efficiency. We have that $\zeta \to 0$ as $\Gamma \to 0$, $\zeta \to 1$ as $\Gamma \to \infty$, and $\zeta \sim 0.5$ when $\Gamma \sim 1$ (we take $\Gamma \sim 1$ as defining the transition region between inefficient ($\Gamma \ll 1$) and efficient ($\Gamma \gg 1$) convection). We shall

regard ζ in this section as the basic unknown. Once its value is determined, the value of the actual gradient ∇ is given by the equation

$$\nabla = (1-\zeta)\nabla_r + \zeta\nabla_{ad}. \tag{14.79}$$

We have that $\nabla \to \nabla_r$ for $\zeta \to 0$ and that $\nabla \to \nabla_{ad}$ for $\zeta \to 1$, so that $\zeta \ll 1$ implies small convective efficiency and $\zeta \simeq 1$ implies large convective efficiency.

We may now express Γ in terms of ζ by making use of (14.71), (14.74), and (14.78):

$$\Gamma = B\zeta^{1/3}, \tag{14.80}$$

where

$$B \equiv [(A^2/a_0)(\nabla_r - \nabla_{ad})]^{1/3}. \tag{14.81}$$

Since ζ is a function only of Γ, it follows that Γ is a function only of B. Hence B may also be regarded as a measure of convective efficiency. Since $\zeta \to 1$ as $\Gamma \to \infty$, then $\Gamma \to B$ in this limit, so that values of $B \gg 1$ imply large efficiency. For $\Gamma \to 0$, we see from (14.78) that $\zeta \to a_0\Gamma^2$. Hence, from (14.80) we have that $\Gamma \to a_0 B^3$, and hence that $B \to 0$ as $\Gamma \to 0$. Hence values of $B \ll 1$ imply convective inefficiency.

We see from the definition (14.81) that B depends on the *two* quantities A and $\nabla_r - \nabla_{ad}$ and that it increases in value as both of these quantities increase. If we require all convective velocities to be subsonic, however, then it is not sufficient merely that $B \gg 1$ for highly efficient convection. It will be shown later in this section that if $A \lesssim 1$, then values of $\nabla_r - \nabla_{ad}$ large enough to render $B \gg 1$ would render $\bar{v}/v_S > 1$, *i.e.*, would require supersonic convective velocities. Hence we may say that if the convection is to be highly efficient and at the same time subsonic, then we must have $A \gg 1$ (large "capacity" of the material to convect efficiently, *cf.* Sect. 14.4 and Sect. 20.5).

The cubic equation for ζ may be obtained by substituting (14.80) into the last member of (14.78). We write this equation in the form

$$\zeta^{1/3} + B\zeta^{2/3} + a_0 B^2 \zeta - a_0 B^2 = 0. \tag{14.82}$$

The left side of (14.82) is seen to increase monotonically from $-a_0B^2$ to $1 + B(>0)$ as ζ varies from 0 to 1, whence it follows that there is one and only one real root of (14.82) for $0 \leq \zeta \leq 1$. We see also that $\zeta \to 0$ as $B \to 0$ and that $\zeta \to 1$ as $B \to \infty$, as must be true from the arguments presented in the preceding paragraph. We warn the reader that (14.82) applies only when turbulent pressure is neglected and when no energy sources are present ($\varepsilon = 0$). It should also be noted that (14.82) applies only when the g appearing

in A is GM_r/r^2 and not g_e (see Sect. 14.5). Hence use of (14.82) implies that the relevant scale height being used here is λ_P and not λ_g.

Equation (14.82) can of course be solved analytically, but we consider here alternative methods of solution. For $B<1$, an iterative procedure can be employed, taking $\zeta = (a_0B^2)^3$, for example, as the initial trial value of ζ and transferring all terms except $\zeta^{1/3}$ in (14.82) to the right side of the equation. Carrying out the iterative procedure analytically, we obtain the following expansion for ζ:

$$\zeta = (a_0B^2)^3\{1 - 3B(a_0B^2) - [3 - (9/a_0)](a_0B^2)^3 + \cdots\} (B \lesssim 1). \quad (14.83)$$

For $B \gtrsim 1$, an iterative solution of (14.82) can also be employed. We write (14.82) as

$$1 - \zeta = (\zeta^{1/3} + B\zeta^{2/3})/a_0B^2$$

and take $\zeta = 1-[(1+B)/a_0B^2]$, for example, as the initial trial value of ζ. Carrying out the iterative procedure analytically, we obtain the following expansion for $1-\zeta$:

$$1 - \zeta = \frac{1+B}{a_0B^2}\left\{1 - \frac{1+2B}{3(1+B)}\left(\frac{1+B}{a_0B^2}\right) + \right.$$

$$\left. + \frac{1}{9}\left[\left(\frac{1+2B}{1+B}\right)^2 - 1\right]\left(\frac{1+B}{a_0B^2}\right)^2 + \cdots\right\} (B \gtrsim 1). \quad (14.84)$$

The solution of (14.82) can also be tabulated numerically. Table 14.2 gives B as a function of ζ, for $a_0 = 9/4$.

Once the value of ζ is known, values of the other convective quantities of interest are readily obtained.

First, the convective efficiency Γ, which depends only on ζ (or B), is obtained from (14.80); some numerical values are presented in Table 14.2. Next, given the value of $\nabla_r - \nabla_{ad}$, the value of the actual gradient ∇ follows from (14.79). The ratio of convective to total flux, F_c/F, is then given, from (14.43) and (14.78), by the relation

$$\frac{F_c}{F} = \frac{\nabla_r - \nabla_{ad}}{\nabla_r}\zeta = [1 - (\nabla_{ad}/\nabla_r)]\zeta. \quad (14.85)$$

Note that F_c/F depends not only on ζ, but also on ∇_r and ∇_{ad}.

Next, given the value of A, the value of $\nabla - \nabla'$ may be obtained from (14.71).

Table 14.2

SOLUTION OF THE CUBIC EQUATION (14.82) AND CONVECTIVE EFFICIENCY

$$\left(a_0 = \frac{9}{4}\right)$$

ζ	B	Γ
0	0	0
0.0001	0.1441	6.689×10^{-3}
0.001	0.2132	2.132×10^{-2}
0.01	0.3216	6.928×10^{-2}
0.05	0.4481	0.1651
0.1	0.5349	0.2473
0.2	0.6729	0.3935
0.3	0.8096	0.5419
0.4	0.9667	0.7123
0.5	1.165	0.9249
0.6	1.441	1.215
0.7	1.871	1.661
0.8	2.684	2.491
0.9	5.001	4.829
0.95	9.509	9.348
0.99	45.12	44.97
0.999	445.1	445.0
0.9999	4445.	4445.
1.	∞	∞

Finally, we may evaluate the average convective velocity from (14.30b) and (14.68). We have

$$\frac{\bar{v}}{v_S} = \frac{Q^{1/2}\alpha}{2\sqrt{2}\Gamma_1^{1/2}}(\nabla - \nabla')^{1/2} \tag{14.86a}$$

$$= \frac{Q^{1/2}\alpha}{2\sqrt{2}\Gamma_1^{1/2}}\frac{\Gamma}{A} \tag{14.86b}$$

$$= \frac{Q^{1/2}\alpha}{2\sqrt{2}\Gamma_1^{1/2}}\left(\frac{\nabla_r - \nabla_{\rm ad}}{a_0 A}\right)^{1/3}\zeta^{1/3}. \tag{14.86c}$$

We assume throughout this section that (14.86) always yield $\bar{v}/v_S \leq 1$; the case where they yield $\bar{v}/v_S > 1$ will be considered in Sect. 14.9.

We may also compute the value of $\nabla-\nabla_{ad}$, which is a measure of the superadiabaticity of the actual gradient. From (14.79) we see immediately that

$$\nabla-\nabla_{ad}=(1-\zeta)(\nabla_r-\nabla_{ad}). \tag{14.87}$$

Making use of (14.78), we may also write (14.87) in the form

$$\nabla-\nabla_{ad}=\frac{1+\Gamma}{a_0\Gamma^2}\zeta(\nabla_r-\nabla_{ad}). \tag{14.88}$$

Note that the ratio $(\nabla-\nabla_{ad})/(\nabla_r-\nabla_{ad})$ depends only on the convective efficiency (Γ, ζ, or B).

For $B\ll 1$ (inefficient convection), we have

$$\zeta\simeq a_0^3B^6=a_0A^4(\nabla_r-\nabla_{ad})^2\ll 1, \tag{14.89}$$

where we have taken only the first term in the expansion (14.83) for ζ. Using (14.89), expressions for some of the convective quantities may be written, in this limit, as follows:

Convective efficiency:

$$\Gamma=B\zeta^{1/3}\simeq A^2(\nabla_r-\nabla_{ad})\ll 1. \tag{14.90}$$

Ratio of convective to total flux:

$$\frac{F_c}{F}=\frac{\nabla_r-\nabla_{ad}}{\nabla_r}\zeta\simeq\frac{a_0A^4(\nabla_r-\nabla_{ad})^3}{\nabla_r}\ll 1. \tag{14.91}$$

Average convective velocity:

$$\frac{\bar{v}}{v_S}=\frac{Q^{1/2}\alpha}{2\sqrt{2}\Gamma_1^{1/2}}\frac{\Gamma}{A}\simeq\frac{Q^{1/2}\alpha}{2\sqrt{2}\Gamma_1^{1/2}}A(\nabla_r-\nabla_{ad}). \tag{14.92}$$

Note that in this limit $\bar{v}/v_S$ is not necessarily small compared to unity. If $\bar{v}/v_S$ is not small compared to unity, then we must have, from the first equality in (14.92), $A\lesssim\Gamma\ll 1$, whence, from the second equality in (14.92), $(\nabla_r-\nabla_{ad})\gg 1$. In this case, then, the "driving force" $(\nabla_r-\nabla_{ad})$ for the convection is very strong, but the "capacity" A for efficient convection is very small. Hence high convective velocities are necessary in this case for the required amount of energy to be transported by convection.

Superadiabaticity of the actual gradient:

$$\nabla-\nabla_{ad}=(1-\zeta)(\nabla_r-\nabla_{ad})\simeq\nabla_r-\nabla_{ad}. \tag{14.93}$$

This result merely states that $\nabla\simeq\nabla_r$ in the case of inefficient convection, as we expect. Also, $\nabla\simeq\nabla_{ad}$ only if $\nabla_r\simeq\nabla_{ad}$, *i.e.*, if the "driving force" for the convection is small.

In the other limiting case of large convective efficiency ($B \gg 1$, $\Gamma \gg 1$), we may simply set $\zeta = 1$ in the formulae or use the first term in the expansion (14.84) for $1-\zeta$ if more accuracy is required. Explicit expressions for some of the quantities of interest are as follows.

Convective efficiency:

$$\Gamma = B\zeta^{1/3} \simeq B = [(A^2/a_0)(\nabla_r - \nabla_{ad})]^{1/3} \gg 1 . \tag{14.94}$$

Ratio of convective to total flux:

$$F_c/F = [(\nabla_r - \nabla_{ad})/\nabla_r]\zeta \simeq 1 - (\nabla_{ad}/\nabla_r) . \tag{14.95}$$

Note that $F_c/F \simeq 1$ in this case only if $\nabla_r \gg \nabla_{ad}$ (*cf.* Sect. 14.5).

Average convective velocity:

$$\frac{\bar{v}}{v_S} = \frac{Q^{1/2}\alpha}{2\sqrt{2}\Gamma_1^{1/2}} \frac{\Gamma}{A} \simeq \frac{Q^{1/2}\alpha}{2\sqrt{2}\Gamma_1^{1/2}} \left[\frac{a_0(\nabla_r - \nabla_{ad})}{A}\right]^{1/3} . \tag{14.96}$$

We note from the first equality in (14.96) that this case ($\Gamma \gg 1$) must correspond to $A \gg 1$ (large "capacity" for efficient convection) if we require that all convective velocities be subsonic, since the factor multiplying Γ/A is normally of order unity. Because $A \gg 1$ in this case, it follows that, usually, $\bar{v}/v_S$ is considerably less than unity in regions of a convection zone where $A \gg 1$. In this case of high convective efficiency the convective velocity does not need to be very large in order that the required energy be transported by convection, because of the high "capacity" for efficient convection of the material.

Superadiabaticity of the actual gradient (see (14.88)):

$$\nabla - \nabla_{ad} = \frac{1+\Gamma}{a_0\Gamma^2}\zeta(\nabla_r - \nabla_{ad}) \simeq (\nabla_r - \nabla_{ad})/a_0\Gamma \ll \nabla_r - \nabla_{ad} . \tag{14.97}$$

Hence $\nabla \to \nabla_{ad}$ in this limit, as we expect.

Comparison of (14.89)–(14.92) with (14.94)–(14.96) shows explicitly that Γ, F_c/F, and $\bar{v}/v_S$ are much more sensitive to the value of the mixing length A for small convective efficiencies ($\Gamma \ll 1$) than for large ($\Gamma \gg 1$). (Recall that $A \propto A^2$, *cf.* (14.70).)

The results of this section are summarized in Fig. 14.3, which shows curves, all corresponding to the value $a_0 = (9/4)$, plotted on the log $(\nabla_r - \nabla_{ad})$ – log A plane.* The straight, parallel, diagonal lines are lines of

* A somewhat similar diagram has also been plotted by Baker [Ba63].

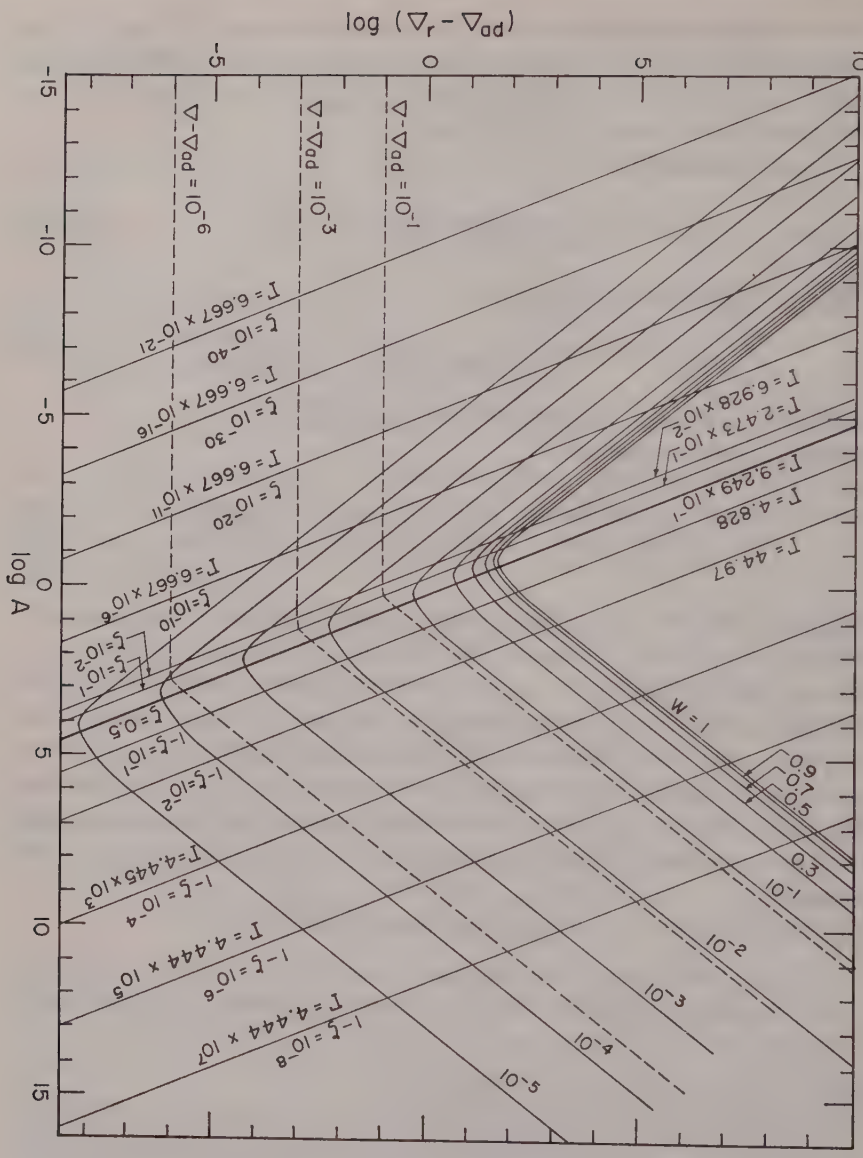


Fig. 14.3 Log $(\nabla_r - \nabla_{ad})$ versus log A.

constant convective efficiency (constant ζ, Γ, or B), labelled with the appropriate values of ζ (or $1-\zeta$) and Γ. The V-shaped lines are lines along which the quantity

$$\left\{\frac{[\Gamma_1/(5/3)]^{1/2}}{Q^{1/2}\alpha}\frac{\bar{v}}{v_S}\right\} \equiv w \qquad (14.97')$$

is constant; they are essentially lines of constant $\bar{v}/v_S$ and are labelled by the appropriate values of $w \simeq \bar{v}/v_S$. We regard as physically significant only that portion of the diagram lying below the curve $w = 1$ ($\bar{v}/v_S \simeq 1$). The diagonal line labelled $\zeta = 0.5$ or $\Gamma = 0.925$ (heavier than the other diagonal lines) approximately marks the transition region between ineffective convection (to the left of the heavy line) and effective convection (to the right). It is seen that, in accordance with our earlier remarks, regions of large convective efficiency are those with $A \gg 1$. Lines along which $\nabla - \nabla_{ad}$ is constant are also shown on the figure (horizontal on the left, bending upward on the right). The region lying below the line labelled $\nabla - \nabla_{ad} = 10^{-3}$ may be considered, for all practical purposes, as the region of "adiabatic" convection (negligible superadiabaticity of the actual gradient). In the region above this line the actual gradient may be superadiabatic to a greater or lesser extent. It is seen that the "transition region" between inefficient and efficient convection (say the region where $0.1 \lesssim \zeta \lesssim 0.9$) corresponds to a range of not quite two powers of ten in A, for given $\nabla_r - \nabla_{ad}$.

Again, the reader is referred to the paper by Henyey, Vardya, and Bodenheimer [He65] for a discussion of the effects on properties of convective envelope models of adjusting the values of the parameters entering into the mixing length theory.

Finally, it is instructive to examine the expression for A, the ratio of the "convective" to "radiative" conductivities. Comparing (14.70) with (14.29), we see that

$$A = \frac{4}{9} \frac{(Q^{1/2}/4\sqrt{2}\,\Gamma_1^{1/2})\rho v_S c_P T \alpha^2}{(4ac/3)(T^4 g/\kappa P)}, \tag{14.98}$$

where we have written $\alpha = \Lambda/\lambda_P$ for the mixing length-pressure scale height ratio. If the perfect gas law, $\rho = (\mu/\mathscr{R})P/T$, is applicable, we have, using (14.29) for v_S,

$$A = \frac{Q^{1/2}(\mu/\mathscr{R})^{1/2} c_P \kappa \alpha^2}{12\sqrt{2}\,ac} \frac{P^2}{g T^{3.5}}. \tag{14.99}$$

A simple, rough order-of-magnitude expression for A, valid in the deep interior of a star, will be obtained at the end of this section.

For purposes of numerical calculation we may also write the general expression (14.70) for A in the form

$$\begin{aligned} A &= \left[\frac{5}{24}\frac{G\mathscr{R}^{1/2}}{ac}\right]\left(\frac{\beta Q}{2\mu}\right)^{1/2}\left(\frac{2\mu c_P}{5\mathscr{R}}\right)\left(\frac{M_r}{r^2}\right)\left(\frac{\kappa p^2 \Lambda^2}{T^{3.5}}\right) \\ &= 0.56\left(\frac{\beta Q}{2\mu}\right)^{1/2}\left(\frac{2\mu c_P}{5\mathscr{R}}\right)\left(\frac{M_r}{r^2}\right)\left(\frac{\kappa p^2 \Lambda^2}{T^{3.5}}\right) \end{aligned} \tag{14.99'}$$

if all quantities are in c.g.s. units, where β is the ratio of gas to total pressure. For example, in the deep interior of a star like the sun we may take $\beta = 1$, $Q = 1$, $\mu = (1/2)$, $c_P = (5/2)(\mathscr{R}/\mu)$, $M_r \approx 1\times10^{33}$ gm, $r \approx 3\times10^{10}$ cm, $\kappa = 100$ cm^2/gm, $\rho = 100$ gm/cm^3, $T = 10^7$ °K, and $\Lambda = 7\times10^9$ cm, to obtain $A \approx 1\times10^{13}$. If $\nabla_r - \nabla_{ad} = 1$, we obtain from Fig. 14.3 or from the approximate formulae the values $1-\zeta \approx 10^{-9}$, $\Gamma \approx 4\times10^8$, $\bar{v}/v_S \approx 10^{-5}$, and $\nabla - \nabla_{ad} \approx 10^{-9}$. In this case the convection is clearly highly efficient and the temperature gradient is only negligibly superadiabatic. In the outer layers of a star like the sun we may take $\beta = 1$, $Q = 1$, $\mu = (1/2)$, $c_P = (5/2)(\mathscr{R}/\mu)$, $M_r = 2\times10^{33}$ gm, $r = 7\times10^{10}$ cm, $\kappa = 100$ cm^2/gm, $\rho = 10^{-6}$ gm/cm^3, $T = 10^4$ °K, and $\Lambda = 2\times10^7$ cm, to obtain $A \approx 1$. If $\nabla_r - \nabla_{ad} = 1$, we obtain from Fig. 14.3 the values $\zeta \approx 0.3$, $\Gamma \approx 0.5$, $\bar{v}/v_S \approx 0.1$, $\nabla - \nabla_{ad} \approx 1$. In this case the convection is somewhat inefficient and the temperature gradient is considerably superadiabatic.

Consider next the question of how A varies with depth in a stellar envelope. We shall neglect the variation of g with depth, as g does not generally vary as much in a stellar envelope as do some of the other factors in (14.99). We shall, in fact, consider only the dependence of A on P and T here and shall regard all other factors as constant just to emphasize the principal way in which A depends on depth. It will be shown in Chap. 20 that for a stellar envelope (radiative or convective) in hydrostatic equilibrium, the relation between P and T, at least below the region of hydrogen ionization, is approximately

$$P = KT^{n_e+1}, \tag{14.100'}$$

where n_e+1 nearly always lies in the range $2.5 \lesssim n_e+1 \lesssim 5$ (in regions where (14.100') is valid) and where K is a constant depending on the luminosity L, mass M, radius R, and chemical composition of the star. Since the exponent $2(n_e+1)-3.5$ of T which results when (14.100') is substituted into (14.99) is practically always greater than unity in such stellar envelopes, it follows that, aside from any effects resulting from variations in c_P, κ, or g with depth, A increases with increasing depth in a stellar envelope (below the region of hydrogen ionization).

The detailed dependence of A on the stellar parameters L, M, R, and chemical composition is complicated and will be considered further in Sect. 20.5, where another interpretation of A will be presented. We note here, however, that a simple, order-of-magnitude expression for A, valid in the deep interior and having the same accuracy as the order-of-magnitude expressions derived at the end of Sect. 14.6, can be obtained. From (14.86b) we see that $\bar{v}/v_S \sim \Gamma/A$, where all quantities are now regarded as representa-

tive, or mean, values for the deep stellar interior. Using (14.69c) and (14.69d′), it follows that

$$A \sim (\Delta T/T)^{-3/2} \tag{14.101}$$

or, since (see (14.69f)) $\Delta T/T \sim (t_{\rm ff}/t_{\rm K})^{2/3}$, that

$$A \sim (t_{\rm ff}/t_{\rm K})^{-1}. \tag{14.101′}$$

This expression yields the value $A \sim 10^{12}$ in the deep interior of a star similar to the sun, in order-of-magnitude agreement with our other computed values. (Another derivation of (14.101′) follows from the considerations in Sect. 20.5.)

14.8 Solution of the Equations When the Actual Gradient is Specified

In this section we assume that the actual gradient ∇ is specified at some point in a convection zone, and we wish to calculate the corresponding value of ∇_r, the fictitious radiative gradient. From this value of ∇_r follows the total flux $L(r)/4\pi r^2 = F_c(r) + F_r(r)$ at the point in question from (14.12). As in the preceding section, we assume that values of $\nabla_{\rm ad}$ and of the other physical variables c_P, κ, A, ρ, T, etc. are all known at this point. This case in which ∇ is specified is just the reverse of that considered in the preceding section, in which it was required to calculate the value of ∇ associated with a given value of ∇_r. The present case is appropriate, for example, to stellar model calculations of the A.N. Cox type (see Sect. 21.7, Sears and Brownlee [Se65], and A. Cox, Brownlee, and Eilers [Co66e]). We shall see that this case is, mathematically, much simpler than the case considered in the preceding section, as only a quadratic, rather than a cubic, equation in the basic unknown is involved in the present case.* Again we neglect turbulent pressure and assume that no energy sources are present ($\varepsilon = 0$) in the convecting region.

Again, the basic equations are (14.71), (14.72), and (14.73). Now, however, ∇ is regarded as a known quantity and the unknowns are ∇_r, ∇', and Γ (values of $\nabla_{\rm ad}$ and A are assumed known). Instead of working directly with (14.71)–(14.73), we shall instead work with the equivalent set of equations (14.71), (14.74), and (14.88), which we write down again for convenience:

$$\Gamma = A(\nabla - \nabla')^{1/2}, \tag{14.102}$$

* The quadratic equation (14.105) that we shall derive is essentially the same as the quadratic derived by Spiegel [Sp63].

$$\nabla_r - \nabla = a_0 \Gamma (\nabla - \nabla'), \tag{14.103}$$

$$\nabla - \nabla_{ad} = \frac{1+\Gamma}{a_0 \Gamma^2} (\nabla_r - \nabla), \tag{14.104}$$

where we have used the definition (14.78) to eliminate ζ from (14.104) and where a_0 would have the value (9/4) in the version of the mixing length theory we are using (Böhm-Vitense [Bö58]). We shall here regard the convective efficiency Γ as the basic unknown.

Equations (14.102)–(14.104) are readily seen to be a quadratic in Γ, which is (for $\Gamma \neq 0$)

$$\Gamma^2 + \Gamma - A^2(\nabla - \nabla_{ad}) = 0. \tag{14.105}$$

The solution of (14.105) of physical interest is

$$\Gamma = \frac{1}{2}\left[\sqrt{1 + 4A^2(\nabla - \nabla_{ad})} - 1\right]. \tag{14.106}$$

It should be noted that, as in the case of (14.82), (14.106) applies only when the g appearing in A (*cf.* (14.70)) is GM_r/r^2 and not g_e (*cf.* Sect. 14.5). Hence use of (14.106) implies that the relevant scale height being used here is λ_P and not λ_g.

It is clear that the quantity $A^2(\nabla - \nabla_{ad})$ depends only on Γ and may therefore be used as an efficiency parameter. Moreover, it is seen from (14.106) that $\Gamma \ll 1$ when $A^2(\nabla - \nabla_{ad}) \ll 1$ (small convective efficiency) and $\Gamma \gg 1$ when $A^2(\nabla - \nabla_{ad}) \gg 1$ (large convective efficiency).

The desired quantity ∇_r is obtained by substituting (14.106) for Γ into the equation which results when $(\nabla - \nabla')$ is eliminated between (14.102) and (14.103):

$$\nabla_r = \nabla + (a_0/8A^2)\left[\sqrt{1 + 4A^2(\nabla - \nabla_{ad})} - 1\right]^3. \tag{14.107}$$

The second term on the right side of (14.107) is proportional to the convective flux and we have, in fact,

$$F_c = \frac{4acT^4 g}{3\kappa P} \frac{a_0}{8A^2}\left[\sqrt{1 + 4A^2(\nabla - \nabla_{ad})} - 1\right]^3 \tag{14.108}$$

(see (14.32) and (14.33a)). The ratio of convective to total flux is given, as before, by the relation

$$\frac{F_c}{F} = \frac{\nabla_r - \nabla}{\nabla_r}. \tag{14.109}$$

The mean convective velocity is given by the relation (see (14.86b))

$$\frac{\bar{v}}{v_S} = \frac{Q^{1/2}\alpha}{2\sqrt{2}\Gamma_1^{1/2}} \frac{\Gamma}{A}$$

$$= \frac{Q^{1/2}\alpha}{2\sqrt{2}\Gamma_1^{1/2}} \cdot \frac{1}{2A}\left[\sqrt{1 + 4A^2(\nabla - \nabla_{ad})} - 1\right]. \tag{14.110}$$

In this section, as in the preceding one, we assume that the computed value of $\bar{v}/v_S$ will never exceed unity. The case in which it does is considered in Sect. 14.9.

Finally, we see from (14.102) that

$$\nabla - \nabla' = (\Gamma/A)^2 = (1/4A^2)\left[\sqrt{1 + 4A^2(\nabla - \nabla_{ad})} - 1\right]^2 \tag{14.111}$$

from which

$$f \equiv \frac{\nabla - \nabla'}{\nabla - \nabla_{ad}} = \frac{\left[\sqrt{1 + 4A^2(\nabla - \nabla_{ad})} - 1\right]^2}{4A^2(\nabla - \nabla_{ad})}. \tag{14.112}$$

We note that (14.110) for $\bar{v}/v_S$ and (14.108) for F_c may be obtained directly by using (14.111) for $\nabla - \nabla'$ in the basic equations

$$\frac{\bar{v}}{v_S} = \frac{Q^{1/2}\alpha}{2\sqrt{2}\Gamma_1^{1/2}} (\nabla - \nabla')^{1/2}, \tag{14.113}$$

$$F_c = \frac{4acT^4 g}{3\kappa P} a_0 A (\nabla - \nabla')^{3/2}. \tag{14.114}$$

Use of (14.112) therefore provides a simple method for taking heat losses due to radiation from the convective elements into account in a calculation in which these losses have been neglected (*i.e.*, in which $\nabla - \nabla_{ad}$ has been used in (14.113) and (14.114) instead of the correct factor $\nabla - \nabla'$): The recipe is simply to multiply the uncorrected value of $\bar{v}$ by $f^{(1/2)}$ and the uncorrected value of F_c by $f^{(3/2)}$: For $A^2(\nabla - \nabla_{ad}) \gg 1$ (large convective efficiency), we see from (14.112) that

$$f \simeq 1, \tag{14.115}$$

which shows that the correct factor $\nabla - \nabla'$ approaches the approximate factor $\nabla - \nabla_{ad}$ at large convective efficiencies. For $A^2(\nabla - \nabla_{ad}) \ll 1$ (small convective efficiency) we see from (14.112) that

$$f \ll 1, \tag{14.116}$$

which shows that use of the approximate factor $\nabla - \nabla_{ad}$ in (14.113) and (14.114) at small convective efficiencies can cause values of $\bar{v}/v_S$ and F_c to be

grossly overestimated. Again we emphasize that all results of this section are valid only if turbulent pressure is neglected; the equation for the basic unknown would otherwise be more complicated than a quadratic. If energy sources were not neglected, the equation in the basic unknown would still be a quadratic, but allowance for a greater number of solutions would have to be made.

Explicit expressions for Γ, $\bar{v}/v_S$, and F_c for $A^2(\nabla-\nabla_{ad})\ll 1$ (small convective efficiency) are as follows:

$$\Gamma \simeq A^2(\nabla-\nabla_{ad})\ll 1\,, \tag{14.117}$$

$$\frac{\bar{v}}{v_S} \simeq \frac{Q^{1/2}\alpha}{2\sqrt{2}\Gamma_1^{1/2}} A(\nabla-\nabla_{ad})\,, \tag{14.118}$$

$$F_c \simeq \frac{4acT^4g}{3\kappa P} a_0 A^4(\nabla-\nabla_{ad})^3\,. \tag{14.119}$$

For $A^2(\nabla-\nabla_{ad})\gg 1$ (large convective efficiency), we have

$$\Gamma \simeq A(\nabla-\nabla_{ad})^{1/2} \gg 1\,, \tag{14.120}$$

$$\frac{\bar{v}}{v_S} \simeq \frac{Q^{1/2}\alpha}{2\sqrt{2}\Gamma_1^{1/2}} (\nabla-\nabla)_{ad}^{1/2}\,, \tag{14.121}$$

$$F_c \simeq \frac{4acT^4g}{3\kappa P} a_0 A(\nabla-\nabla_{ad})^{3/2}\,. \tag{14.122}$$

The above equations (14.117)–(14.122) show explicitly that Γ, $\bar{v}/v_S$, and F_c are much more sensitive to the value α of the mixing length-pressure scale height ratio for small convective efficiencies ($\Gamma\ll 1$) than for large ($\Gamma\gg 1$), in agreement with a conclusion reached in the preceding section. (Recall that $A\propto\alpha^2$, see (14.99).)

14.9 *Solution of the Equations When Supersonic Convective Velocities Are Indicated*

We have assumed in all the preceding sections that the equations of the mixing length theory will always yield $\bar{v}/v_S\leq 1$. If they should indicate that $\bar{v}/v_S>1$ under certain circumstances, then the basic assumptions of the theory are probably invalid under these circumstances. Our whole picture of convective transfer would require extensive modification to take account properly of supersonic convective velocities, if they should actually occur.

In the absence of a detailed theory of supersonic convection, it seems plausible, within the framework of the mixing length theory, simply to make

the rather *ad hoc* (and possibly incorrect) assumption that v_S is an upper limit to the value of $\bar{v}$. Then, if the equations of the mixing length theory should yield values of $\bar{v} > v_S$ in some regions in a convection zone, we would simply set $\bar{v}/v_S = 1$. The solution of the equations of the mixing length theory can then be written down immediately.

First, from (14.86) for $\bar{v}/v_S$, we obtain, setting $\bar{v}/v_S = 1$,

$$(\nabla - \nabla')^{1/2} = \frac{2\sqrt{2}\,\Gamma_1^{1/2}}{Q^{1/2}\alpha}. \tag{14.123}$$

The convective efficiency is then given by (14.102):

$$\Gamma = A(\nabla - \nabla')^{1/2} = \frac{2\sqrt{2}\,\Gamma_1^{1/2} A}{Q^{1/2}\alpha}. \tag{14.124}$$

Next, the relation between ∇ and ∇_r is from (14.103)

$$\nabla_r - \nabla = a_0 A \left(\frac{2\sqrt{2}\,\Gamma_1^{1/2}}{Q^{1/2}\alpha}\right)^3, \tag{14.125}$$

where a_0 would have the value (9/4) in the version of the mixing length theory we are using [Bö58]. Equation (14.125) permits either one of ∇ or ∇_r to be evaluated in terms of the other.

The convective flux is given by the relation (see (14.114))

$$F_c = \frac{4acT^4 g a_0 A}{3\kappa P}\left(\frac{2\sqrt{2}\,\Gamma_1^{1/2}}{Q^{1/2}\alpha}\right)^3 \tag{14.126}$$

and the ratio of convective to total flux is given by

$$\frac{F_c}{F} = \frac{\nabla_r - \nabla}{\nabla_r}. \tag{14.127}$$

Note that (14.123)–(14.126) give upper limits to the values of the quantities on the left sides of the equations corresponding to our assumption that $\bar{v}/v_S \leq 1$.

Thus, assuming that $\bar{v}/v_S = 1$, the equations of the mixing length theory can be used, at least formally, to give values of the various convective quantities in regions of a convection zone where the solutions of the equations presented in Sects. 14.7 and 14.8 indicate supersonic convective velocities. Such values, of course, might be very unrealistic, both in view of the basic crudity of the mixing length theory and of the somewhat *ad hoc* hypothesis of setting $\bar{v}/v_S = 1$ in regions where the convective velocities might actually be supersonic and where the mixing length theory may not be applicable at all.

CHAPTER 15

Ionization of Material in Stellar Interiors

We have seen that the perfect gas equation of state may be written either as

$$P_g = nkT, \tag{15.1}$$

where the subscript g means "gas", k is Boltzmann's constant, and n is the total number of free particles per unit volume; or as

$$P_g = (\mathscr{R}/\mu)\rho T, \tag{15.2}$$

where $\mathscr{R}$ is the gas constant per mole, μ is the mean molecular weight (mass per mole of free particles), and ρ is the mass density. We have shown (*cf.* Sects. 10.5 and 10.6) that (15.1) applies to a gas made up of any kind of material particles which are non-interacting and non-degenerate, whether they are relativistic or not. In the case of relativistic particles we shall understand ρ in (15.2) to mean *not* the actual density, but the product of n and the average *rest mass* per free particle; hence ρ in (15.2) is the density which the gas would have if the relativistic increase in mass with increasing energy were not present. With this interpretation of ρ in (15.2), the mean molecular weight is independent of relativistic effects (the case of equilibrium among electron-positron pairs is discussed in Sect. 24.9). Departures of the equation of state from the perfect gas law form (15.1) or (15.2) in the case of an ionized gas will be discussed in Sect. 15.5b.

It follows from the definition of a mole that μ, the total (rest) mass per mole of free particles, is also equal to the average (rest) mass, in *atomic mass units* (AMU's), per free particle. (The AMU is defined as $M(O^{16})/16$ or as $M(C^{12})/12$ (*cf. NBS Technical News Bulletin*, Oct., 1963), where the M's stand for the rest masses of the neutral atoms O^{16} or C^{12}. We have the identity $1\,\text{AMU} = 1/N_0$, where N_0 is Avogadro's number, and the approximation $1\,\text{AMU} \simeq M(H^1)$, the rest mass of the hydrogen atom.) Hence μ

depends on the number of free particles contained in a fixed (rest) mass of the material.* It follows that the value of μ is affected by processes such as dissociation, ionization, and nuclear reactions, which can alter the number of free particles in a given mass of the gas.

In Sect. 15.1 we shall obtain some general expressions for μ and for the electron density n_e in terms of the relative mass abundances of the respective elements present in the gas and of their states of ionization. We shall then apply these expressions for μ to the special case of complete ionization, which is perhaps the case of greatest interest for the deep interior of a star; and to the case where all atoms are neutral and perhaps combined into molecules, which case is of interest in sufficiently cool stellar atmospheres. The case of partial ionization will be discussed in Sect. 15.3. In Sect. 15.2 we shall apply the general expression for n_e to the two special cases of complete and zero ionization. The excitation and ionization energy of the material will be discussed briefly in Sect. 15.4. Some approximate corrections which are applied in modern equation of state and opacity calculations for electrostatic interactions among the charged particles are discussed in Sect. 15.5. Finally, in Sect. 15.6 numerical results will be presented of recent detailed calculations of the mean molecular weight and the excitation and ionization energy, as functions of density and temperature, for a particular, astrophysically interesting chemical composition.

15.1 Mean Molecular Weight

The most general method of computing the mean molecular weight μ for a mixture of different types of particles of arbitrary nature is to make use of the general definition of μ as the average (rest) mass, in AMU's, per free particle. Let

$$n = \sum_k n_k \tag{15.3}$$

be the total number density of free particles of all types, where n_k is the number density of particles of type k and the summation is extended over all types of particles present in the mixture. The total density of the mixture is then

$$\rho = \sum_k m_k n_k, \tag{15.4}$$

* By "fixed rest mass of material" we mean here a fixed value of the product of the number of free particles in the system and the average *rest* mass per free particle. This average rest mass per free particle is to be computed as if each particle were isolated and not in the presence of any external force fields, which could also affect its mass through the relativistic equivalence between mass and energy.

where m_k is the mass per particle of type k. If we let

$$A_k \equiv m_k/H \tag{15.5}$$

denote the mass, in AMU'S, per particle of type k, where $H = 1/N_0$ is the mass per AMU, then we have from the general definition of μ

$$\mu = (\rho/H)/n = \sum_k A_k n_k / \sum_k n_k . \tag{15.6}$$

We may also express the number density n_k of the k^{th} type of particle in terms of the *relative mass abundance* x_k of the k^{th} type of particle; *i.e.*, x_k is the *number of grams of particles of type k per gram of the mixture*. The relation between x_k and n_k is clearly

$$n_k = \rho x_k / H A_k . \tag{15.7}$$

It is also obviously true that

$$\sum_k x_k = 1 , \tag{15.8}$$

where the summation is extended over all types of particles present in the mixture. Hence (15.6) may also be written in the form

$$\mu = 1/\sum_k (x_k/A_k) . \tag{15.9}$$

Equations (15.6) and (15.9) are exact and perfectly general.

A case of particular interest in astrophysics is where some of the free particles are electrons which have been released by the ionization of some of the other particles present (usually atoms and possibly molecules, at sufficiently low temperatures). In this case we write

$$n = n_e + \sum_i n_i , \tag{15.10}$$

where n_e is the number density of such "ionization electrons"* and where now we use the subscript i to denote particles other than electrons. Let $v_e(i)$ be the number of free electrons that have been contributed by one average particle of type i; then we have, using (15.7),

$$n_{e_i} = \sum_i v_e(i)\, n_i = \frac{\rho}{H} \sum_i v_e(i) \frac{x_i}{A_i} , \tag{15.11}$$

* We shall use the term "ionization electrons" to mean those free electrons which have been added to the mixture as a result of ionization-type processes. The term does not apply, for example, to free electrons which are produced by electron-positron pair formation from photons (*cf.* Sect. 24.9). In this chapter we consider specifically only ionization electrons.

where the summation is now to be extended over all kinds of particles present *except electrons.** We then have, using (15.10),

$$n = \frac{\rho}{H}\sum_i [1+v_e(i)]\frac{x_i}{A_i} = \frac{\rho}{H}\sum_i \bar{n}_i x_i, \tag{15.12}$$

where

$$\bar{n}_i \equiv [1+v_e(i)]/A_i \tag{15.13}$$

is what Chandrasekhar [Ch39, p. 254] has called the "mean ionization per unit atomic weight"; this is the total number of free particles per unit atomic mass contributed by particles of type i of atomic weight A_i. Using (15.12) in (15.6), we obtain

$$\mu = 1/\sum_i \bar{n}_i x_i. \tag{15.14}$$

We note that the left side of (15.8) will be slightly less than unity if the summation is extended over all types of particles *except* electrons (as is usually done), because of the neglect of the electron rest mass.

If the material is completely ionized and hence also completely dissociated (which is usually a good approximation in the deep interior of a star), then the number of free particles released by an atom of atomic number Z_i and atomic weight A_i is Z_i+1, whence

$$\bar{n}_i = (Z_i+1)/A_i. \tag{15.15}$$

For example, for completely ionized hydrogen we have $\bar{n}_{\rm H} = (2/1.008) = 1.984 \simeq 2$. For completely ionized helium we have $\bar{n}_{\rm He} = (3/4.004) = 0.7492 \simeq 0.75$. For all the heavier elements (assumed completely ionized) a rough approximation is $A_i \simeq 2Z_i$, so that $\bar{n}_i \simeq 1/2$ $(Z_i > 2)$. Thus, defining X, Y, and Z: $X \equiv x(\mathrm{H}^1)$, $Y \equiv x(\mathrm{He}^4)$, $Z = 1-X-Y$, we have

$$\mu = \frac{1}{\bar{n}_{\rm H}X + \bar{n}_{\rm He}Y + \bar{n}_{\rm Z}Z}, \tag{15.16}$$

or

$$\mu \simeq \frac{1}{2X+(3/4)Y+(1/2)Z} \tag{15.17a}$$

$$\simeq \frac{2}{1+3X+(1/2)Y}. \tag{15.17b}$$

It should be emphasized that (15.14) or (15.16), with $\bar{n}_i$ given by (15.15) (using accurate values of A_i), is a more accurate expression for evaluating μ for completely ionized material than (15.17).

* In case some of the particles of type i have captured extra electrons and become negative ions, $v_e(i)$ could in principle be slightly negative under certain conditions.

We see that μ is rather insensitive to the detailed distribution of heavy elements (atomic number greater than two) within Z. If a star is composed of pure hydrogen, *i.e.*, if $X = 1$, $Y = 0$, then $\mu \simeq (1/2)$. If a star is composed of pure helium, *i.e.*, if $X = 0$, $Y = 1$, we have $\mu \simeq (4/3)$. If a star contains no hydrogen or helium, *i.e.*, if $X = Y = 0$, we have $\mu \simeq 2$. Thus, in the present approximation of complete ionization we always have

$$(1/2) \lesssim \mu \lesssim 2 \tag{15.18}$$

in stellar interiors.

In the opposite extreme of completely *unionized* material we have

$$\bar{n}_i = 1/A_i \tag{15.19}$$

and

$$\frac{1}{\mu} = \left\langle \frac{1}{A} \right\rangle = \sum_i \frac{x_i}{A_i}. \tag{15.20}$$

This case obtains in sufficiently cool stellar atmospheres. If some of the atoms have combined into molecules, then the x_i and A_i in (15.20) for these molecules would be, respectively, their relative mass abundances and their molecular weights (sum of the atomic weights of the constituents for each type of molecule).

In the intermediate case of partial ionization, of interest for stellar envelopes,* we have (assuming that no negative ions have formed)

$$1/A_i \leq \bar{n}_i \leq (Z_i + 1)/A_i. \tag{15.21}$$

The exact value of $\bar{n}_i$, as a function of density ρ, temperature T, and chemical composition, can be computed by repeated use of the Saha ionization equation in the manner to be described in Sect. 15.3. Hence μ can be computed as a function of ρ, T, and chemical composition by use of the general expression (15.14). Numerical values of μ will be presented for a particular chemical composition in Sect. 15.6.

15.2 Electron Density

The expression (15.11) for the ionization electron density n_e can also be written in terms of $\bar{n}_i$, the mean ionization per unit atomic weight, by use of (15.13). We obtain

$$n_e = \frac{\rho}{H} \sum_i \frac{x_i}{A_i} (A_i \bar{n}_i - 1), \tag{15.22}$$

* By "stellar envelope" we mean those outer stellar layers whose mass is large compared to the mass lying above the photosphere but small compared to the mass of the whole star.

where the summation is extended over all types of particles present except electrons.

The maximum possible value that n_e can have corresponds to the case of complete ionization, in which $\bar{n}_i$ is given by (15.15). We have in this case

$$n_e = \frac{\rho}{H}\sum_i \frac{x_i Z_i}{A_i} = \frac{\rho}{H}\left\langle \frac{Z}{A} \right\rangle, \tag{15.23}$$

where the mean value $\langle Z/A \rangle$ of Z_i/A_i is defined by the second equality in (15.23). An approximation to (15.23) is obtained by letting X and Y denote the relative mass abundances of hydrogen and helium, respectively, taking $A(\mathrm{H}^1) \simeq 1$, $A(\mathrm{He}^4) \simeq 4$, and using the approximation $A_i \simeq 2Z_i$ for $Z_i > 2$. We obtain

$$n_e \simeq (1/2)\frac{\rho}{H}(1+X). \tag{15.24}$$

Thus, in this approximation n_e depends on the chemical composition only as far as the abundance of *hydrogen* is concerned.

In the case of *zero* ionization we have $\bar{n}_i = 1/A_i$, and (15.22) gives $n_e = 0$, as we expect.

15.3 Calculation of $\bar{n}_i$

In this section we shall describe a general procedure for calculating $\bar{n}_i$ as a function, say, of density ρ, temperature T, and chemical composition. Once values are known for $\bar{n}_i$, values of μ and n_e can be obtained from (15.14) and (15.22).

We first define $y_r^{(i)}$, the degree of r^{th} ionization of particles of type i (an r-times ionized particle is one which has lost r electrons):

$$y_r^{(i)} = n_r^{(i)} \Big/ \sum_{s=-q(i)}^{Z(i)} n_s^{(i)}, \tag{15.25}$$

where $n_r^{(i)}$ is the number density of r-times ionized particles of type i, the denominator is the number density of particles of type i in all possible stages if ionization, $Z(i)$ is the total number of electrons which a neutral particle of type i is capable of losing, and $q(i)$ is the maximum possible number of extra bound electrons that can be possessed by a negative ion formed from particles of type i. If no negative ions can be formed from particles of type i (*i.e.*, if the binding energy of all negative ions is negative), then $q(i) = 0$ (*cf.* also Sect. 9.18).

We then see that $v_e(i)$, the number of free electrons contributed by ionization per average particle of type i, is given by the relation

$$v_e(i) = \sum_{r=-q(i)}^{Z(i)} r y_r^{(i)}. \tag{15.26}$$

We may then write (*cf.* (15.13))

$$\bar{n}_i = A_i^{-1}\left[1 + \sum_{r=-q(i)}^{Z(i)} r y_r^{(i)}\right], \tag{15.27}$$

which shows explicitly how $\bar{n}_i$ depends on the state of ionization of the constituents of the mixture.

We note that the expression (15.25) for the degree of r^{th} ionization $y_r^{(i)}$ can be written in the form

$$\frac{1}{y_r^{(i)}} = \frac{n_{-q(i)}}{n_{-q(i)+1}} \cdot \frac{n_{-q(i)+1}}{n_{-q(i)+2}} \cdots \frac{n_{r-1}}{n_r} + \frac{n_{-q(i)+1}}{n_{-q(i)+2}} \cdot \frac{n_{-q(i)+2}}{n_{-q(i)+3}} \cdots \frac{n_{r-1}}{n_r} + \cdots + 1$$

$$+ \frac{n_{r+1}}{n_r} + \frac{n_{r+2}}{n_{r+1}} \frac{n_{r+1}}{n_r} + \cdots + \frac{n_{Z(i)}}{n_{Z(i)-1}} \frac{n_{Z(i)-1}}{n_{Z(i)-2}} \cdots \frac{n_{r+1}}{n_r}, \tag{15.28}$$

where we have omitted the superscripts (i) from all the n's for simplicity in writing. If we assume that the mixture is in chemical equilibrium (*cf.* Sect. 9.12), then the ratios $n_{r+1}^{(i)}/n_r^{(i)}$ of numbers of particles in successive stages of ionization are given by the Saha equation. For generality, we shall assume that the electron gas is partially degenerate and either relativistic or non-relativistic; the special case of a non-degenerate, non-relativistic electron gas is easily treated as a special case of the general procedures we are about to describe. We accordingly write the Saha equation (3.27) in the form

$$\left(\frac{n_{r+1}^{(i)}}{n_r^{(i)}}\right) e^{\eta} = \frac{B_{r+1}^{(i)}}{B_r^{(i)}} \left(\frac{m_{r+1}^{(i)}}{m_r^{(i)}}\right)^{3/2} \exp\left(-\chi_r^{(i)}/kT\right), \tag{15.30}$$

where η is the degeneracy parameter for the free electron gas. In (15.30) the B's denote partition functions, the m's denote the masses of the particles in the stages of ionization indicated, and $\chi_r^{(i)}$ (always assumed positive) is the energy required to remove an electron from the ground state of an r-times ionized particle. Any depression of the continuum or other electrostatic interaction effects are presumed taken into account in $\chi_r^{(i)}$ (see Sect. 15.5). If such interaction effects are taken into account, then $\chi_r^{(i)}$ will in general depend on both n_e and T, among other things (see step (2) in the paragraph after next). Equation (15.30) was derived in Sect. 3.4c, assuming the "ions"

to be non-degenerate and non-relativistic. There is a one-to-one relation between η and the electron density n_e, involving T as a parameter (*cf.* (3.24c)), so that specification of η and T is equivalent to specification of n_e and T. We note that, in general, the B's in (15.30) may depend, among other things, on the particle density (n_e or $\sum_i n_i$) as well as on T, since the number of terms in the summations is determined either by the depression of the continuum or, if the degree of ionization of the gas is very small, by the density $\sum_i n_i$ of atoms and ions. This dependence must be taken into account in a careful calculation. Because the various ratios in (15.28) thus depend, among other things, on n_e (or η) as well as on T, it follows that the $y_r^{(i)}$ also depend on n_e (or η) and T. Given values of n_e (or η) and T, then, the $y_r^{(i)}$ can be calculated from (15.28) and $\bar{n}_i$ from (15.27). Hence μ can also be evaluated from (15.14).

The more usual situation, however, is for ρ and T, instead of n_e (or η) and T, to be specified for a given mixture. In this case the procedure for calculating $\bar{n}_i$ is more elaborate than the procedure described in the preceding paragraph, but no different in principle.

The steps might be as follows:

(1) First select an initial, trial value for n_e and compute the corresponding value of the degeneracy parameter $\eta(n_e,T)$. (2) Using (15.30), evaluate all the relevant ratios of the n's appearing in (15.28) and hence obtain values for all the $y_r^{(i)}$. In case the electrostatic interaction effects (*cf.* Sect. 15.5) are being taken into account, these ratios of the n's, and hence the $y_r^{(i)}$, will depend not only on T, but also on suitable averages of the $v_e(i)$ (*cf.*, for example, (15.45) and (15.49)), which in turn depend on the $y_r^{(i)}$ (*cf.* (15.26)). Since the interaction effects are often small, however, an iterative procedure at this stage may be successful. A first approximation to the values of the $y_r^{(i)}$ may be obtained, for example, by ignoring the interaction effects. Using these values of the $y_r^{(i)}$, the $v_e(i)$, and hence the relevant averages, can be evaluated, and a first approximation to the values of the quantities determining the interaction effects is obtained. Improved values of the $y_r^{(i)}$ can then be calculated, which can be used to obtain improved values of the quantities determining the interaction effects, and so on. The iteration would be continued until two successive values of each of the relevant averages entering into the interaction effects were equal to within some accuracy criterion. Values of the $y_r^{(i)}$, with interaction effects taken into account, would then clearly be available. (3) Using these values of the $y_r^{(i)}$, evaluate the $\bar{n}_i$ for all the different types of particle from (15.27). (4) Using these values of the $\bar{n}_i$, compute n_e from (15.22). This value of n_e will generally differ from the initially assumed trial value, which indicates that the initial value was

incorrect. (5) Use the new value of n_e to obtain an improved value, insert this into step (1), and iterate until a sufficiently accurate value of n_e corresponding to the given values of ρ and T is obtained. Once the value of n_e is known, the corresponding values of $y_r^{(i)}$, $\bar{n}_i$, and μ are obtained as in the preceding paragraph.

If the free electrons are non-degenerate and non-relativistic, we have $e^{\eta} \simeq (1/2)n_e h^3/(2\pi m_e kT)^{(3/2)}$ (*cf.* (3.28)), and the Saha equation (15.29) may be written in the more usual form

$$\frac{n_{r+1}^{(i)} n_e}{n_r^{(i)}} = \frac{2B_{r+1}^{(i)}}{B_r^{(i)}} \frac{(2\pi mkT)^{(3/2)}}{h^3} e^{-\chi_r^{(i)}/kT}, \tag{15.33}$$

where $m = m_e m_r/(m_e+m_r) \simeq m_e$ (electron rest mass). The computational procedure in this case differs from that described in the preceding paragraph only in that it is now not necessary to compute η in step (1) of that procedure.

15.4 Excitation and Ionization Energy

By the *excitation and ionization energy* E_I (per unit mass, say) we shall mean the internal energy per unit mass stored in the form of excitation and ionization energy, with the zero point defined by the state in which all atoms are completely unionized and in their ground states. The *total* internal energy per unit mass would then be given by the sum of the kinetic energy per unit mass and E_I and whatever other forms of internal energy are being taken into account. For example, if the density and temperature ranges were such that molecules could form, a term would have to be included in the total internal energy to take the dissociation energy into account. Also, a term representing the radiant energy could be included.

Assuming the populations of the excited levels of the ions to be given by the Boltzmann equation (*cf.* (3.17)), we clearly have

$$E_I = \frac{1}{H}\sum_i \frac{x_i}{A_i}\left[\sum_{m=1}^{Z(i)} y_m^{(i)} \sum_{r=1}^{m} \chi_{r-1}^{(i)} - \sum_{m=-q(i)}^{-1} y_m^{(i)} \sum_{r=-q(i)}^{m} \chi_{r-1}^{(i)}\right]$$

$$+ \frac{1}{H}\sum_i \frac{x_i}{A_i} \sum_{r=-q(i)}^{Z(i)} \frac{y_r^{(i)}}{B_r^{(i)}} \sum_j \phi_{r,j}^{(i)} g_{r,j}^{(i)} \exp(-\phi_{r,j}^{(i)}/kT) \tag{15.34}$$

(*cf.* (9.140) and (10.60)), where the first term on the right (the one containing the square brackets) is the ionization energy and the second term is the excitation energy, both per unit mass. Because of our choice of zero point,

any negative ions that may be formed make a *negative* contribution to E_I; this negative contribution is represented by the second term in square brackets. The summations over i are to be taken over all elements present in the mixture. In the second term $\phi_{r,j}^{(i)}$ is the excitation energy *above the ground state* of the j^{th} excited level of the r-times ionized atom of type i, $g_{r,j}^{(i)}$ is the corresponding statistical weight of the level j, and the summation over j is to be taken over all relevant excited levels. The $\chi_r^{(i)}$ and the $\phi_{r,j}^{(i)}$ may include electrostatic interaction effects due to surrounding particles.* The remaining symbols in (15.34) have the same meanings as in Sect. 15.3. Hence, once the $y_r^{(i)}$ are all known (*cf.* Sect. 15.3), E_I can be computed as a function of ρ and T for the given mixture, assuming that values of the relevant statistical weights, excitation and ionization potentials, etc., are available.

For equation of state calculations in which molecules may be present see, for example, Vardya [Va65].

15.5 Electrostatic Corrections

In many cases the electrostatic interactions among the charged particles in a plasma may be neglected. In accurate equation of state and opacity calculations, however, such as those whose results are presented in Sects. 15.6 and 16.9, approximate corrections for some of these interactions are applied.

The principal effect of these electrostatic interactions on the internal states of atoms and ions is to produce a *broadening* and a *shift* of atomic energy levels, the effect becoming larger, the closer the level is to the "continuum." The higher levels (*i.e.*, those with large principal quantum numbers), in fact, become smeared together and the result may be described as a "depression of the continuum." The exact extent of the depression may depend, however, on the nature of the particular problem at hand. For example, the depression which determines where the sums in atomic partition functions (*cf.* (3.25)) are terminated (because levels lying above the depressed continuum correspond essentially to "free" particle states) may differ from that which enters into the Saha ionization equation. The whole matter of these electrostatic corrections has been discussed from a unified quantum

* If, however, the $\chi_r^{(i)}$ have been reduced by the electrostatic energy E_0 (see Sect. 15.5a), then a term which gives each free electron the additional energy E_0 should be included to give the *total* internal energy, since E_0 is also the extent to which the energy zero-point for electrons is lowered by depression of the continuum. It would then appear that the *total* internal energy should be computed just as if the $\chi_r^{(i)}$ had not been reduced by E_0. However, opinions are divided on the quantitative methods of dealing with these electrostatic corrections. Approximate expressions for the (small) electrostatic interaction energy (due to charges of both signs) are derived in Sect. 15.5b (see (15.62) and (15.77)).

mechanical standpoint by Ecker and Kroll [Ec65]; see also the review article on opacity and equation of state calculations by A.N. Cox [Co65]. The depression of the continuum will be discussed in Sect. 15.5a.

The electrostatic interactions may also produce a deviation of the pressure from the value given by the perfect gas law. This effect will be discussed in Sect. 15.5b.

There is at present considerable disagreement among workers in the field concerning the quantitative calculation of these electrostatic interactions (see Cooper [Co66b]). Our discussion in Sect. 15.5a is based largely on the work of Ecker and Kroll and of A.N. Cox. Particularly controversial results will be indicated in the appropriate places.

15.5a Depression of the Continuum

As far as the *termination of atomic partition function sums* is concerned, the extent of the depression of the continuum will be called E_0. Thus, if $I_{r,j}^{(i)}$ denotes the energy difference (taken as a positive number) between the *undepressed* continuum and the energy of an electron in the level j of an r-times ionized atom of type i (*i.e.*, $I_{r,j}^{(i)}$ would be the energy required to remove an electron from the level j of the r-times ionized atom of type i, *uncorrected* for depression of the continuum), then one writes

$$\chi_{r,j}^{(i)} = I_{r,j}^{(i)} - E_0 \tag{15.35}$$

for the *actual* energy difference between the *depressed* continuum and the energy of an electron in the level j of the atom. The energy $I_{r,j}^{(i)}$ may itself be corrected for the effects of screening of the nuclear charge by other bound electrons in the atom and of the interaction of the bound electron with other free electrons (*cf.* A.N. Cox [Co65, equation (96)]). This second effect may produce both a broadening and a shift of the level j. If we take the beginning of the depressed continuum as the energy zero point for the electrons, then the total energy of an electron in the state j of an r-times ionized atom of type i is

$$\varepsilon_{r,j}^{(i)} = -\chi_{r,j}^{(i)} = -I_{r,j}^{(i)} + E_0 . \tag{15.36}$$

Thus we may say that the positive energy E_0 is being added to the energy of each bound electron in the atom. Any electron for which $\varepsilon_{r,j}^{(i)} \geq 0$ would be regarded as free, at least in the present context.

The origin of the depression of the continuum may be viewed in the following qualitative way. If there are "free" electrons in the vicinity of an atom or ion, then an electron (free or bound) in the field of the nuclear charge will experience not only the attractive electrostatic force of the nuclear charge (perhaps as modified by screening due to bound electrons), but also the

repulsive electrostatic force of these free electrons. This repulsive force acts as a *positive* potential (for electrons) and tends to counterbalance the *negative* potential due to the nuclear charge. The effect is therefore essentially to add the positive energy E_0 to the energy of each electron in the atom. It is clear that with the beginning of the depressed continuum taken as the energy zero point for the electrons, the effects on the "free" electrons of the electrostatic repulsion by other free electrons are automatically taken into account (however, see footnote in Sect. 15.4). One might say that this positive (repulsive) potential energy of a free electron is just cancelled, on the average, by the negative potential energy of the nuclear charges (however, see Sect. 15.5b).

All partition function sums should be cut off at the highest level for which $\varepsilon_{r,j}^{(i)} < 0$, according to Ecker and Kroll [Ec65]. In the A.N. Cox [Co65] calculations, on which the numerical results presented in Sects. 15.6 and 16.9 are based, the partition function sums are cut off at the continuum depressed not by E_0, but by $E_0 + E_{s,r}$ (see (15.47)). In case most atoms are neutral, E_0 is very small (because of the scarcity of free electrons) and the partition function sums should then be cut off when the orbital radius of a bound electron becomes approximately equal to the mean distance between atoms (A.N. Cox [Co65]).

The energy E_0 has been calculated on the basis of a number of simplified models (see Cooper [Co66b] for a summary and references to the literature). According to Ecker and Kroll [Ec65], E_0 for a fully ionized, pure hydrogen plasma is given by

$$E_0 \simeq \frac{e^2}{2r_0}, \tag{15.37}$$

where e is the electronic charge and r_0 is the radius of a sphere which contains, on the average, just one ion. Note that E_0 is of the order of the electrostatic potential energy of a uniformly charged sphere of radius r_0 having total charge e (for a uniformly charged sphere the factor would be (3/5) instead of (1/2)). E_0 is then evidently a measure of the positive (repulsive) electrostatic potential energy of the free electrons interacting with themselves.

To deal with the case of a mixture of different elements, some of which may be only partially ionized, we follow A.N. Cox [Co65] and define r_i, the "radius" of an atom or ion of type i, as follows. The volume $V_i = (4/3)\pi r_i^3$ contains, on the average, just one nucleus of nuclear charge $Z_i e$ and would also contain just Z_i free ionization electrons if all atoms were completely ionized and if the electron density were uniform. Even if the atoms are not completely ionized, the volume V_i is assumed to possess a total of just Z_i

electrons, some free and some bound, and therefore to be electrically neutral. It follows from the definition of r_i that

$$\frac{4}{3}\pi r_i^3 n_e^* = Z_i, \tag{15.38}$$

where

$$n_e^* = \frac{\rho}{H}\left\langle \frac{Z}{A} \right\rangle \tag{15.39}$$

is the electron density in the case of complete ionization (*cf.* (15.23)) and the average $\langle f \rangle$ is a mass fraction average:

$$\langle f \rangle \equiv \sum_i x_i f_i, \tag{15.40}$$

where the summation is extended over all types of atoms and ions present in the mixture. We then have

$$r_i = \left(\frac{3H}{4\pi\rho}\right)^{1/3} \frac{Z_i^{1/3}}{\langle Z/A \rangle^{1/3}} \tag{15.41a}$$

$$= 0.735 \times 10^{-8} \frac{Z_i^{1/3}}{\rho^{1/3}\langle Z/A \rangle^{1/3}} \text{ cm} \tag{15.41b}$$

if ρ is in gm/cm^3.

We now let $v_e(i)$ denote the number of the Z_i electrons in V_i that are "free" and we write, in analogy to (15.37), for the total (positive) electrostatic energy of these $v_e(i)$ negative charges in V_i

$$E_{es,i} = \alpha \frac{v_e^2(i)e^2}{r_i}, \tag{15.42}$$

where α is a numerical coefficient of order unity ($\alpha = (3/5)$ for a uniform charge density and $\alpha = (1/2)$ as recommended by Ecker and Kroll [Ec65]). Following A.N. Cox [Co65], we now average $E_{es,i}$ over all types of atoms and ions present and divide by $\bar{y}$, the average number of free ionization electrons per atom or ion of all types (the mean degree of ionization, *cf.* (9.133)). We then obtain an expression for E_0, the average electrostatic energy per free electron:

$$E_0 = \frac{1}{\bar{y}} \frac{\sum_i n_i E_{es,i}}{\sum_i n_i} = \alpha e^2 \frac{\sum_i n_i v_e^2(i)/r_i}{n_e}, \tag{15.43}$$

where n_i is the number density of atoms and ions of type i. Using (15.11) for n_e, writing $n_i = (\rho/H)x_i/A_i$, and using (15.41a) for r_i, we obtain

$$E_0 = \alpha e^2 (4\pi\rho/3H)^{1/3}\xi \tag{15.44a}$$

$$= 19.6\,\alpha\,\rho^{1/3}\xi \text{ ev} \tag{15.44b}$$

if ρ is in gm/cm^3, where

$$\xi \equiv \frac{\langle Z/A\rangle^{1/3}\langle v_e^2/AZ^{1/3}\rangle}{\langle v_e/A\rangle}, \tag{15.45}$$

and the averages in ξ are mass fraction averages (*cf.* (15.40)). In the case of complete ionization we have $v_e(i) = Z_i$; if, further, only a single type of atom is present ($x_i = 1$, $v_e(i) = Z_i \equiv Z$), we have $\xi = Z/A^{1/3}$.

We note that an expression for E_0 can also be derived by computing the potential energy of an electron at a distance equal to an average value $\langle r\rangle$ of r_i from an ion having the average charge $\langle v_e\rangle e$. This expression does not differ from (15.44) in any significant way.

In the case of the *ionization equation* (15.30), the results of Ecker and Kroll [Ec65] imply that the energy $\chi_r^{(i)}$ needed to remove an electron from the ground state of an r-times ionized atom (the effective ionization potential) is given, for $r_i < r_D$ (*cf.* (15.48)), by *

$$\chi_r^{(i)} = I_r^{(i)} - E_0 - E_{s,r}\,, \tag{15.46}$$

where $I_r^{(i)}$ is the ionization potential uncorrected for electrostatic effects, E_0 is given by (15.44)†, and

$$E_{s,r} \equiv \frac{(r+1)e^2}{r_D}. \tag{15.47}$$

Here r_D is the *Debye length*, given by

$$r_D = \sqrt{\frac{HkT}{4\pi e^2 \rho\zeta}} \tag{15.48a}$$

$$= 0.89 \times 10^{-8} \sqrt{\frac{T_6}{\rho\zeta}}\,\text{cm}\,, \tag{15.48b}$$

* See (15.56) and (15.57) for alternative expressions for $\chi_r^{(i)}$.

† More recently, Ecker and Kroll (see Cooper [Co66b]) have argued that the E_0 which appears in (15.46) is of the same order of magnitude as $E_{s,r}$, see the paragraph following (15.55).

where $T_6 = T(°K)/10^6$, ρ is in gm/cm^3 in (15.48b), and

$$\zeta \equiv \langle \nu_e(\nu_e+1)/A \rangle , \tag{15.49}$$

the average being a mass fraction average (*cf.* (15.40)).

Qualitatively, the additional term $E_{s,r}$ in (15.46) may be interpreted, approximately, as a shielding of the positive ionic charge by free electrons. This shielding decreases the effective ionic charge and results in a looser binding of an electron to an atom or ion. The details of the shielding process, for the case where the charge density is almost uniform, are worked out in Sect. 17.15. There it is shown that the potential energy of an electron in the field of a shielded positive point charge Ze is given by

$$V(r) = -\frac{Ze^2}{r} e^{-r/r_D} \tag{15.50}$$

$$\simeq -\frac{Ze^2}{r} + \frac{Ze^2}{r_D} \tag{15.51}$$

if $r \ll r_D$, where r_D is given by (15.48). (In (15.49) we have taken $\nu_e(i)e$ to be the effective charge of an ion of type i, where $\nu_e(i)$ is the number of free electrons contributed by an average atom or ion of type i.) The shielding thus has the effect of adding the energy

$$E_s = \frac{Ze^2}{r_D} \tag{15.52}$$

to the energy of an electron in the field of the positive charge, which results in a looser binding of the electron to this charge.

Equation (15.46) is not valid if $r_i > r_D$, for then there would be less than one ion, on the average, in a sphere of radius r_D, and the Debye length would then no longer have physical significance. Moreover, the assumption of almost uniform charge density underlying the theory of the Debye length is violated in this case. In this case of $r_i > r_D$, the implication is that only E_0, and not $E_{s,r}$, should remain in (15.46).

The critical case

$$r_i/r_D = 1 \tag{15.53}$$

defines a "critical density" $\rho_{\text{crit}}^{(i)}$ for ions of type i; this is a density such that each "Debye sphere" of radius r_D contains, on the average, just one ion of type i. We have from (15.41) and (15.48)

$$\frac{r_i}{r_D} = 0.83 \sqrt{\frac{\rho^{1/3}}{T_6}} \frac{\zeta^{1/2} Z_i^{1/3}}{\langle Z/A \rangle^{1/3}}, \tag{15.54}$$

where ρ is in gm/cm^3 and where the last factor on the right side will generally be of order unity. For the sun, for example, the factor $\rho^{1/3}/T_6$ is equal to a few tenths and is almost constant throughout the sun. We then have

$$\rho_{\text{crit}}^{(i)} = 3.1 \frac{\langle Z/A \rangle^2}{\zeta^3 Z_i^2} T_6^3 \text{ gm/cm}^3 . \tag{15.55}$$

The reader should be cautioned that the lowering of the ionization potential is a subject of considerable controversy at present and that the value given by (15.46) is not universally accepted by workers in the field. For example, it has been suggested by several authors (for example, Stewart and Pyatt [St66b]; Cooper [Co66b] and references given there) that (15.46) should be replaced by

$$\chi_r^{(i)} = I_r^{(i)} - E_{s,r} \tag{15.56}$$

for $r_i < r_D$ (or $\rho < \rho_{\text{crit}}^{(i)}$), and by

$$\chi_r^{(i)} = I_r^{(i)} - E_0 \tag{15.57}$$

for $r_i > r_D$ (or $\rho > \rho_{\text{crit}}^{(i)}$). According to this other viewpoint, then, the lowering of the ionization potential for the ionization equation is given, roughly speaking, by the *smaller* of E_0 and $E_{s,r}$. A final choice between these two viewpoints (as well as others*) may ultimately have to be based on experimental results (*cf.* Cooper [Co66b]).

In any event, whenever $\chi_r^{(i)}$ as computed by (15.46), (15.56), or (15.57) turns out to be zero or negative, then effectively *no* energy is required to remove the r^{th} electron from the (r-1)-times ionized atom and the atoms should be considered "completely ionized", *i.e.*, all at least r-times ionized. This state of "complete ionization" can be realized formally by setting the partition function B_{r-1} equal to zero in the Saha equation (15.30).

The numerical results presented in Sects. 15.6 and 16.9 are based on (15.46) rather than on (15.56) or (15.57).

15.5b Electrostatic Corrections to the Pressure Equation of State

We consider here corrections to the "perfect" gas pressure equation of state arising from the electrostatic interactions among the charges in a partially or fully ionized plasma. By " 'perfect' gas pressure equation of state" we mean here the gas pressure equation of state computed as if there were no electrostatic or other interactions among the particles, without regard to the kind of statistics obeyed by the particles. Thus, from the present

* A method for calculating pressure ionization at high densities has been described by Rouse [Ro64].

standpoint even the degenerate electron pressure equation of state (*cf.* Chap. 24), for example, would be regarded as "perfect" if interactions among the particles were not taken into account.

Since corrections to the "perfect" gas pressure equation of state over the whole range of densities, temperatures, and compositions of astrophysical interest have not yet been worked out, we consider here approximate electrostatic corrections in only two extreme cases. The first case corresponds to low densities and high temperatures, the second to high densities and low (strictly speaking, zero) temperatures. Our treatment in the first case is based on the Debye-Hückel theory (*cf.*, for example, Eddington [Ed26, Chap. 10] and Landau and Lifshitz [La58, Sect. 74]), which is conceptually simple but limited in scope. Our treatment in the second case is based on a paper by E.E. Salpeter [Sa61].

In neither case do we consider other factors which tend to make a gas non-perfect, such as the finite size of the particles, short-range intermolecular forces, and exchange effects (see, for example, Fowler and Guggenheim [Fo39, Chap. 7]; Landau and Lifshitz [La58, Chap. 7]; and Clarke and McChesney [Cl64] for discussions of general procedures).

We consider now the low density, high temperature case. The equations of the Debye-Hückel theory, for the case of an almost uniform charge density, are worked out in Sect. 17.15 and have already been applied in Sect. 15.5a (*cf.* (15.50) and (15.51)). According to this theory the potential of a shielded point charge Ze is given (*cf.* (17.242)) by

$$\phi(r) = \frac{Ze}{r} \cdot e^{-r/r_D} \tag{15.58}$$

$$\approx \frac{Ze}{r} - \frac{Ze}{r_D} \tag{15.59}$$

if $r \ll r_D$, where r_D is the Debye radius (*cf.* Sect. 17.15 and (15.48)). The first term on the right side of (15.59) is the potential at distance r from a bare charge Ze. The second term must therefore be the potential in the vicinity of the charge Ze due to the interaction of the charge Ze with all the other charges in the gas. The potential *energy* of the charge Ze due to electrostatic interaction with all the other charges is then

$$\varepsilon_{es} = -\frac{Z^2 e^2}{r_D}. \tag{15.60}$$

Note that (15.60) is approximately valid only as long as

$$\frac{|\varepsilon_{es}|}{kT} \ll 1, \tag{15.61}$$

i.e., only as long as $|\varepsilon_{es}|$ is small compared to the mean particle kinetic energy $\sim kT$. The requirement (15.61) is a consequence of the assumption of an almost uniform charge density (*cf.* Sect. 17.15).

We note that ε_{es} is negative regardless of the sign of the charge Ze (Z would here be (-1) for an electron). This fact may be interpreted as follows. Consider, for example, a particular particle of charge Ze in the plasma. As long as the temperature of the plasma is not infinite and the density is not zero (*i.e.*, r_D not infinite), this charge will "polarize" its immediate surroundings to some extent, producing an excess density of charge of the opposite sign around it (its "Debye sphere"). The radius of the charge cloud is approximately r_D, and the total electric charge within the cloud itself (not counting the particle charge Ze) is approximately $(-Ze)$. The original charge Ze and its charge cloud of total charge $\sim(-Ze)$ form a bound, almost electrically neutral system with *negative* energy, since positive work would be required to separate the cloud from its charge. Hence every charge, regardless of its sign, is "bound" to its charge sphere, and hence to the plasma as a whole. The "binding energy" of a charge Ze is, according to the present theory, the same as that of two point charges $(+Ze)$ and $(-Ze)$ a distance r_D apart. In the limit of infinite temperature or of vanishing density (*i.e.*, $r_D \to \infty$), the charge density in the plasma becomes uniform and $\varepsilon_{es} \to 0$.

We let $\sum_i N_{0i}$ and N_{0e} denote, respectively, the total numbers of ions and electrons in a system of volume V (zero subscripts mean that the N's are to be computed as if there were no interactions among the particles), and let $\nu_e(i)e$ denote the effective ionic charge of an average ion of type i (we would have $\nu_e(i) = Z_i$ in the case of complete ionization). The total electrostatic interaction energy of the whole plasma is then from (15.60)

$$E_{es} = -\frac{1}{2}\frac{e^2}{r_D}\left[\sum_i N_{0i}\nu_e^2(i) + N_{0e}\right], \tag{15.62}$$

where the factor (1/2) takes account of the fact that the electrostatic interaction energy of each particle has actually been counted twice in the summation (*cf.* Sect. 17.2). Writing the expression for r_D in the form (*cf.* Sect. 17.15)

$$r_D = \left\{\frac{kTV}{4\pi e^2\left[\sum_i N_{0i}\nu_e^2(i) + N_{0e}\right]}\right\}^{1/2}, \tag{15.63}$$

we obtain

$$E_{es} = -\frac{\pi^{1/2}e^3}{V^{1/2}(kT)^{1/2}}\left[\sum_i N_{0i}\nu_e^2(i) + N_{0e}\right]^{3/2}. \tag{15.64}$$

We note that (15.64) is approximately valid only if $|E_{es}|$ is small compared to the thermal kinetic energy $E_{th} \sim (3/2)\ (\sum_i N_{0i} + N_{0e})kT$, because of condition (15.61).

We can obtain the contribution F_{es} to the free energy of the plasma, corresponding to E_{es}, by integrating the thermodynamic identity (*cf.* (9.51))

$$E = \left[\frac{\partial(F/T)}{\partial(1/T)}\right]_{V,N}, \tag{15.65}$$

where the subscript N means that all particle numbers appearing in F are to be held fixed during the differentiation. Integrating (15.65), holding V and all the N's (and hence all the $v_e(i)$) constant during the integration, we obtain for the total gas free energy of the plasma

$$F_g = F_p - \frac{2}{3}\frac{\pi^{1/2}e^2}{(kT)^{1/2}V^{1/2}}\left[\sum_i N_{0i}v_e^2(i) + N_{0e}\right]^{3/2}, \tag{15.66}$$

where "g" stands for "gas" and F_p is the "perfect gas" contribution to F_g. We have set the "constant" of integration equal to zero so that there will be no electrostatic contribution to F_g in the limit of infinite temperature or volume, when $E_{es} \to 0$. The pressure equation of state can now be obtained from (15.66) by making use of the thermodynamic identity (*cf.* (9.47))

$$P = -(\partial F/\partial V)_{T,N}. \tag{15.67}$$

Before explicitly evaluating (15.67) for the pressure, let us note that (15.66), which was derived on the assumption of Maxwellian statistics (nondegenerate ions and electrons), can be generalized to apply to any kind of statistics, as Landau and Lifshitz [La58, Sect. 74] have shown. The appropriate generalization of (15.66) is

$$F_g = F_p - \frac{2}{3}\frac{\pi^{1/2}e^3}{V^{1/2}}\left[\sum_i N_{0i}v_e^2(i) + N_{0e}\right]\left[\sum_i v_e^2(i)\left(\frac{\partial N_{0i}}{\partial \mu_i}\right)_{V,T} + \left(\frac{\partial N_{0e}}{\partial \mu_e}\right)_{V,T}\right]^{1/2}, \tag{15.68}$$

where μ_i and μ_e are the chemical potentials for, respectively, ions of type i and electrons (*cf.* Sect. 9.12). The corresponding generalization of (15.63) for the Debye radius r_D for arbitrary statistics is

$$r_D = \left\{\frac{V}{4\pi e^2\left[\sum_i (\partial N_{0i}/\partial \mu_i)v_e^2(i) + \partial N_{0e}/\partial \mu_e\right]}\right\}^{1/2}. \tag{15.69}$$

If the electrons are arbitrarily degenerate, we must use the relation (*cf.* (3.8))

$$N_{0e} = V \cdot \int_0^\infty \frac{8\pi p^2 \mathrm{d}p/h^3}{e^{-\eta+\varepsilon(p)/kT}+1}, \tag{15.70}$$

where $\varepsilon(p)$ is the kinetic energy of an electron whose momentum is p, and where we have set $\eta \equiv \mu_e/kT$. We then have

$$\frac{\partial N_{0e}}{\partial \mu_e} = \frac{1}{kT}\frac{\partial N_{0e}}{\partial \eta} = \frac{V}{kT}\int_0^\infty \frac{8\pi p^2 \mathrm{d}p/h^3}{e^{-\eta+\varepsilon/kT}+1} \cdot \frac{1}{1+e^{\eta-\varepsilon/kT}} \leq \frac{N_{0e}}{kT}, \tag{15.71}$$

the equality sign applying to the non-degenerate ($-\eta \gg 1$) case. If the ions are non-degenerate, we clearly have $\partial N_{0i}/\partial \mu_i = N_{0i}/kT$. Degeneracy of the electron gas is thus seen to result in an increase in r_D, but not by a very large factor (unless the gas is nearly unionized, in which case the gas would probably be non-degenerate anyway). Clearly, (15.68) for F_g and (15.69) for r_D reduce to their non-degenerate forms (15.66) and (15.63) when $(-\eta) \to \infty$.

We shall evaluate (15.67) for the pressure only for the case where both the ions and the electrons are non-degenerate. In this case the differentiation of F_g as given by (15.66) is trivial, and we obtain form (15.67)

$$P_g = P_p - \frac{\pi^{1/2}e^3}{3(kT)^{1/2}V^{3/2}}\left[\sum_i N_{0i}v_e^2\,(i) + N_{0e}\right]^{3/2}, \tag{15.72}$$

where P_p is the "perfect gas" contribution to P_g (see the first paragraph of this subsection).

We note that the electrostatic interactions give a negative contribution to the pressure. Thus the plasma tends to "hold itself together," as is implied also in our interpretation of the sign of ε_{es} a few paragraphs back.

Letting $n_{0i} \equiv N_{0i}/V$ and $n_{0e} = N_{0e}/V$ denote particle number densities, we obtain

$$P_g = P_p - \frac{e^3}{3}\left(\frac{\pi}{kT}\right)^{1/2}\left[\sum_i n_{0i}v_e^2\,(i) + n_{0e}\right]^{3/2}. \tag{15.73}$$

Since the plasma is electrically neutral in the absence of any electrostatic potentials, we have $n_{0e} = \Sigma v_e(i) n_{0i}$, and the quantity in square brackets in (15.73) may be written in the form

$$\left[\sum n_{0i}v_e^2\,(i) + n_{0e}\right] = \sum n_{0i}v_e(i)\left[v_e(i)+1\right] = n\mu\zeta, \tag{15.74}$$

where $n \equiv \sum_i n_{0i} + n_{0e}$ is the total particle number density, μ is the mean molecular weight (*cf.* Sect. 15.1), and ζ is defined in (15.49) (the relations $n_{0i} = \rho x_i / A_i H$, $n = \rho/\mu H$ have been used in the last equality in (15.74)).

We write the non-degenerate perfect gas contribution to P_g in the form

$$P_p = nkT. \tag{15.75}$$

Using (15.74), (15.75), and the relation $n = \rho/\mu H$ in (15.73), we finally obtain the desired expression for the gas pressure for the case of interest:

$$P_g = nkT\left[1 - \frac{e^3}{3}\left(\frac{\pi}{H}\right)^{1/2} \frac{\rho^{1/2}}{(kT)^{3/2}} \mu \zeta^{3/2}\right] \tag{15.76a}$$

$$= nkT\left[1 - 0.032 \frac{\rho^{1/2}}{T_6^{3/2}} \mu \zeta^{3/2}\right], \tag{15.76b}$$

where $T_6 \equiv T(°\mathrm{K})/10^6$ and ρ is in gm/cm^3 in (15.76b).

For example, at the center of a star like the sun we may take $\mu \approx (1/2)$, $\zeta \approx 2$ (values appropriate for a predominantly hydrogen composition), $\rho \sim 10^2$ gm/cm^3, and $T_6 \sim 10$. We then obtain from (15.76b) $P_g \approx P_p(1–0.015)$. Deviations from the perfect gas law arising from electrostatic interactions among the charges in the material of stars roughly similar to, or hotter and less dense than the sun are thus normally expected to be rather small. In stars somewhat more dense and cooler than the sun, however, the electrostatic interactions may not be negligible (see later in this subsection). Note, incidentally, that ρ/T^3 is often a slowly varying function of depth in a star (*cf.* Chap. 23); when this is the case, the electrostatic correction term in (15.76) is also a slowly varying function of depth, as was already pointed out by Eddington [Ed26, Chap. 10].

We note that the second term in brackets in (15.76) is of the order of magnitude of $|\varepsilon_{es}|/kT$. Since $|\varepsilon_{es}|/kT \ll 1$ for validity of the above theory, it follows that (15.76) for P_g is approximately valid only as long as the correction term in this equation is small compared to unity. A more elaborate theory (*cf.*, for example, Landau and Lifshitz [La58, Sect. 74]) would be needed in case this correction term were not small compared to unity.

We note, moreover, that the second term in brackets in (15.76) is also of the order of magnitude of the ratio $((4/3)\pi\langle r\rangle^3)/((4/3)\pi r_D^3)$, where $\langle r\rangle$ is the average inter-ionic separation. Hence we must have, equivalently, $\langle r\rangle^3 \ll r_D^3$ for approximate validity of (15.76).

We note, finally, that nowhere in our derivation of (15.76) has the electron gas been assumed non-relativistic. Moreover, we know (*cf.* Sect. 10.5) that the

perfect gas law form $P_p = nkT$ is valid for non-degenerate, non-interacting particles regardless of how relativistic they may be. Hence (15.76) is as valid for a relativistic gas as for a non-relativistic one, other things being equal.

In the opposite extreme of very low temperatures and very high densities, we are likely to have $\langle r \rangle^3 \gg r_D{}^3$, in which case the theory leading up to (15.76) is completely invalid. The limiting case of *zero* temperature and very high densitites (*i.e.*, *complete* degeneracy) has been studied by Salpeter [Sa61], as well as by others. This case is a good approximation for the material in *white dwarf* stars (*cf.* Chap. 25). We shall follow Salpeter's derivation of the electrostatic correction to the pressure equation of state in this case. (Other corrections are also discussed by Salpeter [Sa61].)

Salpeter shows that under most conditions of interest for white dwarf stars* at essentially zero temperature the ions can be treated as if they formed a regular lattice structure (as in a crystalline solid) embedded in a gas of free electrons of (practically) uniform density. (A necessary condition for validity of this treatment is that the mean distance between ions be small compared to the Thomas-Fermi "mean radius" of a free neutral ion, *cf.* Sa61. A good discussion of the reasons for formation of a lattice structure by the ions is given by Mestel [Me65a].)

Pressure ionization under these conditions guarantees essentially complete ionization. In this case the electrostatic interaction energy between the electrons and ions can be computed to adequate accuracy by adopting a simple model in which each (stationary) ion of charge $+Z_i e$ is surrounded by a uniform negative charge cloud of total charge $(-Z_i e)$ and of radius r_i, where r_i, the "ion radius", is defined by (15.38). (Recall that use of r_i allots to each free electron an equal volume of space under conditions of complete ionization.)

The total electrostatic energy of such an "ion sphere" may easily be shown to be $(-(9/10)Z_i{}^2 e^2/r_i)$. Neglecting the interaction energy of different ion spheres with one another, we may write in this case for the total electrostatic energy in a volume V of the gas, instead of (15.62),

$$E_{es} = -\frac{9}{10} e^2 \sum_i \frac{N_i Z_i{}^2}{r_i}, \tag{15.77}$$

where N_i is the number of ions of type i in the volume V and where the summation extends over all types of ions present in the mixture.

* As is stated in Chaps. 24 and 25, the term "white dwarf stars" excludes objects having densities greater than some 10^{11}–10^{12} gm/cm^3, at which the *ions* may begin to be partially degenerate.

Using (15.41) for r_i, the relation $N_i = V\rho x_i/A_i H$ (the notation here is the same as in other sections in this chapter), and proceeding just as before, we obtain for the pressure, with electrostatic effects included,

$$P_g = P_p - \frac{3}{10}\left(\frac{4\pi}{3}\right)^{1/3} \frac{e^2}{H^{4/3}} (\rho\langle Z/A\rangle)^{4/3} \frac{\langle Z^{5/3}/A\rangle}{\langle Z/A\rangle}, \tag{15.78}$$

where angular brackets denote a mass fraction average (*cf.* (15.40)) and where P_p is the "perfect gas" contribution to the pressure. Note (*cf.* (15.23)) that $(\rho\langle Z/A\rangle)$ is proportional to n_e, the density of free electrons. If only a single type of ion is present ($Z_i = Z$), the ratio of angular brackets in (15.78) reduces to $Z^{2/3}$, and (15.78) is then identical to Salpeter's [Sa61] result.

In the case of zero temperature the contribution of the ions to the pressure can be considered completely negligible in cases of interest (see the immediately preceding footnote), so that P_g and P_p in (15.78) refer to the *electron* pressure. In the extreme non-relativistic (N.R.) approximation P_p is given by (24.141), so that (15.78) becomes in this case

$$P_g = P_p\left[1 - \frac{4\pi m_e e^2 H^{1/3}}{h^2} \frac{1}{(2\rho\langle Z/A\rangle)^{1/3}} \frac{\langle Z^{5/3}/A\rangle}{\langle Z/A\rangle}\right] \text{ (N.R.)} \tag{15.79a}$$

$$= P_p\left[1 - \frac{0.71}{(2\rho\langle Z/A\rangle)^{1/3}} \frac{\langle Z^{5/3}/A\rangle}{\langle Z/A\rangle}\right] \text{ (N.R.)}, \tag{15.79b}$$

where ρ is in gm/cm^3 in (15.79b). (Note that $2\langle Z/A\rangle = 1 + X$ in the approximation used in (15.24).)

According to Salpeter [Sa61], (15.79b) is tolerably accurate under the assumed conditions even when the correction term in square brackets is as large as 0.4, and quite accurate if this term is small compared to unity. On the other hand, the non-relativistic assumption breaks down for too large densities (*cf.* Chap. 24). These considerations then lead to the following range of densities for approximate validity of (15.79) (under the assumed conditions):

$$5(\langle Z^{5/3}/A\rangle/\langle Z/A\rangle)^3 \lesssim (2\rho\langle Z/A\rangle) \lesssim 4\times 10^6, \tag{15.80}$$

where ρ is in gm/cm^3.

Note that, according to (15.79), the gas becomes more "perfect" with increasing density, as Landau and Lifshitz [La58, Sect. 56] have also pointed out. This circumstance arises from the fact that $P_p \propto \rho^{5/3}$ for a completely degenerate, non-relativistic electron gas (*cf.* Chap. 24), whereas the electrostatic correction term in (15.78) varies only as $\rho^{4/3}$. We may say, equivalently, that the Fermi energy of an electron (*cf.* Chap. 24) increases with increasing density faster than the average electrostatic interaction energy per electron.

In the extreme relativistic approximation (E.R.) P_p is given by (24.147). Since in this approximation $P_p \propto (\rho\langle Z/A\rangle)^{4/3}$, the correction factor to P_p reduces to a constant, independent of ρ:

$$P_g = P_p\left[1-\frac{9}{5\pi^2}\left(\frac{2\pi}{3}\right)^{5/3}\frac{e^2}{\hbar c}\frac{\langle Z^{5/3}/A\rangle}{\langle Z/A\rangle}\right] \text{ (E.R.)} \tag{15.81a}$$

$$= P_p(1-0.00456\langle Z^{5/3}/A\rangle/\langle Z/A\rangle) \text{ (E.R.)}, \tag{15.81b}$$

where $e^2/\hbar c \approx 1/137$ is the fine structure constant. Note that the correction factor is close to unity even for quite large values of the Z_i; for example, taking $Z_i = Z = 100$, this correction factor has the value 0.91. (Other corrections, as well as numerical results, are given in this E.R. case by Salpeter [Sa61].)

In order that the E.R. approximation be valid, we must have (*cf.* Chap. 24).

$$2\rho\langle Z/A\rangle \gg 4\times 10^6 \text{ gm/cm}^3 \tag{15.82}$$

for approximate validity of (15.81) (under the assumed conditions).

We may summarize this subsection by noting that electrostatic corrections to the "perfect" gas pressure equation of state are small both for very low densities and very high temperatures and also for very high densities and very low temperatures. We may therefore expect the electrostatic corrections to be largest at some "intermediate" density for fixed temperature, this "intermediate" density generally increasing with increasing temperature. It may be that neither of the approximations used in this subsection is adequate at these "intermediate" densities. Electrostatic corrections are probably significant for low-mass stars (much less than the solar mass) lying on or near the main sequence and possibly for other types of stars (*cf.*, for example, Hamada and Salpeter [Ha61]).

15.6 Numerical Results for a Particular Chemical Composition

We present in this section some numerical values of μ and E_I for a grid of values of density ρ and temperature T, of relevance for stellar envelopes, for an "Aller Mix" chemical composition (A.N. Cox [Co65])*. The "Aller Mix" composition may be representative of a "Population I" type composition, and the detailed composition is given in Table 15.1. In all these calculations the electron gas has been assumed non-degenerate and

* With the kind permission of Dr. Arthur N. Cox of the Los Alamos Scientific Laboratory, where the calculations were carried out.

non-relativistic, and special care has been used in computing the partition functions (*cf.* A.N. Cox [Co65] for detailed methods of calculation and Sect. 15.5a). The value $\alpha = (3/5)$ (*cf.* (15.42)) has been used in all these calculations.

Table 15.1

COMPOSITION OF THE ALLER MIX

Element	*Fractional Mass Abundance*
H^1	0.5960
D^2	0
T^3	0
He^4	0.3839
C^{12}	2.827×10^{-3}
N^{14}	9.294×10^{-4}
O^{16}	8.432×10^{-3}
Ne^{20}	5.993×10^{-3}
Al	2.648×10^{-5}
Si	5.294×10^{-4}
Fe	1.707×10^{-4}

Figure 15.1 shows the quantity $\mathcal{R}/\mu$ vs. T for several values of ρ. The units of T here are Kev (1 Kev = 10^3 electron volts = 11.60×10^6 °K) and the units

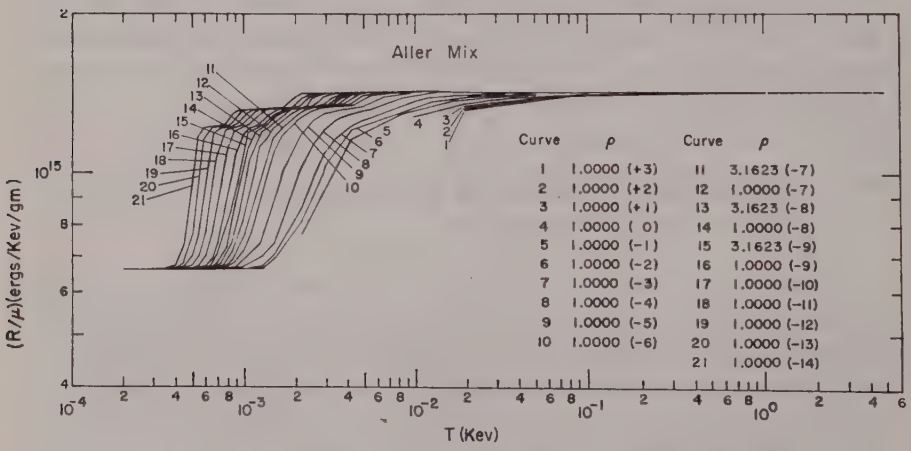


Fig. 15.1 $\mathcal{R}/\mu$ as a function of temperature for various densities. The numbers in parentheses are the powers of ten by which the corresponding entries are to be multiplied.

of $\mathscr{R}/\mu$ are ergs/Kev/gm (10^{16} ergs/Kev/gm $= 0.8621 \times 10^9$ erg/degree Centigrade/gm). The first "step" in the $\mathscr{R}/\mu$ curve (for given ρ), at $T \sim 10^{-3}$ Kev, is due to hydrogen ionization; the second and third "steps" (proceeding to higher temperatures, for given ρ) are due to the ionization of, respectively, neutral and singly ionized helium.

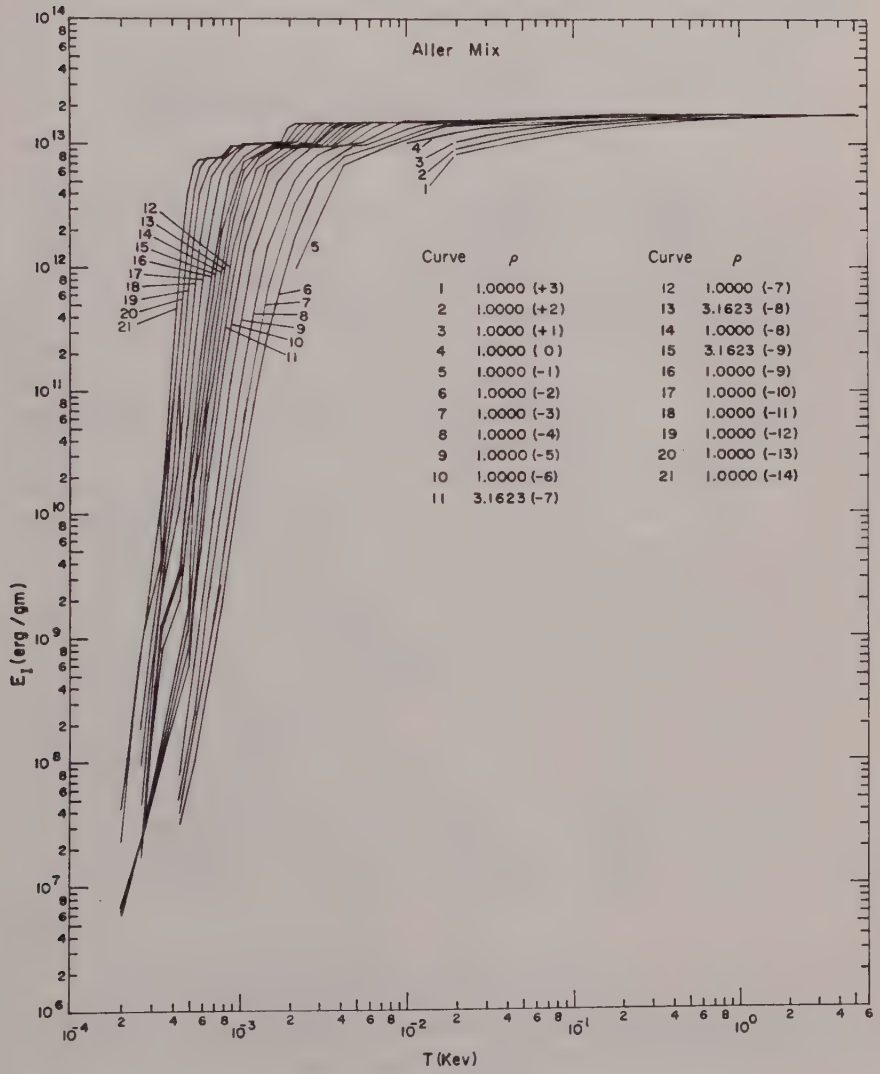


Fig. 15.2 E_I as a function of temperature for various densities. The numbers in parentheses are the powers of ten by which the corresponding entries are to be multiplied.

Figure 15.2 shows E_I vs. T, again for several values of ρ. The units of T are again Kev and those of E_I are ergs/gm. The "steps" in the curves of E_I vs. T have the same interpretation as do those in the $\mathscr{R}/\mu$ curves.

CHAPTER 16

Stellar Opacity

In order to be able to treat properly the transfer of radiation in stars, it is necessary to have values of the opacity for all conditions that may be expected to obtain not only in stellar interiors but also in stellar atmospheres. As yet opacities are not available for the entire range of stellar conditions. Most effort in opacity calculations has been devoted to those stellar conditions which we may call "typical": say $10^4\,°\mathrm{K} \lesssim T \lesssim 30 \times 10^6\,°\mathrm{K}$, $\rho \lesssim 10^4$ gm/cm^3. Reasonably reliable opacities for these conditions have become available in recent years. We may note that, within these "typical" conditions, most of the mass of a typical star is at temperatures of, say, $(1-30) \times 10^6\,°\mathrm{K}$. At these temperatures the peak of the Planck function varies from 29 Å to 0.9 Å.* Hence the nature of the radiation in this temperature range is principally that of soft to moderately hard x-rays. At these temperatures all elements will be ionized to such an extent that only a few electrons, at most, will remain in the M (and inner) shells. Also, at these temperatures hydrogen and helium will be essentially completely ionized and will therefore appear as free protons, alpha particles, and electrons.

Absorption under our "typical" conditions may take place by the following three mechanisms:

(1) "True" absorption by atoms and ions. (We are here using the term "true" in the sense of Sect. 2.7 rather than in the sense of Sect. 2.10b). This "true" absorption in most cases occurs as one of the following types of process:

(a) Bound-bound transitions (line absorption);
(b) Photo effect (bound-free transitions);
(c) Free-free transitions (bremstrahlung) (a free electron in the field of a nucleus or ion may absorb an arbitrary amount of energy and depart

* From the Wien law, we have $\lambda_{\max} T = 0.290 \times 10^8$ Å-deg, from which these numbers follow.

with increased kinetic energy; classically, one may say that the electron switches from one "hyperbolic" orbit to another of higher energy with the absorption of the radiation).

(2) Thomson scattering of photons by free electrons (coherent Compton effect) (a free electron is made to oscillate with the same frequency as that of the impinging radiation and thus the electron re-radiates (or scatters) this energy into other directions. This process, while not a true absorption, nevertheless results in an attenuation of a beam of radiation).

(3) Attenuation with only negligible absorption resulting from effects of dispersion (see Sect. 2.10b). This process, which can be superposed on processes 1 and 2, will be completely ignored in this chapter (*i.e.*, we shall always take the refractive index to be unity), as effects of dispersion on the values of the Rosseland mean opacity have not yet been examined exhaustively. (Some general discussion on how a non-unity refractive index affects the Rosseland mean is given in Sect. 8.2b.) These effects are probably negligible under most conditions of interest in stellar interiors (however, see next-to-last paragraph in this introduction).

In Sects. 16.1 through 16.4 the processes 1b, 1c, and 2 (which are more important than 1a in stellar interiors) will be discussed in some detail; some mention of process 1a will be made in Sect. 16.8. In Sect. 16.5 the evaluation of the Rosseland mean opacity is discussed briefly. In Sect. 16.6 some simple but approximate formulas are derived for the opacity due separately to bound-free absorption, free-free absorption, and Thomson scattering by free electrons (some corrections for Compton scattering are also included there). Thermal conduction by free electrons is discussed briefly in Sect. 16.7. Our treatment of these topics is not intended to be exhaustive or definitive. It will be, rather, somewhat schematic, so that certain basic principles and methods may be emphasized.

In the "low-temperature" region (say $T \lesssim 10^4$ °K) other physical processes may become important. We shall only list these here and shall comment further on some of them in Sect. 16.8:

(4) Negative ion absorption;
(5) Molecular absorption;
(6) Rayleigh scattering;
(7) Raman scattering;
(8) Photo-excitation to auto-ionizing states.

In the "high-temperature" region (say $T \gtrsim 10^9$ °K) the following processes (to be commented on in Sect. 16.8) may be important:

(9) Pair production;
(10) Compton scattering (*cf.* Sect. 16.6a);
(11) Nuclear absorption;

(12) Photon-photon scattering;
(13) Photo-neutrino processes.

Finally, in the "high density" region (say $\rho > 10^4$ gm/cm^3) thermal conduction by free electrons (*cf.* Sect. 16.7) may become important (primarily as a result of electron degeneracy).

Moreover, at these high densities an appreciable fraction of the photons in a Planck distribution may have frequencies less than the plasma frequency (see (2.157) and (2.163)) and therefore may not propagate. The refractive index μ_ν for such photons will be essentially zero if we neglect dissipative effects. According to the discussion in Sect. 8.2b, the effect of these non-propagating photons is to increase the mean mass absorption coefficient. This effect is not considered in this chapter and, as was pointed out a few paragraphs back, we assume throughout this chapter that $\mu_\nu = 1$.

The whole question of stellar opacities has been discussed thoroughly in a recent survey paper by Arthur N. Cox [Co65]. Atmospheric Rosseland mean opacities have also been computed by Vardya [Va64]. Other (less recent) summaries on opacities are given in Chandrasekhar [Ch39, pp. 261ff]; Chandrasekhar [Ch51, pp. 614–621]; and Schwarzschild [Sc58b, Chap. 2]. An excellent and very thorough discussion of opacity, based on fundamental quantum mechanical principles, is given by Frank-Kamenetskii [Fr62, Chap. 6]. See also A.N. Cox [Co66c].

16.1 Photo Effect

The photo effect will be taken here to mean the absorption of a photon by an atom, ion, or molecule and the attendant emission of an atomic electron. We shall use the term "atom" to represent any of these kinds of photon-absorbing particles unless it is necessary to distinguish between them explicitly. (It should be noted that the term "photo-effect" is sometimes also applied to the dissociation of a molecule, or the dissociation of a nucleus, by absorption of a photon.) It is clear that the photo effect can occur only when radiation having frequencies greater than that required for removal of an electron from a particular shell of an atom of a particular element is present. Thus, if $\chi_n^{(i)}$ is the ionization potential of an atom or ion of type i from the shell whose principal quantum number is n, then the photo effect can occur only for those photon frequencies for which

$$\nu \geq \nu_n^{(i)} = \chi_n^{(i)}/h, \tag{16.1}$$

where h is Planck's constant. The excess energy of the photon, $h(\nu - \nu_n^{(i)})$, appears as kinetic energy of the emitted electron. H. A. Kramers [Kr23] first derived formulae for the absorption coefficient for atomic bound-free and

free-free transitions for x-rays. He did this by first computing the energy radiated, according to classical electrodynamics, by an electron being accelerated in the Coulomb field of an ion. Using the Bohr correspondence principle and applying the principle of detailed balancing, he obtained his formulae, which have been extensively applied in astrophysics. A good discussion of the approximations underlying the Kramers formulae is given by Frank-Kamenetskii [Fr62, Chap. 6, Sect. 14].

Kramers' formula for the photo effect may be described as follows. Consider an electron in the shell whose principal quantum number is n of an ion whose nuclear charge is $Z_i e$ and whose *effective* nuclear charge is $S_{n,i}Z_i e$ ($S_{n,i} \leq 1$). The screening factor $S_{n,i}$ corrects the nuclear charge for screening by electrons in shells inside the shell being considered (if there is only one electron in this shell), and also for the screening by other electrons in the shell n if they are present (see Sect. 16.4). Kramers' formula for the *atomic absorption coefficient* (or *cross section*) for bound-free transitions *per electron in shell n* may be written, for $\nu \geq \nu_n^{(i)} = \chi_n^{(i)}/h$, in the form

$$a_0(\nu; i; n) = \frac{64\pi^4}{3\sqrt{3}} \frac{Z_i^4 m_e e^{10}}{ch^6} \frac{1}{n^5} \frac{1}{\nu^3} S_{n,i}^4 g_i(\nu; n) \tag{16.2a}$$

$$= 2.815 \times 10^{29} \frac{Z_i^4 S_{n,i}^4 g_i(\nu; n)}{n^5 \nu^3} \text{ cm}^2 \tag{16.2b}$$

(ν in sec^{-1} in (16.2b)), where $g_i(\nu;n)$ is the *Gaunt factor*. This factor is computed by use of quantum theory and provides the quantum mechanical correction to Kramers' original semi-classical formula. Extensive tabulations of Gaunt factors are given by W.J. Karzas and R. Latter [Ka61]. For the values of ν and n of interest, $g_i(\nu;n)$ is of order unity and is a slowly varying function of ν and n. It is near unity in value near an absorption edge and decreases slowly toward zero as ν increases beyond $\nu_n^{(i)}$. The variation of $g_i(\nu;n)$ is taken into account in modern opacity calculations, but $g_i(\nu;n)$ in early calculations was often replaced by a constant, average value near unity. The screening factor $S_{n,i}$ is also computed and included in modern opacity calculations but was set equal to unity in some of the earlier calculations. We shall formally retain $S_{n,i}$ in our formulae but shall not discuss methods for its calculation (see A.N. Cox [Co65] for some of these computational methods). We emphasize that a_0 in (16.2) is the cross section *per electron in the shell n* of the atom or ion of type i.

For a single type of atom, a_0 would have the schematic appearance shown in Fig. 16.1. The heights of the peaks are roughly proportional to the principal quantum number n. This may be seen from (16.2) by noting that

$h\nu = h\nu_n^{(i)} = \chi_n^{(i)}$ at an absorption edge and that $\chi_n^{(i)} \propto 1/n^2$ if the levels are hydrogen-like. Also, we see from (16.2) that *between* absorption edges (n = constant) a_0 varies approximately as $1/\nu^3$.

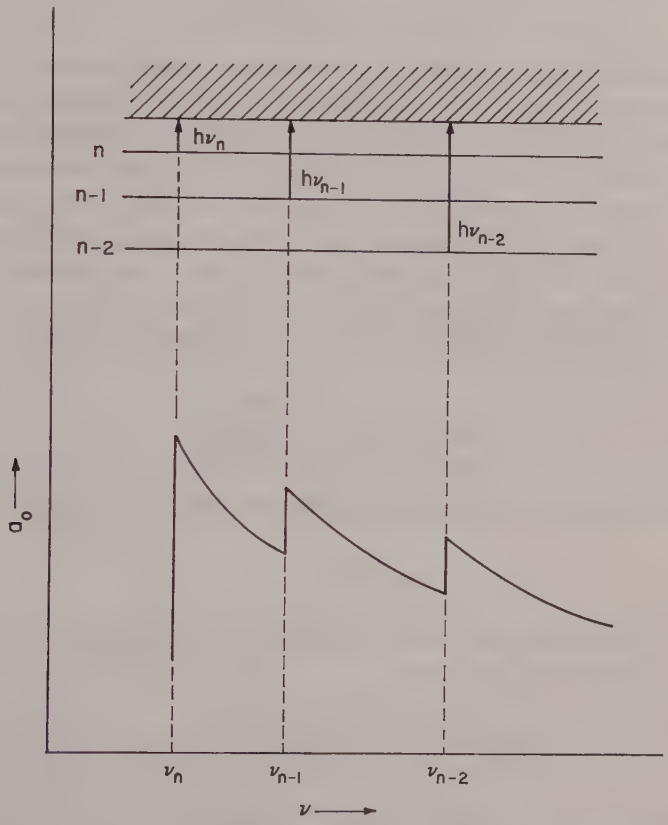


Fig. 16.1 Schematic dependence of a_0 on frequency.

If the electron gas is partially degenerate ($-\eta$ not large compared to unity (see Sect. 3.2)), an additional factor, say $q_{bf,\,i}$, must be introduced into (16.2). This factor accounts for the fact that the unit cell in phase space corresponding to the final state of the (free) electron may already be occupied by an electron. According to the Pauli Exclusion Principle, not more than one fermion of given spin (electron, in the present case) can occupy a given unit cell of "volume" h^3 in phase space. This principle, in fact, is already embodied in the formula for the "occupation number" $\tilde{n}$, which is the number of fermions which occupy a given unit cell in phase space in statistical

equilibrium, corresponding to a certain energy, direction of motion, and spin state. According to Sect. 3.2 $\tilde{n}$ is given by the relation

$$\tilde{n} = \frac{1}{\exp(-\eta + \varepsilon_f/kT) + 1}, \tag{16.3}$$

where η is the degeneracy parameter ($-\eta \gg 1$ for small degeneracy, $\eta \gg 1$ for large degeneracy) and ε_f is the kinetic energy of the free electron (relative to the ion, say). Clearly, $\tilde{n} \leq 1$, which is equivalent to the statement of the Pauli principle given above. Because of the statistical nature of our considerations, it is clear that $\tilde{n}$ as given by (16.3) may also be interpreted as the *probability* that a given unit cell in phase space is occupied by an electron. Hence the probability that the given unit cell is *not* occupied by an electron is $1-\tilde{n}$. Since no more than one electron of given spin can occupy the given unit cell, then clearly the factor $q_{bf,i}$ must be equal to this latter probability:

$$q_{bf,i} = \frac{1}{\exp(\eta - \varepsilon_f/kT) + 1}. \tag{16.4}$$

We relate ε_f to the frequency ν of the absorbed photon by writing

$$h\nu = \varepsilon_f + \chi_n^{(i)}, \tag{16.5}$$

where $\chi_n^{(i)}$ is the ionization potential from the level n from which the electron originated. Then (16.4) becomes

$$q_{bf,i} = \frac{1}{\exp(\eta - h\nu/kT + \chi_n^{(i)}/kT) + 1}. \tag{16.6}$$

We note that $q_{bf,i} \leq 1$, so that the effect of degeneracy is to *reduce* the bound-free absorption coefficient. It is clear that, because $q_{bf,i}$ depends on η, it also depends on the electron density n_e (see Sect. 3.2). If the electron gas is non-degenerate, then $q_{bf,i} = 1$.

With this factor $q_{bf,i}$ included, (16.2a) may be written in the form

$$a_0(\nu; i; n) = \frac{64\pi^4}{3\sqrt{3}} \frac{Z_i^4 m_e e^{10}}{ch^6} \frac{1}{n^5} \frac{1}{\nu^3} S_{n,i}^4 g_{bf,i}(\nu, n), \tag{16.7}$$

where

$$g_{bf,i}(\nu,n) \equiv g_i(\nu,n) q_{bf,i}(\nu,n) \tag{16.8}$$

may depend on n_e and T if the electron gas is partially degenerate. The factor $q_{bf,i}$ is included in modern opacity calculations (A.N. Cox [Co65]).

16.2 Free-Free Transitions (Bremstrahlung)

Kramers' formula for the absorption coefficient for free-free transitions may be described as follows. Consider a single nucleus (or ion) of nuclear charge $Z_i e$ and of *effective* charge $S_{f,i} Z_i e (S_{f,i} \leq 1)$ in an environment in which there are $dn_e(p)$ free electrons per unit volume having momenta, relative to the nucleus (or ion), between p and $p+dp$. The *free-free* absorption coefficient per nucleus (or ion) of the kind under consideration, for absorption of radiation of frequency ν by the $dn_e(p)$ free electrons in the relevant momentum range, is given by Kramers' formula:

$$a'_0(\nu; i; p) dn_e(p) = \frac{4\pi Z_i^2 e^6}{3\sqrt{3} hcm_e^2 v^3 v(p)} S_{f,i}^2 g_{ff,i}(\nu, p) dn_e(p), \tag{16.9}$$

where $v(p)$ is the speed corresponding to the momentum p and $g_{ff,i}(\nu,p)$ is the Gaunt factor for free-free transitions. In order for a free electron to absorb a photon, there must be a nucleus (or ion) in the vicinity of the electron in order that momentum may be conserved during the process. The screening factor $S_{f,i}$ depends, in general, on the electron density n_e and the temperature T, but is unity for complete ionization. One possible expression for $S_{f,i}$ is (*cf.* A.N. Cox [Co65])

$$S_{f,i} = \nu_e(i)/Z_i, \tag{16.10}$$

where $\nu_e(i)$ is the number of free electrons contributed per average atom of nuclear charge $Z_i e$ (see Chap. 15).

Just as in the case of bound-free absorption, a factor, q_{ff}, must be applied to (16.9) when the electron gas is partially degenerate. This factor is the probability that the unit cell in phase space corresponding to the final state of the electron is not already occupied by another electron and is given by (*cf.* (16.4))

$$q_{ff} = \frac{1}{\exp(\eta - \varepsilon_f/kT) + 1}, \tag{16.11}$$

where ε_f is the kinetic energy of the electron (say relative to the ion) in its final state. If ε is the kinetic energy of the electron in its initial state (before absorption of the photon), we have

$$\varepsilon_f = h\nu + \varepsilon, \tag{16.12}$$

so that (16.11) becomes

$$q_{ff}(\nu, p) = \frac{1}{\exp[\eta - h\nu/kT - \varepsilon(p)/kT] + 1}. \tag{16.13}$$

Note that $q_{ff} \leq 1$, so that degeneracy *reduces* the free-free absorption coefficient; and that $q_{ff} = 1$ if the electron gas is non-degenerate ($-\eta \gg 1$).

In order to obtain the free-free absorption coefficient per atom (or ion), averaged over all electron momenta, we must multiply the right side of (16.9) by $q_{ff}(v,p)$ and integrate the resulting equation over all electron momenta. We have

$$a_0(\nu;i) \equiv \int_0^\infty a_0'(\nu;i;p) q_{ff}(\nu,p) \mathrm{d}n_e(p)$$

$$= \frac{4\pi Z_i^2 e^6 S_{f,i}^2}{3\sqrt{3} h c m_e^2 \nu^3} \int_0^\infty \frac{g_{ff,i}(\nu,p) q_{ff}(\nu,p) \mathrm{d}n_e(p)}{v(p)}. \tag{16.14}$$

For the number of free electrons per unit volume having momenta between p and $p+\mathrm{d}p$, we have (see (3.8) and Sect. 3.3b)

$$\mathrm{d}n_e(p) = \frac{8\pi p^2 \mathrm{d}p}{h^3} \frac{1}{\exp\left[-\eta + \varepsilon(p)/kT\right] + 1}. \tag{16.15}$$

Equation (16.14) then becomes

$$a_0(\nu;i) = \frac{4\pi Z_i^2 e^6 S_{f,i}^2 \bar{g}_{ff,i}(\nu)}{3\sqrt{3} h c m_e^2 \nu^3} \int_0^\infty \frac{8\pi p^2 \mathrm{d}p}{v(p) h^3} \frac{1}{\exp\left[-\eta + \varepsilon(p)/kT\right] + 1}, \tag{16.16}$$

where (following A.N. Cox [Co65]) $\bar{g}_{ff,i}(\nu)$ is an average Gaunt factor which includes in its definition the degeneracy factor $q_{ff}(\nu,p)$. In the non-relativistic case, which is the only case of interest when free-free absorption is important, we have $\varepsilon(p) = p^2/2m_e$ and $v(p) = p/m_e$. The integral in (16.16) can then be evaluated analytically and its value is

$$(8\pi m_e^2 kT/h^3) \ln(1+e^\eta). \tag{16.17}$$

We may then write (16.16) in the form

$$a_0(\nu;i) = \frac{16\pi^2 Z_i^2 e^6 S_{f,i}^2 \bar{g}_{ff,i}(\nu) F}{3\sqrt{3} h c (2\pi m_e)^{(3/2)} \nu^3} \frac{n_e}{(kT)^{(1/2)}}, \tag{16.18a}$$

where

$$F \equiv [G(T)/n_e] \ln(1+e^\eta) \tag{16.19}$$

and

$$G(T) \equiv 2(2\pi m_e kT)^{(3/2)}/h^3 \tag{16.20a}$$

$$= 4.830 \times 10^{15} T^{(3/2)} \,\mathrm{cm}^{-3} \tag{16.20b}$$

if T is in degrees Kelvin. Note that F depends, in general, on n_e and T. If ν is in $\sec^{-1}$ and T is in degrees Kelvin, we have

$$a_0(\nu;i) = 3.692\times 10^8 \frac{Z_i^2 S_{f,i}^2 \bar{g}_{ff,i}(\nu) F n_e}{T^{(1/2)}\nu^3}\,\mathrm{cm}^2. \tag{16.18b}$$

A good, comprehensive discussion of the cross section for free-free transitions has been given by Oster [Os61].

It is useful at this point to write down some explicit relations between η, n_e, and T in the case of non-relativistic electrons, which is the case of interest in connection with atomic free-free and bound-free absorption processes. Integrating (16.15), with $\varepsilon(p) = p^2/2m_e$, over all values of the electron momenta, we obtain the total number density of free electrons:

$$n_e = \frac{8\pi}{h^3}\int_0^\infty \frac{p^2 dp}{\exp(-\eta + p^2/2m_e kT)+1} = \frac{4\pi(2m_e kT)^{(3/2)}}{h^3} F_{1/2}(\eta), \tag{16.21}$$

where

$$F_w(\eta) \equiv \int_0^\infty \frac{u^w du}{e^{u-\eta}+1} \tag{16.22}$$

is the Fermi-Dirac integral of order w (to be discussed in more detail in Chap. 24). For small degeneracy ($-\eta \gg 1$), we have

$$F_w(\eta) \simeq e^\eta \Gamma(w+1), \tag{16.23}$$

where Γ denotes the gamma function. In particular, we have (for $-\eta \gg 1$)

$$F_{1/2}(\eta) \simeq (\pi^{(1/2)}/2) e^\eta. \tag{16.24}$$

For large degeneracy ($\eta \gg 1$), we have

$$F_w(\eta) \simeq \frac{1}{w+1}\eta^{w+1} \tag{16.25}$$

and in particular (for $\eta \gg 1$)

$$F_{1/2}(\eta) \simeq \frac{2}{3}\eta^{(3/2)}. \tag{16.26}$$

Consequently, we have for small degeneracy ($-\eta \gg 1$)

$$e^\eta \simeq \frac{n_e h^3}{2(2\pi m_e kT)^{(3/2)}} = \frac{n_e}{G(T)} \ll 1. \tag{16.27}$$

For large degeneracy ($\eta \gg 1$) we have

$$\eta \simeq \left(\frac{3\pi^{(1/2)}}{4} \frac{n_e}{G(T)} \right)^{(2/3)} \gg 1 . \tag{16.28}$$

Limiting values of the function F can now easily be obtained. For $\eta < 0$, we have

$$\ln(1+e^{\eta}) = e^{\eta}(1-(1/2)e^{\eta}+\ldots)$$

whence, for small degeneracy ($-\eta \gg 1$) we have

$$F = [G(T)/n_e]e^{\eta}(1-\ldots) \simeq {}^{1}(1-\ldots) \simeq 1 . \tag{16.29}$$

For large degeneracy ($\eta \gg 1$) we have $\ln(1+e^{\eta}) \simeq \eta$, whence

$$F \simeq \left[\frac{9\pi}{16} \frac{G(T)}{n_e} \right]^{(1/3)} \ll 1 . \tag{16.30}$$

Hence $F \leq 1$, the equality sign applying to the non-degenerate case.

16.3 Thomson Scattering (Coherent Compton Effect)

We may easily compute the Thomson scattering cross section (valid for $h\nu/m_e c^2 \ll 1$ or $\lambda \gg 0.02$Å) by considering the electromagnetic radiation emitted by an oscillating electron, the oscillations being induced by and having the same frequency as the impinging electromagnetic wave. The total energy radiated in unit time by an electron whose acceleration is of magnitude a and whose speed is much less than c is given by classical electrodynamics as

$$\frac{d\varepsilon}{dt} = \frac{2}{3} \frac{e^2}{c^3} a^2 . \tag{16.31}$$

In the case of our oscillating electron we will have

$$a = -eE/m_e = -(eE_0/m_e)\sin \omega t , \tag{16.32}$$

where E is the electric field strength of the impinging electromagnetic wave and E_0 and ω are its amplitude and angular frequency, respectively. Thus we have

$$\frac{d\varepsilon}{dt} = \frac{2}{3} \frac{e^4}{m_e^2 c^3} {E_0}^2 \sin^2 \omega t . \tag{16.33}$$

On the other hand, the rate of flow of incoming electromagnetic energy across unit area is given by the *Poynting* vector (see (2.143)):

$$\mathbf{j} = \frac{c}{4\pi}\mathbf{E}\times\mathbf{H}. \tag{16.34}$$

Since in vacuo $\mathbf{E}$ and $\mathbf{H}$ are perpendicular to each other and of equal magnitude (in Gaussian units, *cf.* Sect. 2.10a), we have ($j \equiv |\mathbf{j}|$)

$$j = \frac{c}{4\pi}E_0{}^2 \sin^2 \omega t. \tag{16.35}$$

From the definition of a cross section we then have for the *Thomson scattering cross section*

$$\begin{aligned}\sigma_0 &= \frac{d\varepsilon/dt}{j} \\ &= \frac{8\pi}{3}\left(\frac{e^2}{m_e c^2}\right)^2 \\ &= \frac{8\pi}{3}a_0{}^2 \quad (= 0.6652\times 10^{-24}\,\mathrm{cm}^2),\end{aligned} \tag{16.36}$$

where $a_0 = e^2/m_e c^2$ is the "classical radius of the electron." Note that σ_0 is independent of frequency.

For more energetic radiation one must use the general *Klein-Nishina* formula (*cf.* Heitler [He54, p. 221]):

$$\frac{\sigma_e}{\sigma_0} = \frac{3}{4}\left\{\frac{1+\alpha}{\alpha^3}\left[\frac{2\alpha(1+\alpha)}{1+2\alpha} - \ln(1+2\alpha)\right] + \frac{\ln(1+2\alpha)}{2\alpha} - \frac{1+3\alpha}{(1+2\alpha)^2}\right\}, \tag{16.37}$$

where $\alpha \equiv h\nu/m_e c^2$. Two limiting cases are of interest:
Non-relativistic: Here $\alpha \ll 1$, whence (16.37) can be expanded to give

$$\sigma_e = \sigma_0(1 - 2\alpha + (26/5)\alpha^2 + \ldots). \tag{16.38}$$

Extreme Relativistic: Here $\alpha \gg 1$, whence (16.37) becomes

$$\sigma_e = \sigma_0 \frac{3}{8\alpha}\left[\ln(2\alpha) + \frac{1}{2}\right]. \tag{16.39}$$

Note that $\sigma_e < \sigma_0$ for all values of α and that $\sigma_e \to 0$ as $\alpha \to \infty$.

We recall that the scattering is coherent in the limit where $\alpha \ll 1$ (Thomson scattering), since the recoil of the scattering electron is negligible. In this limit degeneracy of the electron gas has no effect on the cross section, since in the

(virtual) absence of recoil the electron occupies the same cell in phase space after the scattering as before. In the general case (α not necessarily small compared to unity) the recoil of the electron is not negligible and the scattering is non-coherent (see the remarks in Sect. 2.8a3). Moreover, in this case degeneracy of the electron gas would *reduce* the cross section by reducing the number of cells in phase space available to the electron after the scattering (see Sect. 16.6a).

16.4 Monochromatic Mass Absorption Coefficient

To obtain the *mass* absorption coefficient, κ, from an *atomic* absorption coefficient (or cross section), we proceed as follows (we assume for the moment that only a single type of "absorber" is present): Let N/ρ be the number of "absorbers" per unit mass, where ρ is the mass density of the material. Then, if a_0 denotes the atomic absorption coefficient or cross section, $a_0 N/\rho$ is the total "absorbing area" per unit mass, *i.e.*, κ. But $N/\rho = N_0/A$, where A is the atomic mass of the atom and $N_0 = 1/H$ is Avogadro's number. Hence

$$\kappa = a_0/AH\,. \tag{16.40}$$

Using (16.18a) we then have for the *mass absorption coefficient* for *free-free* transitions (for atoms or ions of a single kind)

$$\kappa_\nu^{(i)}(ff) = \frac{1}{A_i H}\frac{16\pi^2}{3\sqrt{3}}\frac{Z_i{}^2 e^6}{hc(2\pi m_e)^{3/2}}\frac{n_e}{(kT)^{1/2}}\left(\frac{1}{\nu^3}\right) S_{f,i}^2 F \bar{g}_{ff,i}(\nu)\,. \tag{16.41}$$

Letting

$$u \equiv h\nu/kT, \tag{16.42}$$

we can write (16.41) in the form

$$\kappa_\nu^{(i)}(ff) = D_{ff}^{(i)}/u^3\,, \tag{16.43}$$

where

$$D_{ff}^{(i)} = \frac{1}{A_i H}\frac{16\pi^2}{3\sqrt{3}}\frac{Z_i{}^2 e^6 h^2}{c(2\pi m_e)^{3/2}}\frac{n_e}{(kT)^{7/2}} S_{f,i}^2 F \bar{g}_{ff,i}(u) \tag{16.44a}$$

$$= 24.59\frac{Z_i^2}{A_i}\frac{n_e}{T^{3.5}} S_{f,i}^2(n_e,T) F(n_e,T)\bar{g}_{ff,i}(u)\,\mathrm{cm}^2/\mathrm{gm} \tag{16.44b}$$

if n_e and T are in c.g.s. units. Thus, aside from the slowly varying function $\bar{g}_{ff,i}(u)$, $D_{ff}^{(i)}$ is independent of u. One sometimes sets $\bar{g}_{ff,i}(u) = constant \simeq 1$ in approximate calculations.

To obtain the mass absorption coefficient for the *bound-free* transitions from the level n of an ion whose nuclear charge is $Z_i e$, we must multiply the right side of (16.40) by the average number of electrons per nucleus in the shell n, $\bar{n}_n^{(i)}$:

$$\kappa_\nu^{(i)}(n) = \frac{a_0(\nu; i; n)}{A_i H} \bar{n}_n^{(i)}. \tag{16.45}$$

Using (16.7), we obtain for the *mass absorption coefficient* for *bound-free* transitions (for atoms or ions of a single kind)

$$\kappa_\nu^{(i)}(n) = \frac{1}{A_i H} \frac{64\pi^4}{3\sqrt{3}} \frac{Z_i{}^4 m_e e^{10}}{ch^6} \frac{1}{n^5} \frac{1}{\nu^3} g_{bf,i}(\nu, n) S_{n,i}^4 \bar{n}_n^{(i)}. \tag{16.46}$$

Some methods of evaluating $\bar{n}_n^{(i)}$ are described by A.N. Cox [Co65]. One of these methods, which works best if the atoms are rather highly ionized and which is only approximate, is the following. Let V_i be the volume which contains, on the average, just one nucleus of nuclear charge $Z_i e$ and would also contain just Z_i free ionization electrons if all atoms were completely ionized and if the electron density were uniform. We assume, in fact, that the volume V_i always contains just Z_i electrons, some bound and some free, so that the net electric charge in V_i is always zero. We now assume that these Z_i electrons may be treated as if independent of each other and we apply Fermi-Dirac statistics to the electrons to determine the occupation of the various bound energy levels. This assumption of independence would appear to mean that we are neglecting the electrostatic interactions of the electrons with each other, both for the bound and the free electrons. In the case of the bound electrons, this would also seem to imply that the energy levels of the atom or ion are exactly the same as if the atom or ion were strictly hydrogen-like, *i.e.*, as if it contained only one bound electron.

We saw in Sect. 15.5a, however, that the electrostatic interactions of the *free* electrons with themselves could be approximately taken into account by introducing a depression of the continuum and by taking the zero of the electron energy to be at the beginning of this depressed continuum. The electrostatic interactions of the *bound* electrons with each other and with free electrons can also be taken into account approximately by suitably modifying the energies of the hydrogen-like levels for the unscreened nuclear charge $Z_i e$ by applying appropriate screening factors to account for the interaction of bound electrons with each other, and by adding terms to these energies to account for the interaction of bound electrons with free electrons (*cf.* A.N. Cox [Co65]). In other words, all electrostatic interactions of electrons with each other can be approximately taken into account by a suitable definition of $\chi_n^{(i)}$, the energy required to remove an electron from the

level n of an atom or ion. Taking, then, the zero of the electron energy to be at the beginning of the depressed continuum, we have that the total energy of a free electron (non-relativistic) is $p^2/2m_e$, and of a bound electron is $(-\chi_n^{(i)})$, where $(-\chi_n^{(i)})$ includes the depression of the continuum and other relevant electrostatic corrections. The number of free electrons in the volume V_i having momenta (relative to the ion) between p and $p+\mathrm{d}p$ is then given by the relation (see Sects. 3.2 and 3.3)

$$\mathrm{d}N_e(p) = \frac{V_i}{h^3} \frac{8\pi p^2 \mathrm{d}p}{\exp(-\eta + p^2/2m_e kT)+1}, \tag{16.47}$$

where η is the degeneracy parameter. The total number of free electrons per unit volume is then given by dividing both sides of (16.47) by V_i and integrating over all momenta. The result is given by (16.21).

For the bound electrons, now, we have for the number of electrons bound to the level n of the atom or ion of type i

$$\bar{n}_n^{(i)} = \frac{g_n^{(i)}}{\exp[-\eta - \chi_n^{(i)}/kT]+1}, \tag{16.49}$$

where $g_n^{(i)}$ is the statistical weight of the level n ($= 2n^2$ if assumed hydrogen-like) and the η here is the same as the η appearing in (16.47), since bound electrons are assumed to differ from free electrons only in their energies. We saw in Sect. 15.4 how, given the mass density ρ and the temperature T for a given mixture, the ionization electron density n_e could be calculated. Assuming, then, that the values of n_e and T are known, the corresponding value of η can be computed from (16.21), and this value can be used in (16.49) to evaluate $\bar{n}_n^{(i)}$.

It is clear that this method, being essentially a statistical one, may not give very accurate results if Z_i is not considerably larger than unity.

We note that $\bar{n}_n^{(i)} \ll g_n^{(i)}$, *i.e.*, that the level n is *unsaturated,* if $\exp[-\eta - \chi_n^{(i)}/kT] \gg 1$, or if

$$-\eta > 1 + \chi_n^{(i)}/kT, \tag{16.50}$$

say. Since $\chi_n^{(i)}/kT$ is non-negative, it follows that a level can be unsaturated only if $(-\eta) > 1$, *i.e.*, only if the electron gas is only partially or slightly degenerate or non-degenerate. In the latter case $(-\eta) \gg 1$ and we have $e^{\eta} \simeq n_e/G(T) \ll 1$ (see (16.27)). Equation (16.49) may then be written in the form

$$\bar{n}_n^{(i)} = \frac{g_n^{(i)}}{1 + [G(T)/n_e] \exp(-\chi_n^{(i)}/kT)} \quad (-\eta \gg 1), \tag{16.51}$$

which shows that $\bar{n}_n^{(i)}$ increases with increasing electron density and with decreasing temperature.

It follows from (16.49) that the level n is *saturated*, *i.e.*, that $\bar{n}_n^{(i)} \simeq g_n^{(i)*}$, if $\exp[-\eta - \chi_n^{(i)}/kT] \ll 1$, or if

$$\eta > 1 - \chi_n^{(i)}/kT, \tag{16.52}$$

say. Since $\chi_n^{(i)}/kT$ may be large compared to unity, it follows that saturation of levels can occur even if the electron gas is non-degenerate ($-\eta \gg 1$). It is clear that in the case of a saturated level, $\bar{n}_n^{(i)}$ cannot be increased beyond the maximum possible number $g_n^{(i)}$ of electrons that the level n can accommodate, regardless of how large n_e is or how small T is (provided, of course, that the depression of the continuum has not eliminated the level n). An example of saturation in a non-degenerate electron gas would be provided by the ground state of hydrogen at a temperature of, say, 5000 °K. In this case $\chi_n^{(i)}/kT \simeq 31$ and saturation of this level could occur for $\eta > -30$, say; *i.e.*, the electron gas could be quite non-degenerate and saturation could still occur. Another example would be atoms and molecules at room temperature.

If the electron gas is highly degenerate ($\eta \gg 1$), (16.49) shows that *all* levels would be saturated, independently of temperature. In this case, however, most bound levels of most atoms and ions would probably have been eliminated by the depression of the continuum, *i.e.*, by pressure ionization (see Sect. 15.5).

We now let $u \equiv h\nu/kT$ and write (16.46) in the form

$$\kappa_\nu^{(i)}(n) = \frac{1}{A_i H} \frac{16\pi^2}{3\sqrt{3}} \frac{Z_i^2 e^6 h^2}{c(2\pi m_e)^{3/2}} \frac{2(2\pi m_e kT)^{3/2}}{h^3} \frac{2\pi^2 e^4 m_e Z_i^2}{n^2 h^2} g_{bf,i} S_{n,i}^4 \frac{1}{n^3} \frac{\bar{n}_n^{(i)}}{u^3 (kT)^{9/2}}. \tag{16.53}$$

We note that the ionization potential of the level n of a hydrogen-like atom of nuclear charge $Z_i e$ is given by

$$I_n^{(i)} = \frac{2\pi^2 e^4 m_e Z_i^2}{n^2 h^2} \tag{16.54}$$

(see any book on modern or atomic physics). If the atoms of type i are highly ionized and if the density of the material is not too great, then $I_n^{(i)}$ may be a fairly good approximation to $\chi_n^{(i)}$, the actual ionization potential from the

* Assuming, of course, that $Z_i \gg g_n^{(i)}$.

level n. Using (16.20a), (16.54), and (16.44a), we see that we may write (16.53) in the forms

$$\kappa_\nu^{(i)}(n) = D_{ff}^{(i)} \frac{I_n^{(i)}}{kT}\left(\frac{g_{bf,i}}{\bar{g}_{ff,i}}\right)\left(\frac{S_{n,i}^4}{S_{f,i}^2}\right)\frac{1}{n^3 u^3 F(n_e,T)}\left[\frac{G(T)\bar{n}_n^{(i)}}{n_e}\right] = D_n^{(i)}/u^3, \qquad (16.55)$$

where

$$D_n^{(i)} \equiv D_{ff}^{(i)} \frac{I_n^{(i)}}{kT}\left(\frac{g_{bf,i}}{\bar{g}_{ff,i}}\right)\left(\frac{S_{n,i}^4}{S_{f,i}^2}\right)\frac{1}{n^3 F(n_e,T)}\left[\frac{G(T)\bar{n}_n^{(i)}}{n_e}\right]. \qquad (16.56)$$

We now write (16.56) explicitly for the two limiting cases of unsaturated and saturated levels. We have (*cf.* (16.49))

$$\left[\frac{G(T)\bar{n}_n^{(i)}}{n_e}\right] = \left(\frac{G(T)}{n_e}\right)\frac{g_n^{(i)}\exp(\eta+\chi_n^{(i)}/kT)}{1+\exp(\eta+\chi_n^{(i)}/kT)}, \qquad (16.57)$$

where $g_n^{(i)}$ is the statistical weight of the level n. For *unsaturated* levels the exponential in (16.57) is small compared to unity. This can only be true if $(-\eta)$ is large compared to unity, *i.e.*, if the electron gas is non-degenerate. In this case we have $e^\eta \simeq n_e/G(T) \ll 1$ (see (16.27)) and $F(n_e,T) \simeq 1$ (see (16.29)). We may also write $I_n^{(i)} = I_1^{(i)}/n^2$, so that (16.56) becomes in this case $(\eta+\chi_n^{(i)}/kT < -1$, say)

$$D_n^{(i)} \simeq D_{ff}^{(i)} \frac{I_1^{(i)}}{kT}\left(\frac{g_{bf,i}}{\bar{g}_{ff,i}}\right)\left(\frac{S_{n,i}^4}{S_{f,i}^2}\right)\frac{2}{n^3}\left(\frac{g_n^{(i)}}{2n^2}\right)\exp(\chi_n^{(i)}/kT). \qquad (16.58)$$

This equation shows that, because of the presence of the exponential factor, $D_n^{(i)}$ decreases rapidly with increasing n, for given temperature (note that $g_n^{(i)} \simeq 2n^2$ if the levels are approximately hydrogen-like). Physically, this decrease of $D_n^{(i)}$ with increasing n is just a reflection of the fact that, because of the presence of the Boltzmann factor, most electrons tend to occupy the low-lying energy levels.

In the case of *saturated* levels (where the exponential in (16.57) is large compared to unity) we have

$$D_n^{(i)} \simeq D_{ff}^{(i)} \frac{I_1^{(i)}}{kT}\left(\frac{g_{bf,i}}{\bar{g}_{ff,i}}\right)\left(\frac{S_{n,i}^4}{S_{f,i}^2}\right)\frac{2}{n^3 F}\left(\frac{g_n^{(i)}}{2n^2}\right)\frac{G(T)}{n_e}. \qquad (16.59)$$

Using (16.44a) for $D_{ff}^{(i)}$ and regarding $g_{bf,i}$ and $S_{n,i}^4$ as constant, we have

$$D_n^{(i)} \simeq \text{const.}\left(\frac{g_n^{(i)}}{2n^2}\right)\frac{1}{n^3 T^3}, \qquad (16.60)$$

which shows that in this case $D_n^{(i)}$ is independent of n_e, just as we expect for saturated levels.

We must now sum over all levels which are capable of photoelectric absorption of radiation of the frequency ν. Thus we write

$$\kappa_\nu^{(i,a)} = D^{(i)}(u)/u^3\,, \tag{16.61}$$

where the superscript a stands for "absorption" and where, if we include also the contribution from free-free transitions, we have

$$D^{(i)}(u) \equiv D_{ff}^{(i)}(u) + \sum_n D_n^{(i)}(u) \tag{16.62}$$

$$= D_{ff}^{(i)} \left\{ 1 + \sum_n \frac{I_n^{(i)}}{kT} \left(\frac{g_{bf,i}}{\bar{g}_{ff,i}} \right) \left(\frac{S_{n,i}^4}{S_{f,i}^2} \right) \frac{1}{n^3 F} \left[\frac{G(T)\bar{n}_n^{(i)}}{n_e} \right] \right\}, \tag{16.63}$$

where the summation over n is over all levels for which $\chi_n^{(i)} \leq h\nu$. We note that $D^{(i)}(u)$ is a function of u not only because of the (weak) dependence of $g_{bf,i}$ on u, but also because the number of terms in the summation depends on u. In practice, only a few terms in the summation in (16.63) may be important, because the levels having the largest values of $\chi_n^{(i)}$ (*i.e.*, the lowest-lying levels consistent with the requirement that $\chi_n^{(i)} \leq h\nu$) will contribute most to the sum (this will be especially true if the levels are unsaturated). Additional terms would of course be included in (16.62) if other physical sources of absorption, such as bound-bound transitions (see A.N. Cox [Co65]), were taken into account. Note that, aside from the factor in curly brackets in (16.63), $D^{(i)}(u)$ has the same dependence on n_e and T as does $D_{ff}^{(i)}$.

For a *mixture* of different elements, now, we must average over all the elements present that contribute to the absorption of radiation of the frequency ν. Thus we have, letting x_i stand for the relative mass abundance of element i,

$$\kappa_\nu^{(a)} = \sum_i x_i \kappa_\nu^{(i,a)} = D(u)/u^3\,, \tag{16.64}$$

where

$$D(u) \equiv \sum_i x_i D^{(i)}(u)\,, \tag{16.65}$$

and the summations are extended over all relevant elements.

We note that the function D(u) is almost a step function, such that

$$D(u) = D'(u, u_j, u_{j+1}) \approx \text{const. for } u_j < u < u_{j+1}\,, \tag{16.66}$$

where (u_j, u_{j+1}) denotes the frequency interval between two successive absorption edges j and $j+1$. The slow variation of D with u between successive absorption edges is caused by the (small) dependence of the Gaunt factors on u. The function $D(u)$ increases discontinuously with increasing u whenever

an absorption edge is crossed (*i.e.*, when one more term is included in the sum in (16.63)). We may also write (16.64) in the form

$$\kappa_\nu^{(a)} = \frac{f(\text{composition}, u, n_e, T)}{u^3} \frac{n_e}{T^{3.5}}, \tag{16.67}$$

where f is a step function of u, the values of f on the "steps" depending in a complicated way on n_e and T. For a mixture of different elements the monochromatic absorption coefficient might appear schematically as shown in Fig. 16.2. For some actual plots of $\kappa_\nu^{(a)}$ versus u (or ν), see A. Cox, Stewart, and Eilers [Co65a].

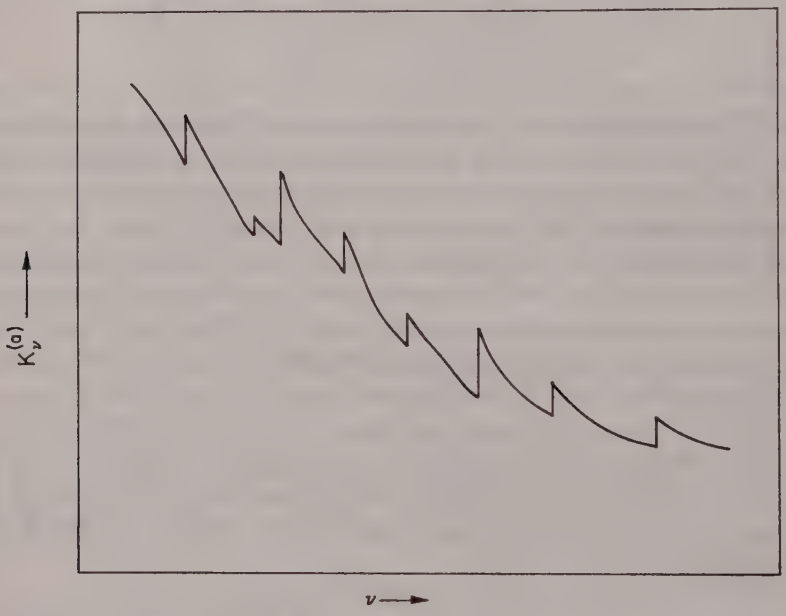


Fig. 16.2 Schematic dependence of the monochromatic mass absorption coefficient (for a mixture of elements) on frequency.

The *monochromatic mass absorption coefficient* for *Thomson scattering* by free electrons is clearly given by the relation

$$\kappa_\nu^{(s)} = \sigma_0 n_e/\rho, \tag{16.68}$$

where σ_0 is the Thomson scattering cross section (see Sect. 16.3) and is frequency-independent because σ_0 is.

Finally, the *total monochromatic mass absorption coefficient*, with coherent scattering and induced emission taken into account (see Chap. 2), is

$$\kappa_\nu = \kappa_\nu^{(a)}(1-e^{-u}) + \kappa_\nu^{(s)}, \tag{16.69}$$

where $u = h\nu/kT$ and where the form of the induced emission factor in (16.69) is valid if LTE obtains (see Chaps. 4 and 7).

16.5 Rosseland Mean Opacity

We recall (see (8.10)) that the Rosseland mean of the monochromatic absorption coefficient was defined by the equation

$$\frac{1}{\kappa} \equiv \frac{\int_0^\infty \frac{1}{\kappa_\nu} \frac{\partial B_\nu(T)}{\partial T} d\nu}{\int_0^\infty \frac{\partial B_\nu(T)}{\partial T} d\nu} \tag{16.70}$$

if the refractive index μ_ν is unity. $B_\nu(T)$ is the Planck function (see (3.31)) and κ_ν is the total monochromatic absorption coefficient with electron scattering and induced emission both taken into account. (See Chap. 8 for an expression for κ when $\mu_\nu \neq 1$.) The Rosseland mean is a good approximation to the mean defined by (8.7) in regions where $\tau_\nu \gg 1$. We use (16.69) for κ_ν, let $u \equiv h\nu/kT$, write the denominator of (16.70) in terms of σ, the Stefan-Boltzman constant, and express σ in terms of basic atomic constants (see (3.41)). We obtain, finally,

$$\frac{1}{\kappa} = \frac{15}{4\pi^4} \int_0^\infty \frac{1}{\kappa_\nu^{(a)}(1-e^{-u}) + \kappa_\nu^{(s)}} \frac{u^4 e^{-u}}{(1-e^{-u})^2} du, \tag{16.71}$$

where the quantity

$$(15/4\pi^4) u^4 e^{-u}/(1-e^{-u})^2 \equiv W(u) \tag{16.72}$$

may be regarded as a weighting factor. If the term e^{-u} in the denominator of $W(u)$ were neglected, then $W(u)$ would have its maximum value at $u_{\max} = 4$, since the function $u^n e^{-u}$ attains maximum at $u_{\max} = n$. Since $e^{-4} \ll 1$, $W(u)$ must attain its maximum at $u_{\max} \simeq 4$, *i.e.*, $(h\nu/kT)_{\max} \simeq 4$. We see then that the high frequencies (*i.e.*, those for which $h\nu$ is several times kT) are weighted most heavily in the Rosseland mean.

We note also that, because the Rosseland mean is a harmonic mean, it is *incorrect* to say that the Rosseland mean of the total monochromatic absorption coefficient is equal to the sum of the respective Rosseland means of the individual contributions (from different elements or physical mechanisms) to the monochromatic absorption coefficient. However, an *approximate* result may be obtained by evaluating the quantity $[\kappa_\nu^{(a)}(1-e^{-u}) + \kappa_\nu^{(s)}]^{-1}$ at

the maximum ($u_{\max} \simeq 4$) of the weighting factor $W(u)$ and removing this quantity from under the integral sign. Since $\int_0^\infty W(u)\mathrm{d}u = 1$, we obtain

$$\kappa \simeq \kappa_{\nu(\max)}^{(a)}[1-\exp(-u_{\max})] + \kappa_{\nu(\max)}^{(s)}. \tag{16.73}$$

It must be remembered, however, that (16.73) is only a rough approximation.

When $\kappa_\nu^{(a)} \ll \kappa_\nu^{(s)}$ and when the scattering is due to Thomson scattering by free electrons, so that $\kappa_\nu^{(s)} = \kappa_e$, say, and is frequency-independent, we have that

$$\kappa = \kappa_e, \tag{16.74}$$

so that the Rosseland mean of the Thomson scattering absorption coefficient is exactly equal to the scattering absorption coefficient itself. Note also that in this case the maximum of the integrand of (16.71) occurs at the maximum ($u_{\max} \simeq 4$) of the weighting function $W(u)$.

When $\kappa_\nu^{(a)} \gg \kappa_\nu^{(s)}$, we have from (16.71), writing $\kappa_\nu^{(a)} = D(u)/u^3$,

$$\frac{1}{\kappa} = \frac{15}{4\pi^4}\int_0^\infty \frac{1}{D(u)}\frac{u^7 e^{-u}}{(1-e^{-u})^3}\mathrm{d}u. \tag{16.75}$$

Note that the weighting factor, $W'(u)$, say, for the quantity $D^{-1}(u)$ is now given by the relation

$$W'(u) = \frac{15}{4\pi^4}\frac{u^7 e^{-u}}{(1-e^{-u})^3}, \tag{16.76}$$

which has a maximum at $u_{\max} \simeq 7$ and whose integral over all u is

$$\int_0^\infty W'(u)\mathrm{d}u = \frac{15}{4\pi^4}\int_0^\infty \frac{u^7 e^{-u}}{(1-e^{-u})^3}\mathrm{d}u = 196.5, \tag{16.77}$$

according to A.N. Cox [Co65]. Hence, a rough value of the Rosseland mean, when scattering is negligible, is given by evaluating the step function $D(u)$ at $u_{\max} \simeq 7$ and removing it from under the integral sign. Thus we have

$$\kappa \simeq D(u_{\max} \simeq 7)/196.5 = [\kappa_\nu^{(a)} u^3]_{(u \simeq 7)}/196.5, \tag{16.78}$$

where it is to be emphasized that this is only a rough approximation.

16.6 Approximate Formulae

Here we shall derive approximate formulae for calculating the opacity arising separately from Thomson scattering by free electrons, free-free

transitions, and bound-free transitions. We want to express these formulae for the opacity as functions of chemical composition, *mass density* ρ (rather than electron number density n_e), and temperature T.

In Sect. 15.1 we wrote the expression for the ionization electron density in the form

$$n_e = (\rho/H)\langle v_e/A\rangle, \tag{16.79}$$

where H^{-1} is Avogadro's number;

$$\langle f\rangle \equiv \sum_i x_i f_i \tag{16.80}$$

is a mass-fraction average of the quantities f_i over a mixture of different elements, x_i being the relative mass abundance of the element i; $v_e(i)$ is the number of free ionization electrons contributed per average atom or ion of type i; and A_i is the atomic mass of an atom of type i. In general, the $v_e(i)$ depend in a complicated way on both n_e and T for a given mixture, and the methods of evaluating the $v_e(i)$ are discussed in Sect. 15.3. In the case of complete ionization, however, we have $v_e(i) = Z_i$, where Z_i is the atomic number of a particle of type i, and (16.79) becomes simply

$$n_e = (\rho/H)\langle Z/A\rangle. \tag{16.81}$$

An approximation to (16.81) was shown in Sect. 15.2 to be

$$n_e \simeq \frac{1}{2}(\rho/H)(1+X), \tag{16.82}$$

where X is the relative mass abundance of hydrogen.

16.6a Thomson Electron Scattering Opacity

We had (see (16.68) and (16.74)) that the opacity resulting from Thomson scattering by free electrons was given by the relation

$$\kappa_e = \sigma_0 n_e/\rho, \tag{16.83}$$

where $\sigma_0 = (8\pi/3)(e^2/m_e c^2)^2$ is the Thomson scattering cross section (*cf.* (16.36)). Using (16.81) for n_e, we have for *the case of complete ionization*

$$\kappa_e = (\sigma_0/H)\langle Z/A\rangle \tag{16.84a}$$

$$= 0.4008\,\langle Z/A\rangle\ \mathrm{cm^2/gm} \tag{16.84b}$$

$$\simeq 0.2004\,(1+X)\ \mathrm{cm^2/gm}, \tag{16.84c}$$

where the average is a mass-fraction average (see (16.80)) and where the last equality is obtained from the approximation (16.82). We recall (*cf.* Sect. 16.5) that κ_e as given by equations (16.84) is also the Rosseland mean opacity when true absorption is negligible compared to scattering.

Rosseland mean opacities for *Compton* scattering by free electrons have been computed in the non-degenerate limit by Sampson [Sa59a] and with effects of partial electron degeneracy included by Chin [Ch65b].

Sampson's [Sa59a] results are expressed in terms of a dimensionless function $G(T)$ only of temperature T, defined by the following expression:

$$\kappa_C = (n_{e\pm}/\rho)G(T)\sigma_0, \tag{16.84d}$$

where κ_C is the Rosseland mean opacity for Compton scattering by free electrons, $n_{e\pm}$ is the total number density of electrons and positrons (see Sect. 24.9), ρ is the mass density (rest mass per unit volume of atoms, ions, and *ionization* electrons (see Sect. 15.1)), and σ_0 is the Thomson scattering cross section (see Sect. 16.3). On the other hand, the Rosseland mean opacity κ_e for Thomson scattering by free electrons is given by (16.83), where in this equation n_e is the number density, say $n_e^{(m)}$, of *ionization* electrons. Consequently, we have

$$\frac{\kappa_C}{\kappa_e} = \frac{n_{e\pm}}{n_e^{(m)}} G(T). \tag{16.84e}$$

Table 16.1

$G(T)$ AS A FUNCTION OF T

	kT(Kev)					
	1	2	4	6	9	14
kT/m_ec^2	0.001957	0.003914	0.007828	0.01174	0.01761	0.02740
$G(T)$	–	0.95005	0.9044	0.8626	0.8067	0.7279

	kT(Kev)				
	20	30	50	80	125
kT/m_ec^2	0.03914	0.05871	0.09785	0.1566	0.2446
$G(T)$	0.6525	0.5590	0.4408	0.3411	0.2579

Hence $G(T)$ is the ratio of the Rosseland frequency-averaged cross section for Compton scattering to σ_0. Table 16.1 [from Sa59a] gives some values of $G(T)$. The following fitting formula is given by Sampson [Sa59a]:

$$G(T) = -0.13887 + 4.9871(kT)^{-1/2} - 5.9479\,kT)^{-1} - 2.362(kT)^{-3/2}$$

$$(20\ \text{Kev} \lesssim kT \lesssim 125\ \text{Kev}). \quad (16.84f)$$

It is seen that $G(T)$ decreases with increasing T. This behavior arises in part from the decrease in the Compton scattering cross section (see (16.37)–(16.39)) with increasing photon energy, and in part from the non-coherence of Compton scattering: the stimulated emission factor begins to drop below unity as soon as the recoil of the electron becomes non-negligible (*cf.* Chap. 2).

Because $n_{e\pm}/n_e^{(m)}$ can become very large at very high temperatures (provided that the density is small enough that the electron gas is not degenerate, see Sect. 24.9 and Table A.4.1), however, the ratio κ_C/κ_e can be several orders of magnitude larger than unity at sufficiently high temperatures and for not-too-large densities.

Chin's [Ch65b] results are expressed in terms of a factor

$$G(kT, \eta) \equiv \kappa_C/\kappa_e \quad (16.84g)$$

(η = degeneracy parameter) by which the opacity due to Thomson scattering (see (16.83)) is to be multiplied to obtain the opacity due to Compton scattering, with effects of partial electron degeneracy included. This factor is tabulated in Table 16.2. The positron contribution to $G(kT, \eta)$ has been omitted by Chin since it is small ($\lesssim 1$ per cent) over the range of interest,

Table 16.2

$G(kT,\eta)$ AS A FUNCTION OF (kT/m_ec^2) AND η

kT/m_ec^2	$T_8 \equiv T(°K)10/^8$	η					
		$-$"∞"	-1	0	1	2	4
0.01	0.5930	–	0.8294	0.7368	0.6112	0.4864	0.3094
.03	1.779	–	.6732	.6060	.5094	.4076	.2554
.05	2.965	0.5967	.5699	.5189	.4410	.3553	.2217
.15	8.895	.3496	.3416	.3194	.2816	.2322	.1423
.25	14.82	0.2542	.2529	.2389	.2135	.1776	.1066
0.35	20.76	–	0.2028	0.1923	0.1730	0.1443	0.0843

according to Chin. Since $n_{e\pm}/n_e^{(m)} = 1$ when electron-positron pairs are absent, then $G(kT, \eta)$ is also the ratio of the Rosseland frequency-averaged Compton scattering cross section to σ_0.

Chin [Ch65b] gives the following polynomial representation of $G(kT, \eta)$ between tabulated entries in Table 16.2 (accurate to within about 5 per cent, according to Chin):

$$\ln G(kT,\eta) = -0.3037 - 6.89757\left(\frac{kT}{m_e c^2}\right) + 8.89771\left(\frac{kT}{m_e c^2}\right)^2 \cdot$$
$$-0.158737\eta + 0.392553\left(\frac{kT}{m_e c^2}\right)\eta - 0.0146867\eta^2$$
$$-0.451961\left(\frac{kT}{m_e c^2}\right)^2\eta - 0.0523759\left(\frac{kT}{m_e c^2}\right)\eta^2 . \qquad (16.84h)$$

Values of electron density or mass density corresponding to the tabulated values of T and η can be found by interpolation in Table A.3.1 (Appendix 3) if electron-positron pairs are neglected or in Tables A.4.1 and A.4.2 (Appendix 4) if electron-positron pairs are included.

The (expected) decrease in $G(kT, \eta)$ with increasing degeneracy (increasing η) at given T is clearly evident (see Sect. 16.3).

16.6b Free-Free Opacity

For negligible electron scattering opacity we had for the Rosseland mean (see (16.75))

$$\frac{1}{\kappa} = \int_0^\infty \frac{W'(u)\mathrm{d}u}{D(u)}, \qquad (16.85)$$

where $D(u) \equiv \kappa_\nu^{(a)} u^3$, $u \equiv h\nu/kT$, and $W'(u)$, defined in (16.76), is the appropriate weighting function. For pure free-free transitions we have $D(u) = D_{ff} = \sum_i x_i D_{ff}^{(i)} =$ constant if we ignore the slow variation of the Gaunt factor. Hence (16.85) yields the result

$$\kappa_{ff} = D_{ff} / \int_0^\infty W'(u)\mathrm{d}u$$
$$= D_{ff}/196.5\ \mathrm{cm}^2/\mathrm{gm}. \qquad (16.87)$$

Using the expression (16.44b) for $D_{ff}^{(i)}$, we obtain

$$\kappa_{ff} = \frac{24.59}{196.5}\left\langle \frac{Z^2 S_f{}^2 \bar{g}_{ff}}{A} \right\rangle \frac{F(n_e,T)n_e}{T^{3.5}} \text{cm}^2/\text{gm}, \tag{16.88}$$

where the average is a mass-fraction average (see (16.80)) and where the function $F(n_e,T)$, which is unity if the electron gas is non-degenerate and smaller otherwise, is defined in (16.19).

For the case of essentially complete ionization we have $S_{f,i} \simeq 1$ (see (16.10)) and $n_e = (\rho/H)\langle Z/A\rangle$ (see (16.81)), so that (16.88) becomes for this case

$$\kappa_{ff} = \frac{24.59}{196.5}\frac{1}{H}\left\langle \frac{Z^2 \bar{g}_{ff}}{A} \right\rangle \left\langle \frac{Z}{A} \right\rangle \frac{F'(\rho,T)\rho}{T^{3.5}} \text{cm}^2/\text{gm}, \tag{16.89}$$

where $F'(\rho,T) \equiv F(n_e(\rho,T),T)$. To obtain an approximate form of (16.89,) we define another average free-free Gaunt factor for the mixture as a whole:

$$\bar{g}_{ff,\text{av}} \equiv \frac{\langle Z^2 \bar{g}_{ff}/A\rangle}{\langle Z^2/A\rangle}$$

and evaluate $\langle Z^2/A\rangle$ and $\langle Z/A\rangle$ approximately in terms of X and Y, the relative mass abundances of hydrogen and helium. Using $A(\text{H}^1) \simeq 1$, $A(\text{He}^4) \simeq 4$, and setting $Z_i/A_i \simeq (1/2)$ for $Z_i > 2$, we have

$$\left\langle \frac{Z}{A} \right\rangle \simeq \frac{1}{2}(1+X), \tag{16.90}$$

$$\left\langle \frac{Z^2}{A} \right\rangle \simeq X + Y + B(1-X-Y), \tag{16.91}$$

where

$$B \equiv \sum_{i(Z_i>2)} \frac{c_i Z_i{}^2}{A_i} \tag{16.92}$$

and

$$c_i \equiv x_i/(1-X-Y) \tag{16.93}$$

is the mass abundance of element i relative to the mass abundance of all elements heavier than helium. We use the approximation $Z_i \simeq A_i/2$ for $Z_i \geq 2$, so that

$$B \simeq \bar{A}_{Z_i>2}/4, \tag{16.94}$$

where $\bar{A}_{Z_i>2}$ is the mean atomic weight (mass-fraction average) of all elements heavier than helium ($\bar{A}_{Z_i>2} = \sum_{Z_i>2} c_i A_i$). We may finally write (16.89) in the form

$$\kappa_{ff} \simeq 37.6[X+Y+B(1-X-Y)](1+X)\bar{g}_{ff,\,\mathrm{av}} F'(\rho,T)\rho/T_6^{3.5}\,\mathrm{cm}^2/\mathrm{gm}\,, \qquad (16.95)$$

where $T_6 \equiv T(^\circ\mathrm{K})/10^6$. Since $1-X-Y \ll 1$ in most stars and since B is usually in the range, say, 5–10, then the term in (16.95) involving B is usually negligible compared to the term $X+Y$. Hence most of the free-free opacity in stars of "normal" composition is contributed by hydrogen and helium, essentially because of their great relative abundance.

It is instructive to pause for a moment to review briefly how the proportionality of κ_{ff} to the quantity $\rho/T^{3.5}$ comes about when the electron gas is non-degenerate ($F = F' = 1$). We recall that, according to the original Kramers formula, the monochromatic absorption coefficient for free-free transitions satisfied the proportionality

$$\kappa_\nu(ff) \propto \frac{n_e}{\nu^3} \int_0^\infty \frac{1}{v(p)} \frac{\mathrm{d}n_e(p)}{n_e}$$

(regarding $g_{ff}(\nu,p)$ as a constant). Replacing the integral by $1/\bar{v}$, where $\bar{v}$, an average electron velocity, is proportional to $T^{(1/2)}$ if the electron gas is non-degenerate, and using $u = h\nu/kT$, we obtain

$$\kappa_\nu(ff) \propto \frac{n_e}{T^{3.5}} \frac{1}{u^3}. \qquad (16.96)$$

Forming the Rosseland mean of $\kappa_\nu(ff)$ is equivalent, roughly, to setting $u = const. \simeq 7$. Since, further, $n_e \propto \rho$ in regions where the material is fully ionized, we have $\kappa(ff) \propto \rho/T^{3.5}$, where the factor of proportionality does not depend on ρ and T if the electron gas is non-degenerate (aside from the very weak dependence on the Gaunt factor).

16.6c Bound-Free Opacity

To derive an approximate expression for pure bound-free opacity, we must use the appropriate D in (16.85). This is $D_{bf}(u)$ and it may be written in the form (see (16.63))

$$D_{bf}(u) = \sum_i \frac{x_i D_{ff}^{(i)}}{F\bar{g}_{ff,i} S_{f,i}^2} \sum_n \frac{I_n^{(i)}}{kT} \frac{g_{bf,i}(u,n) S_{n,i}^4}{n^3} \left[\frac{G(T)\bar{n}_n^{(i)}}{n_e}\right], \qquad (16.97)$$

where the first summation is over all relevant elements in the mixture, x_i is the relative mass abundance of the element of type i, and the second sum-

mation is over all levels of the atom or ion of type i for which $\chi_n^{(i)} < h\nu$. We introduce an average Gaunt factor $\bar{g}_{bf,i}$ (averaged over all relevant levels) for the element of type i by the writing (16.97) in the form

$$D_{bf}(u) = \sum_i \frac{x_i D_{ff}^{(i)} \bar{g}_{bf,i} f^{(i)}(u)}{F \bar{g}_{ff,i} S_{f,i}^2}, \tag{16.98}$$

where

$$f^{(i)}(u) \equiv \sum_n \frac{I_n^{(i)}}{kT} \frac{S_{n,i}^4}{n^3} \left[\frac{G(T)\bar{n}_n^{(i)}}{n_e}\right]. \tag{16.99}$$

An explicit expression for $\bar{g}_{bf,i}$ can be found by comparing (16.97) and (16.98). We also use (16.44b) for $D_{ff}^{(i)}$ in (16.98) and obtain

$$D_{bf}(\bar{g}) = 24.59 \left[\sum_i \frac{x_i Z_i^2 \bar{g}_{bf,i} f^{(i)}(u)}{A_i}\right] \frac{n_e}{T^{3.5}} \text{cm}^2/\text{gm} \tag{16.100}$$

$$= 24.59 \bar{g}_{bf,\text{av}} \left[\sum_i \frac{x_i Z_i^2 f^{(i)}(u)}{A_i}\right] \frac{n_e}{T^{3.5}} \text{cm}^2/\text{gm} \tag{16.101}$$

if n_e is in cm^{-3} and T is in degrees Kelvin, where $\bar{g}_{bf,\text{av}}$ is an average bound-free Gaunt factor for the mixture as a whole, defined by comparison of (16.100) and (16.101).

We write the quantity in square brackets in (16.101) in the form

$$\left[\sum_i \frac{x_i Z_i^2 f^{(i)}(u)}{A_i}\right] \simeq X f^{(\text{H})}(u) + Y f^{(\text{He})}(u) + \left[\sum_{i(Z_i>2)} \frac{c_i Z_i^2 f^{(i)}(u)}{A_i}\right](1-X-Y)$$

$$= X f^{(\text{H})}(u) + Y f^{(\text{He})}(u) + B\bar{f}(u)(1-X-Y), \tag{16.102}$$

where as usual X and Y denote relative mass abundances of hydrogen and helium,

$$B \equiv \sum_{i(Z_i>2)} \frac{c_i Z_i^2}{A_i} \simeq \bar{A}_{Z_i>2}/4, \tag{16.103}$$

and $\bar{f}(u)$ is the average of $f^{(i)}(u)$ over all relevant elements. Under conditions for which hydrogen and helium are almost completely ionized (either by "temperature ionization" at high temperatures, say $T > 10^6$ °K, or by pressure ionization at high densities, say $\rho > 10$ gm/cm^3), the first two terms on the right side of (16.102) may be small compared to the third and so may be neglected under these conditions. Physically, hydrogen and helium cannot contribute to the bound-free opacity if they are completely ionized (essentially

complete ionization implies that all $\bar{n}_n^{(i)}$ are small or zero). Under these conditions, then, which may be expected to obtain throughout most of a stellar interior (assuming a non-negligible heavy element abundance $(1-X-Y)$), we have

$$D_{bf}(u) \simeq 24.59 \bar{g}_{bf,\mathrm{av}} B(1-X-Y) \bar{f}(u) n_e / T^{3.5} \mathrm{cm}^2/\mathrm{gm}. \qquad (16.104)$$

The Rosseland mean opacity for bound-free transitions in elements heavier than helium is then given by the relation

$$\frac{1}{\kappa_{bf}} = \int_0^\infty \frac{W'(u)\mathrm{d}u}{D_{bf}(u)} = \frac{1}{24.59 \bar{g}_{bf,\mathrm{av}} B(1-X-Y)} \frac{T^{3.5}}{n_e} \int_0^\infty \frac{W'(u)\mathrm{d}u}{\bar{f}(u)} \mathrm{gm}/\mathrm{cm}^2$$

or

$$\kappa_{bf} = 24.59 \bar{g}_{bf,\mathrm{av}} B(1-X-Y) \frac{n_e}{T^{3.5}} \frac{1}{t} \mathrm{cm}^2/\mathrm{gm}, \qquad (16.105)$$

where

$$t \equiv \int_0^\infty \frac{W'(u)\mathrm{d}u}{\bar{f}(u)} \qquad (16.106)$$

is essentially the so-called "guillotine factor," which is a complicated but slowly varying function of the relative abundances of elements other than hydrogen and helium, and of density and temperature. This factor must be computed numerically and tabulated for each chemical composition under consideration. Numerically, t is normally around 10 or so and may vary between, say, 1 and 100. The guillotine factor is falling out of use nowadays because detailed numerical calculations of opacities usually give values of the opacity directly as a function of, say, chemical composition, density, and temperature. Tables of the guillotine factor, for a particular composition, may be found in Allen [Al63a, p. 94].

Under the conditions being considered here (complete ionization of hydrogen and helium), we may use the approximation (16.82) for n_e, so that we obtain, finally,

$$\kappa_{bf} \simeq 7.40 \times 10^3 B(1-X-Y)(1+X) \frac{\bar{g}_{bf,\mathrm{av}}}{t} \frac{\rho}{T_6^{3.5}} \mathrm{cm}^2/\mathrm{gm}. \qquad (16.107)$$

Typical values for B are, say, about 5 to 7, since most of the contribution to bound-free opacity in the deep interior is made by elements of "medium" atomic weight (say A ~ 12–40). Also, $\bar{g}_{bf,\mathrm{av}}$ will typically be of the order of unity.

We consider now some qualitative features of the dependence of κ_{bf} on ρ and T. Consider first the density dependence. For *very low* densities, the discrete levels which contribute most to the bound-free opacity (to order of magnitude, those for which $\chi_n^{(i)} \sim (1\text{–}10)kT$) may be expected to be unsaturated. For unsaturated energy levels (16.57) shows that the population is approximately proportional to n_e and hence (assuming hydrogen and helium to be completely ionized) also to the mass density ρ of the stellar material. Since the bound-free opacity is proportional to the population of these levels out of which electrons must be lifted in order for photoelectric absorption to occur, it follows that κ_{bf} is approximately proportional to ρ under conditions of very low density.

For *very high* densities, now, the discrete levels which contribute most to the bound-free opacity may tend to become saturated with the maximum number of electrons which they are capable of accommodating, again in accordance with (16.57). Hence κ_{bf} may become almost independent of n_e and also of ρ under conditions of very high density. Detailed calculations (*cf.* A.N. Cox [Co65]) confirm these qualitative expectations.

Consider now the dependence of κ_{bf} on temperature T. Aside from the factor $n_e T^{-3.5}$ in the expression for κ_{bf}, it is difficult to make any very precise or general statement about the temperature dependence of κ_{bf}. On the one hand, an increase in temperature will decrease the population of any given energy level (unless the level is highly saturated, *cf.* (16.57)) and hence the contribution of that level to the opacity. On the other hand, however, increasing temperature means increasingly energetic photons, for which lower-lying energy levels become available for photoelectric absorption. These two effects compete against each other for a given trend in temperature and therefore tend to cancel. The factor multiplying $n_e T^{-3.5}$ in the expression (16.105) for κ_{bf} should then be only a slowly varying function of T.

The fact that κ_{bf}, for a given value of the electron density n_e, decreases strongly with increasing temperature (roughly as T^{-3}) is due mainly to the dependence of $\kappa_{bf}(\nu)$ on $\nu(\kappa_{bf}(\nu) \propto \nu^{-3})$ and to the fact that taking the Rosseland mean is roughly equivalent to setting $u = h\nu/kT = const.$ Under conditions in which a predominantly abundant element is midway between two successive stages of ionization, n_e will be an increasing function of T in the case of "temperature ionization" and of ρ in the case of pressure ionization. An important example of the first case is the region of a stellar envelope or atmosphere of "normal" hydrogen-rich composition where the temperature is so low that hydrogen is not fully ionized. Because of the nature of the Saha equation, $n_e(T)$ is a *very* strongly increasing function of T in regions of partial hydrogen ionization (barring degeneracy of the electron gas). Since κ_{bf} increases with increasing n_e (except in the case of saturated

energy levels), κ_{bf} will generally *increase* strongly with increasing temperature in these low temperature regions. In addition, at these low temperatures absorption by the negative hydrogen ion makes a fairly strong contribution.

16.6d Relative Magnitudes

We first compare κ_{bf} and κ_{ff} under stellar interior conditions in which hydrogen and helium are assumed fully ionized and the electron gas is assumed non-degenerate. We note that, because the numerical coefficient in (16.107) for κ_{bf} is much larger than that in (16.95) for κ_{ff}, bound-free absorption will predominate over free-free absorption in the stellar interior if the heavy-element abundance $Z = 1-X-Y$ is sufficiently large. To find the values of Z such that

$$\kappa_{bf} \gtrsim \kappa_{ff}$$

under the assumed conditions, we set the quantity in square brackets, $\bar{g}_{ff,\mathrm{av}}$, and $F'(\rho,T)$ all equal to unity in (16.95) and set $\bar{g}_{bf,\mathrm{av}} = 1$, $B = 5$, and $t = 10$ in (16.107). We then have

$$7.40\times10^{24}\frac{5}{10}Z(1+X)\rho T^{-3.5} \gtrsim 3.76\times10^{22}(1+X)\rho T^{-3.5},$$

whence

$$Z = 1-X-Y \gtrsim 0.01\,. \tag{16.108}$$

Thus bound-free absorption will predominate over free-free absorption throughout most of the mass of a non-degenerate star if Z exceeds, say, 0.02, as will usually be the case with Population I stars. If Z is less than, say, 0.005, however, as is probably the case with Population II stars, then free-free absorption will predominate over bound-free absorption. For $Z\sim0.01$, both sources of absorption must normally be included.

We now examine approximately the conditions under which electron scattering will predominate over bound-free and free-free absorption. We require first that

$$\kappa_e \gtrsim \kappa_{bf}\,.$$

Again setting $\bar{g}_{bf,\mathrm{av}}B/t = (1/2)$, we obtain (see (16.84) and (16.107))

$$0.20(1+X) \gtrsim 7.40\times10^{24}\cdot\frac{1}{2}\cdot(1+X)\cdot Z\rho T^{-3.5}$$

or

$$T \gtrsim 1.66\times10^{7}\left(\frac{\rho}{100}\cdot\frac{Z}{0.01}\right)^{2/7}{}^{\circ}\mathrm{K}\,. \tag{16.109}$$

Since $\kappa_{bf} \simeq \kappa_{ff}$ for $Z = 0.01$, the condition that $\kappa_e > \kappa_{ff}$ (computed under the same assumptions as in the preceding paragraph) would have led to (16.109) except that the factor $Z/0.01$ would have been missing. From (16.109) we have that $\kappa_e \simeq \kappa_{ff}$, or $\kappa_e \simeq \kappa_{bf}$ for $Z = 0.01$, along the line defined by the relation

$$\log \rho = 3.5 \log T - 23.27. \tag{16.110}$$

This line divides the $\rho - T$ plane into two regions, corresponding to electron scattering on the one hand, and *either* bound-free (for $Z \simeq 0.01$) *or* free-free absorption, on the other (see Fig. 16.3).

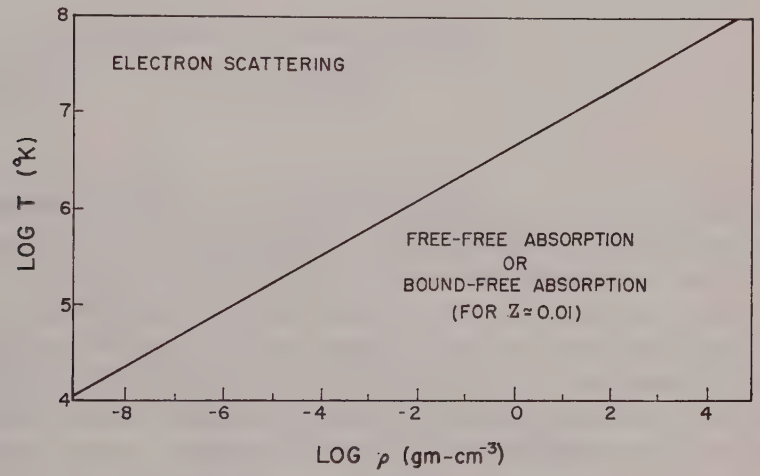


Fig. 16.3 Regions on the ρ-T plane where the indicated mechanisms provide the dominant contribution to the Rosseland mean opacity. Hydrogen and helium are assumed completely ionized and the electron gas is assumed non-degenerate in the vicinity of the demarcation line.

Thus, for $\rho \sim 10$–100 gm/cm^3 (typical stellar interior densities) and $Z \simeq 0.01$, electron scattering will predominate over bound-free and free-free absorption for temperatures in excess of, say, some 10^7 °K.

We recall that (16.84), (16.95), and (16.107), respectively giving the *individual* Rosseland means of the monochromatic absorption coefficients for pure electron scattering, free-free transitions, and bound-free transitions, may be added to give the Rosseland mean of the *total* monochromatic absorption coefficient only if two of the individual opacities are small compared with the third (see comments made in Sect. 16.5). Thus, if we are considering the total Rosseland mean opacity in the region of a star which falls near the "demar-

cation" line of Fig. 16.3, we may *not* properly compute this opacity by simply adding the opacities as given by (16.84), (16.95), and (16.107), even though these equations were used to define the demarcation line. Such a sum might, however, give a good enough approximation to the actual opacity in approximate work. A slightly more accurate way of computing the opacity in the vicinity of the demarcation line might be to use (16.73) if monochromatic absorption coefficients are available. Direct use of tabulated opacities is sometimes the most accurate and simplest procedure and this procedure is quite common in modern stellar structure calculations.

16.6e Interpolation Formulae

One frequently uses for approximate work an interpolation formula of the form

$$\kappa = \kappa_0 \rho^n T^{-s}, \tag{16.111}$$

where κ_0 is a function of chemical composition, and n and s are parameters whose values are adjusted so that (16.111) fits tabulated opacities (including all sources of opacity included in the tables) over the temperature and density range of interest. Formulae such as (16.111) are not accurate but are nevertheless useful in many cases for approximate work. The case $n = 1$, $s = 3.5$ is often referred to as the "Kramers opacity." As we have seen, this case strictly applies only to free-free absorption in a non-degenerate gas in which most elements are fully ionized. The case $n = 0.75$, $s = 3.5$ is sometimes called the "Schwarzschild approximation." The case $n = s = 0$ might correspond to the case of pure electron scattering opacity.

16.7 Electron Thermal Conduction

The flux $\mathbf{F}_{\text{cond}}$ carried by thermal conduction is usually expressed in terms of a coefficient of thermal conduction ν_c and the temperature gradient, as follows:

$$\mathbf{F}_{\text{cond}} = -\nu_c \nabla T, \tag{16.112}$$

where the dimensions of ν_c are ergs/cm/sec/degree in the c.g.s. system. It is convenient in astrophysical applications to define a "conductive opacity," κ_c, in analogy to the case of radiative transfer (*cf.* (8.14)):

$$\mathbf{F}_{\text{cond}} = -\frac{c}{\kappa_c \rho} \nabla \left(\frac{1}{3} a T^4 \right) = -\frac{4ac}{3\kappa_c \rho} T^3 \nabla T. \tag{16.113}$$

Comparison with (16.112) then yields the relation

$$\kappa_c = \frac{4acT^3}{3\nu_c \rho}, \tag{16.114}$$

so that $\kappa_c \propto 1/\nu_c$.

The "total" effective opacity κ_T, when energy is being carried by both radiation and thermal conduction (but with no contribution from convection), may be obtained in terms of κ_c and κ_R, the ordinary "radiative" opacity, by writing for the total flux

$$\mathbf{F}_{\text{total}} = -\frac{c}{\kappa_T \rho} \nabla \left(\frac{1}{3} a T^4 \right) = \mathbf{F}_{\text{rad}} + \mathbf{F}_{\text{cond}} . \tag{16.115}$$

Using (16.113) and a similar equation for the radiative flux, we obtain

$$\frac{1}{\kappa_T} = \frac{1}{\kappa_R} + \frac{1}{\kappa_c}, \tag{16.116}$$

so that κ_T is a harmonic mean of κ_R and κ_c. When $\kappa_c \gg \kappa_R$, we have $\kappa_T \simeq \kappa_R$ and, when $\kappa_c \ll \kappa_R$, we have $\kappa_T \simeq \kappa_c$.

Calculations of thermal conductivity have been carried out by L. Mestel [Me50], K.S. Singwi and M.K. Sundaresan [Si51], E. Schatzman [Sc58], A.N. Cox [Co65], and Wyller [Wy63, 63a, and 64]. We may, however, obtain a crude order-of-magnitude estimate of the value of κ_c for the case of a non-degenerate electron-ion plasma, by use of the elementary mean-free-path, or diffusion, approximation. We shall later give the results of Mestel's more accurate calculations.

In terms of this approximation the conductive flux is given, to order of magnitude, by the relation

$$F_c \sim -n_e \bar{v} \ell \partial \varepsilon / \partial x , \tag{16.117}$$

where n_e is the total number density of free electrons, $\bar{v}$ is the mean electron speed, ℓ is the "mean free path," or the average distance travelled by an electron between successive encounters with nuclei, and ε is the average translational kinetic energy per electron (see, for example, Chapman and Cowling [Ch52, p. 103]). We neglect electron-electron encounters and assume, for simplicity, that the gas is completely ionized. For non-relativistic, non-degenerate electrons we may take, assuming equipartition (see Sect. 10.7a),

$$\varepsilon = \frac{1}{2} m_e \bar{v}^2 = \frac{3}{2} kT; \tag{16.118}$$

so that

$$F_c \sim -\frac{3}{2} n_e \bar{v} \ell k \partial T/\partial x,$$

whence

$$v_c \sim \frac{3}{2} k n_e \bar{v} \ell. \tag{16.119}$$

For the mean free path we may take

$$\ell = 1/(n_i \sigma), \tag{16.120}$$

where n_i is the ion number density and σ is an average cross section for collision of an electron with an ion. To evaluate σ, we suppose that a "collision" has taken place when an electron has approached sufficiently close to a nucleus that the mean kinetic energy of the electron is equal to its electrostatic potential energy in the field of the nucleus:

$$\frac{1}{2} m_e \bar{v}^2 \simeq Ze^2/r_0, \tag{16.121}$$

where Ze may be taken as an average nuclear charge. Thus we have

$$r_0 \simeq Ze^2/\left(\frac{1}{2} m_e \bar{v}^2\right),$$

so that

$$\sigma = \pi {r_0}^2 \simeq \pi Z^2 e^4/\left(\frac{1}{2} m_e \bar{v}^2\right)^2 = \frac{4\pi}{9} Z^2 e^4/(kT)^2 \tag{16.122}$$

from (16.118). We use the relation $\bar{v} \simeq \sqrt{3kT/m_e}$ and write $n_e = (1/2)(\rho/H)(1+X)$ (approximately valid for complete ionization), X denoting the mass fraction of hydrogen, and $n_i = (\rho/H)/A$, where A may be taken as the average atomic mass of the nuclei. Substituting in (16.119), we have

$$v_c \sim \frac{27\sqrt{3}}{16\pi} \frac{k^{(7/2)}}{m_e^{(1/2)} e^4} \frac{1+X}{Z^2/A} T^{(5/2)}, \tag{16.123}$$

which shows that v_c does not depend on ρ in this approximation. Using (16.114), we obtain

$$\kappa_c \sim \frac{4ac}{3} \frac{16\pi}{27\sqrt{3}} \frac{m_e^{(1/2)} e^4}{k^{(7/2)}} \frac{Z^2/A}{1+X} \frac{T^{(1/2)}}{\rho}. \tag{16.124}$$

Inserting numbers, we obtain

$$\kappa_c \sim 5 \times 10^3 \frac{Z^2/A}{1+X} \frac{T_7^{(1/2)}}{(\rho/10)} \text{ cm}^2/\text{gm}, \tag{16.125}$$

where $T_7 \equiv T(°\text{K})/10^7$ and ρ is in gm/cm^3. It turns out that (16.125) is correct, for this non-degenerate case, to within a factor of 2 or 3 in spite of the crudity of our derivation.

We shall not attempt to derive an approximate formula for κ_c for the case of great degeneracy (however, formulae are presented later in this section). We note, however, that in this case $\bar{v}$ and ℓ will both be considerably larger than in the non-degenerate case. This is true for $\bar{v}$ because most electrons are moving with speeds appropriate to the maximum momentum p_0 (the "Fermi momentum") in a highly degenerate electron gas (see Sect. 24.1), and for ℓ because the unavailability of arbitrary final states for the electron, after collision, greatly increases ℓ. Hence we expect κ_c to become very small when the electron gas is highly degenerate. In this case most of the energy should be carried by electron conduction; as we shall see, this is indeed the case.

From Mestel's [Me50] work, in which the electron thermal conductivity was computed for non-relativistic electrons, we may write the conductive opacity in the general form

$$\kappa_c = 1.158 \times 10^3 \frac{\sum_i Z_i^2 x_i \Theta_i / A_i}{T_7 f(\eta)} \text{ cm}^2/\text{gm}, \tag{16.126}$$

where x_i is the relative mass abundance of element i and the summation is extended over all elements present; $T_7 \equiv T(°\text{K})/10^7$; and $f(\eta)$ is a monotonically increasing function only of the degeneracy parameter η. Equation (16.126) applies to the case of complete ionization, which is the case of interest when thermal conduction may be important. The quantity Θ_i is a slowly varying logarithmic term which essentially accounts for the effects of "distant" encounters between electrons and ions and is given by the relation

$$\Theta_i = \ln [2/(1 - \cos\theta_i)]^{(1/2)}, \tag{16.127}$$

where $Z_i^{(1/3)}\theta_i$ is also a function only of η. Numerically, Θ_i is nearly always close to unity. For partial electron degeneracy (say $-4 < \eta < +8$), $f(\eta)$ and $Z_i^{(1/3)}\theta_i$ must be obtained from numerical tabulations; Table 16.3 gives values of these functions for $-4 < \eta < +8$. Analytic expressions exist for small ($-\eta \gg 1$) and large ($\eta \gg 1$) degeneracy.

The function $f(\eta)$ increases from about 0.4 at $\eta = -4$ to about 2000 at $\eta = +8$, so that the effects of degeneracy reduce κ_c by a factor of about 5000 over this range of η; theoretically, $\kappa_c \to 0$ as $\eta \to \infty$ (complete degeneracy). In

this case $\kappa_T = \kappa_c \to 0$ and a star with such a vanishing opacity would necessarily become almost isothermal in a steady state if a finite flux of energy were being supported by conduction.

Table 16.3

TABULATION OF $f(\eta)$, $f(\eta)/e^{\eta}$, AND $Z_i^{(1/3)}\theta_i$

η	$Z_i^{(1/3)}\theta_i$	$f(\eta)$*	$f(\eta)/e^{\eta}$
−4.0	0.15479	0.4369	23.85
		(0.4396)	(24)
−3.0	0.21466	1.186	23.81
−2.0	0.29766	3.184	23.53
−1.0	0.39518	8.383	22.79
−0.2	0.48637	17.67	21.58
+0.4	0.54945	30.10	20.17
+1.0	0.60840	49.75	18.30
+1.6	0.65825	79.47	16.04
+2.2	0.69820	122.5	13.57
+2.8	0.72929	182.4	11.09
+3.4	0.75305	262.9	8.770
+4.0	0.77132	367.9	6.738
+5.0	0.79263	608.5	4.100
+6.0	0.80670	947.8	2.349
+7.0	0.81615	1406.	1.282
+8.0	0.82284	2001.	0.6713
		(1999.)	(0.6708)

* $f(\eta)$ is here computed by the relation:

$$f(\eta) = 3.828\ e^{\eta}\ [\Theta_{\mathrm{H}} I^{\mathrm{H}}(\eta)],$$

where the quantity in brackets is tabulated as a function of η by A.N. Cox [Co65]. The numbers in parantheses are found by the approximate analytical formulae (16.128) and (16.134).

For *small* degeneracy (say $\eta < -4$) we have

$$f(\eta) \simeq 24 e^{\eta}, \tag{16.128}$$

so that (16.126) becomes

$$\kappa_c \simeq 48.2 \frac{\sum_i Z_i^2 x_i \Theta_i / A_i}{T_7}\, e^{-\eta}\, \mathrm{cm}^2/\mathrm{gm}\ (\eta < -4). \tag{16.129}$$

Also

$$\theta_i \simeq 0.589(e^{\eta})^{(1/3)}/Z_i^{(1/3)}. \tag{16.130}$$

For $(-\eta) \gg 1$ we had (see (16.27))

$$e^{\eta} \simeq \frac{n_e h^3}{2(2\pi m_e kT)^{(3/2)}} = \frac{h^3 \rho(1+X)}{4H(2\pi m_e kT)^{(3/2)}} \tag{16.131}$$

if we use (16.82) for the electron density n_e. Inserting numbers, we obtain

$$e^{\eta} \simeq 1.973 \times 10^{-2}(1+X)\frac{\rho/10}{T_7^{(3/2)}} \ll 1, \tag{16.132}$$

where ρ is in gm/cm^3. Thus (16.129) becomes

$$\kappa_c \simeq 2450 \frac{\sum_i Z_i^2 x_i \Theta_i / A_i}{1+X} \frac{T_7^{(1/2)}}{(\rho/10)} \text{ cm}^2/\text{gm} \quad (\eta < -4), \tag{16.133}$$

which is in reasonable agreement with (16.125). Taking a typical value such as 5 for the value of the factor in (16.133) involving composition, we see that ρ would have to exceed 10^5 gm/cm^3 for $T_7 = 1$, in order that κ_c be of order unity and therefore possibly comparable to κ_R.* However, for $\rho = 10^5$ gm/cm^3 and $T_7 = 1$, (16.132) is grossly violated, *i.e.*, the electron gas would be degenerate at these densities and temperatures. In general, by setting $\kappa_c \sim 1$, taking 5 for the value of the factor in (16.126) involving composition, and requiring that $T_7 \lesssim 10$, say, we see from (16.126) that $f(\eta) \gtrsim 500$, *i.e.*, that $\eta \gtrsim 4$–5 (see Table 16.3). In other words, if thermal conduction is to be competitive with radiative transfer at temperatures less than some 10^8 °K, say, the material must be partially or highly degenerate. On the other hand, if the electron gas is non-degenerate ($\eta < -4$, say), then $f(\eta) < 1$ and (16.126) shows that $\kappa_c \lesssim 1$ only if $T \gtrsim 10^{10}$ °K. At such high temperatures the electron gas would be relativistic and the Mestel theory would no longer be valid. We may conclude that electron thermal conduction can essentially always safely be neglected as long as the electron gas is non-degenerate and non-relativistic.

For *large* degeneracy (say $\eta > +8$) we have

$$f(\eta) \simeq \frac{\pi^2(\eta^6 + 21.71\eta^4 + 136.4\eta^2 - 628.1)}{3\eta(\eta^2 + 9.870)} \simeq \frac{\pi^2}{3}\eta^3. \tag{16.134}$$

* If we neglect the possibility of the occurrence of Compton scattering (see Sects. 16.3 and 16.6a), then Thomson scattering sets a lower limit to the radiative absorption coefficient of appreciably ionized stellar matter of $0.2 \lesssim \kappa$ (cm^2/gm) $\lesssim 0.4$, the lower limit corresponding to X (relative mass abundance of hydrogen) = 0, the upper limit to $X = 1$ (see (16.84)).

Also, for $\eta \gg 1$, we have from (16.20), (16.28), and (16.82)

$$\frac{2}{\pi^{(1/2)}} \frac{2}{3} \eta^{(3/2)} \simeq 1.973 \times 10^{-2} (1+X) \frac{(\rho/10)}{T_7^{(3/2)}}$$

or

$$\eta^{(3/2)} \simeq 2.623 \times 10^{-2} (1+X) \frac{(\rho/10)}{T_7^{(3/2)}} \gg 1 . \tag{16.135}$$

Thus (16.126) gives the result

$$\kappa_c \simeq 5.12 \times 10^{-3} \frac{\sum_i Z_i^2 x_i \Theta_i / A_i}{(1+X)^2} \frac{T_7^2}{(\rho/10^5)^2} \text{ cm}^2/\text{gm} \quad (\eta > +8). \tag{16.136}$$

Also ($\eta > +8$),

$$\theta_i \simeq 0.848(1 - 2.06\eta^{-2})/Z_i^{(1/3)} . \tag{16.137}$$

We see then that, for $T_7 = 1$ and $\rho \gtrsim 10^4$ gm/cm^3, κ_c may become smaller than κ_R (assumed to be of order unity) and then electron conduction would predominate. Note from (16.135) that $\eta \gg 1$ under these conditions, so that the electron gas would indeed be highly degenerate.

16.8 Other Effects

Here we shall briefly discuss some of the more important sources of absorption (other than absorption by atoms, Compton and Thomson scattering, and electron thermal conduction) which were listed at the beginning of this chapter. Most of our comments have been based on the A.N. Cox opacity summary [Co65].

16.8a Bound-Bound Absorption

Efforts are currently being made by the Cox group to include the effects of bound-bound absorption. The problem is exceedingly complex owing to the large number of possible transitions in an atom. The absorption lines are broadened considerably in the stellar interior and this enhances their contribution to the opacity. It is clear (see Fig. 16.4) that large numbers of broadened absorption lines can conceivably increase the opacity considerably.

The sources of broadening of the lines are natural, collision, Doppler, and Stark broadening, of which collision broadening is the most important (see, for example, Aller's text [Al63b, Chap. 7] for a discussion of these

mechanisms of line broadening). According to A.N. Cox [Co65], bound-bound absorption is important for $T < 10^6$ °K and can increase the opacity by a factor of 2 or more. At $T \sim 10^7$ °K, the effects of lines probably increase the opacity by only 10 per cent or less.

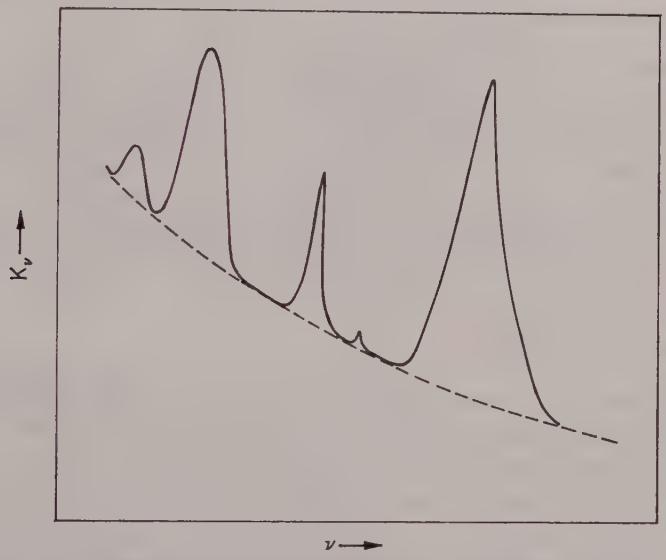


Fig. 16.4 Monochromatic mass absorption coefficient as a function of frequency when bound-bound transitions (absorption lines) are present (schematic).

Among the sources of absorption that are important at *low* temperatures (say $T < 10^4$ °K) are those described in Sects. 16.8b through 16.8f below.

16.8b Negative Ion Absorption

At low temperatures certain types of atoms or molecules can acquire one or more additional electrons each and thus become negative ions. Photo-ionization then produces a source of continuous opacity. Of most importance in astrophysics is probably the H^- ion, which is the principal source of absorption in the solar atmosphere and probably in most stars whose surface temperatures are less than about 7000 °K and greater than, say, 2000 ° or 3000 °K. The H^- ion has only one discrete energy level and the ionization potential from this level is 0.75 ev, which corresponds to a wavelength of $\lambda = 16{,}500$ Å. Thus bound-free absorption can occur for $\lambda \leq 16{,}500$ Å and free-free absorption in the field of H^- ions can occur for any wavelength, but

predominantly for those longer than 16,500 Å (since $\kappa_{ff}(\nu) \propto \nu^{-3}$, see (16.41).) Since the ionization potential of most negative ions is around 1 ev, then these ions are important only at the low temperatures that may be encountered in certain stellar atmospheres.

16.8c Molecular Absorption

Molecules may exist in stellar atmospheres with $T < 5000\,°\mathrm{K}$ and possibly in somewhat hotter atmospheres. Molecules may absorb either by means of photo-dissociation or by molecular band absorption. The importance of molecular absorption in a stellar atmosphere depends significantly on the chemical composition of the atmosphere as well as the temperature and density. Effects of molecular absorption have been investigated, for example, by M.S. Vardya [Va64]; Vardya and Böhm [Va65a]; and Yamashita [Ya62].

16.8d Rayleigh Scattering

We may derive the expression for the cross section for Rayleigh scattering by utilizing the classical picture of an electron in an atom, discussed in Sect. 2.10d. We represent such an electron as a harmonic oscillator of angular frequency ω_0. We consider such an oscillator in the field of an electromagnetic wave of angular frequency ω, whose electric field strength is $E = E_0 \sin \omega t$, where E_0 is the amplitude of the field strength. The equation of motion for the electron (assumed non-relativistic) of charge $-e$ is (neglecting the Lorentz force, *cf.* (2.152))

$$\ddot{r} = -\Gamma\dot{r} - \omega_0^2 r - (e/m_e)E_0 \sin \omega t, \tag{16.138}$$

where r is the distance of the electron from its equilibrium position and the term $-\Gamma\dot{r}$ is the "self-force" of the electron, *i.e.*, the dissipation via radiative damping due to energy losses which accompany the acceleration of the electron, see (2.153). We shall neglect this dissipative term in the discussion, but we shall retain it for the time being. Because of the presence of this dissipative term, the only steady state solution of (16.138) is the *particular solution* $r \propto \sin \omega t$, where the factor of proportionality does not depend on the time t. This solution is (*cf.* (2.154))

$$r = \frac{(e/m_e)E_0 \sin \omega t}{(\omega^2 - \omega_0^2) - i\omega\Gamma}, \tag{16.139}$$

from which we find

$$\ddot{r} = -\frac{\omega^2 (e/m_e)E_0 \sin \omega t}{(\omega^2 - \omega_0^2) - i\omega\Gamma}. \tag{16.140}$$

The total rate of radiation of energy by the electron (*i.e.*, the atom) is then, according to classical electrodynamics (see Sect. 16.3),

$$\frac{d\varepsilon}{dt} = \frac{2}{3}\frac{e^2}{c^3}|\ddot{r}|^2$$

$$= \frac{2}{3}\frac{e^4}{m_e{}^2c^3}\frac{\omega^4 E_0{}^2 \sin^2 \omega t}{(\omega^2-\omega_0{}^2)^2+(\omega\Gamma)^2}. \tag{16.141}$$

On the other hand, the "current density" of electromagnetic energy incident on the atom is (*cf.* (16.35))

$$j = (c/4\pi)E_0{}^2 \sin^2 \omega t. \tag{16.142}$$

We then obtain the expression for the cross section from the relation

$$\sigma = \frac{d\varepsilon/dt}{j},$$

which gives the result

$$\sigma = \sigma_0 \frac{\omega^4}{(\omega^2-\omega_0^2)^2 + (\omega\Gamma)^2}, \tag{16.143}$$

where $\sigma_0 = (8\pi/3)r_0^2 (r_0 = e^2/m_e c^2)$ is the Thomson scattering cross section, calculated in Sect. 16.3.

The case where $\omega^2 \ll \omega_0^2$ and $\Gamma^2 \ll \omega_0^2$ is called *Rayleigh scattering*, for which, from (16.143), the cross section is

$$\sigma_{\text{Rayleigh}} = \sigma_0(\nu^4/\nu_0^4) \ll \sigma_0, \tag{16.144}$$

where $\nu = \omega/2\pi$ and $\nu_0 = \omega_0/2\pi$ is the "resonance" frequency. This resonance frequency may be taken, for example, as the frequency corresponding to the energy of excitation of one of the excited levels or as the frequency at an absorption edge. Strictly speaking, ν_0 may be the frequency corresponding to the energy difference between any two discrete levels of the atom (see Heitler [He54, Sect. 19]).

In the opposite case where $\omega^2 \gg \omega_0^2$ and $\omega^2 \gg \Gamma^2$, (16.143) shows that σ reduces to σ_0, the Thomson scattering cross section. In this case the frequency ω of the electromagnetic wave is so high compared to the resonance frequency ω_0 that the scattering electron behaves as if it were free. We thus see that Rayleigh scattering differs from Thomson scattering only in that in the former process a *bound* electron (which has a resonance frequency ω_0) is driven by the impinging radiation at the frequency ω, where $\omega^2 \ll \omega_0^2$; in the latter process the scattering electron is either free (*i.e.*, it has no resonance frequency) or it behaves as if it were.

If the temperature in an atmosphere is low enough, then Rayleigh scattering may be the only significant source of continuous absorption in some important spectral regions. Bound-free absorption and line absorption would occur predominantly in the violet and ultraviolet regions of the spectrum, because most atoms would be in their ground states. On the other hand, there is very little radiation present in these spectral regions in a very cool stellar atmosphere. Because of the small degree of ionization in such a cool stellar atmosphere, there would be essentially no free electrons present. Hence no important contributions to the opacity from free-free absorption, H^- absorption, and electron scattering would be expected. Since σ_{Rayleigh} is very small (say $\sim 10^{-26} - 10^{-27}$ cm^2), such an atmosphere would be expected to be quite transparent in those spectral regions where molecular absorption is not important (compare with the earth's atmosphere, for which Rayleigh scattering is the major source of absorption at visible wavelengths).

16.8e Raman Scattering

Raman scattering is a form of non-coherent scattering (see Heitler [He54, Sect. 19]), and may in principle contribute to the absorption in stellar atmospheres, but its importance has not been investigated thoroughly (see A.N. Cox [Co65]). The cross section for Raman scattering is of the same order of magnitude as for Thomson scattering.

*16.8f Photo-Excitation to Auto-Ionizing States**

It is well known (see, *e.g.*, Condon and Shortley [Co35, pp. 369ff]) that in some multiple-electron atoms or ions the phenomenon of "auto-ionization" can take place. In this phenomenon two electrons initially are simultaneously in discrete excited states, and the sum, say E, of the excitation energies of the two electrons is greater than the energy required to remove one of the two electrons entirely from the atom or ion. In some cases it is possible for the atom or ion to make a "radiationless" transition to another state of the same total energy E, in which one of the two electrons is in its ground state and the other electron has been removed entirely from the atom or ion and is moving with a kinetic energy equal to the difference between E and the energy required to remove the one electron from the atom or ion. This process of auto-ionization must clearly be considered as a possible absorption mechanism, since the excitation energy of the two electrons, some or all of which may have been supplied by photon-induced transitions, is ultimately transformed entirely into ionization and kinetic energy.

* We are indebted to John C. Stewart (*private communication*, 1966) for suggesting this process to us as a possible mechanism of photon absorption in stars.

While the importance of this process for photon absorption in stellar interiors may at first sight seem small because *double* excitation is involved, it should nevertheless be kept in mind that at temperatures of several thousands or tens of thousands of degrees a non-negligible fraction of atoms or ions of the appropriate kind may at any given time already be *once* excited. Detailed calculations, however, have not yet been carried out to determine the quantitative importance of this absorption process in stellar interiors or atmospheres.

This process may be considered either as a special kind of "bound-bound" process or as a special kind of "bound-free" process. The first viewpoint is based on the fact that the *second* excitation of the atom or ion occurs between two *discrete* levels. It should, incidentally, be noted that, because of the finite probability of the atom or ion making a radiationless transition to the continuum, the upper of these two levels must be regarded as somewhat broadened. This broadening of the auto-ionizing level is analogous to the natural width of the upper level of an "ordinary" absorption line (in that the broadening occurs without external perturbers and therefore does not depend on density), but the magnitude of the broadening can be very much larger than for natural broadening. The second viewpoint is based on the fact that excitation of the second electron by absorption of a photon may ultimately result in the liberation of an electron.

We now consider some high temperature contributions to opacity.

16.8g Pair Production

Perhaps the most important source of absorption at *high* temperatures (say $T \gtrsim 10^9\,^\circ\mathrm{K}$) is the conversion of a photon into a positron-electron pair. The threshold energy is $2m_e c^2$, *i.e.*, we must have $h\nu > 2m_e c^2 \simeq 1.02$ Mev for production of a pair (this energy corresponds, setting $h\nu \sim kT$, to $T \sim 10^{10}\,^\circ\mathrm{K}$). Pair production can occur in the field of a nucleus of charge Ze or in the field of an electron or positron of charge $\mp e$.To order of magnitude, the cross section for pair production for $\alpha \equiv h\nu/m_e c^2 \sim 10 - 10^2$ (see Heitler [He54, Sect. 26]) is

$$\sigma_{\text{pair}} \sim \sigma_0 \frac{Z^2}{137}, \tag{16.145}$$

where σ_0 is the Thomson scattering cross section, $(1/137) = e^2/\hbar c$ is the fine structure constant, and Z is the atomic number of the appropriate nucleus if the pair is produced in the field of this nucleus, or is ± 1 if the pair is produced in the field of, respectively, a positron or an electron. According to Heitler [He54, Sect. 26], σ_{pair} increases rapidly with increasing $\alpha = h\nu/m_e c^2$

for $2 \lesssim \alpha \lesssim 10^3$ and increases only very slowly with increasing α for $\alpha \gtrsim 10^3$, say. Heitler gives in the extreme relativistic case ($\alpha \gg 1$) the formula

$$\sigma_{\text{pair}} \simeq \frac{3\sigma_0}{8\pi} \frac{Z^2}{137} \left[\frac{28}{9} \ln(2\alpha) - \frac{218}{27} \right] \quad (\alpha \gg 1). \tag{16.146}$$

On the other hand, the cross section for *Compton scattering* is given (for $\alpha \gg 1$) by

$$\sigma_{\text{Compton}} \simeq \frac{3\sigma_0}{8\alpha} \left[\ln(2\alpha) + \frac{1}{2} \right] \quad (\alpha \gg 1) \tag{16.147}$$

(*cf.* (16.39)). Hence, for a given value of Z, σ_{pair} becomes much larger than σ_{Compton} for $\alpha \gg 1$. However, as will be shown in Sect. 24.9, the number density of positron-electron pairs present under conditions of thermodynamic equilibrium increases rapidly (essentially exponentially) with increasing temperature and becomes very large for $kT/m_e c^2 \gg 1$, *i.e.*, for $T \gg 5.9 \times 10^9 \text{K}$. The Compton scattering from these very large numbers of positron-electron pairs should then predominate, in the *total* absorption, over the direct absorption due to pair production at such very high temperatures (*cf.* Sect. 16.6a).

Other "high-temperature" effects, such as *nuclear absorption* (for example, photo-disintegration at photon energies of several Mev) and *photon-photon scattering* (Milford [Mi57]; Heitler [He54, Sect. 32]), are probably dominated at high temperatures by the effects (both direct and indirect) of pair production.

16.9 A. N. Cox Opacity Results

In this section we summarize some of the results of the A.N. Cox Rosseland mean opacity calculations (A.N. Cox, Stewart, and Eilers [Co65a]) for a particular mixture of chemical elements, for a range of densities and temperatures of interest in problems of stellar structure. This mixture is the "Aller Mix," presumably representative of a Population I type chemical composition, and the detailed composition is given in Table 15.1. The effects of bound-bound (line) absorption, absorption by the H^- ion, atomic bound-free and free-free absorption, and absorption due to Thomson scattering have all been included in these calculations, but effects of molecular absorption have not. Further numerical results are given by A.N. Cox, Stewart, and Eilers [Co65a].

Figure 16.5 shows the Rosseland mean opacity κ, in units of cm^2/gm, as a function of temperature T, in units of Kev, for a number of densities ρ, in

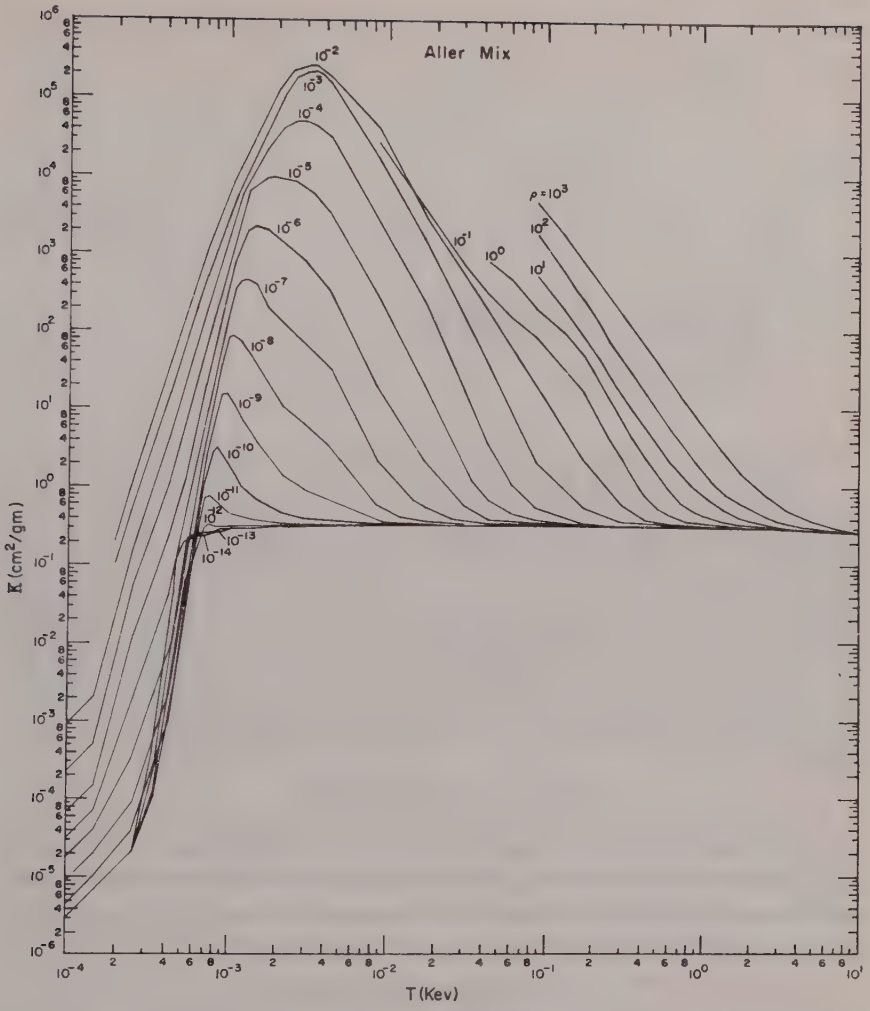


Fig. 16.5 Opacity as a function of temperature for various densities.

units of gm/cm^3. The steep initial increase in the value of κ with increasing temperature, for given density, results primarily from the increase of electron density with increasing temperature in the region of partial hydrogen ionization (*cf.* Sect. 16.6c). Beyond the maximum of κ (*i.e.*, at higher temperatures) for a given density, hydrogen has become almost fully ionized and κ then decreases with increasing T, *i.e.*, the opacity becomes, roughly speaking, "Kramers-like."

CHAPTER 17

Stellar Energy Sources

We are here concerned with the problem of accounting for the tremendous energy output of the stars. We are not concerned so much here with the question of the great *rate* of energy release by the stars, as this question is not necessarily directly related to the problem of energy sources. We are primarily concerned with the maintenance of this rate of energy release over very long periods of time. Geologic evidence tells us that the sun, for example, has been radiating at approximately its present rate for several billions of years. Now the average rate of loss of energy by radiation per gram for the sun is $\bar{\varepsilon} = L/M \approx 2 \text{erg/gm/sec}$. For stars on the main sequence we have

$$0.1 \lesssim \bar{\varepsilon}(\text{erg/gm/sec}) \lesssim 1000 .*$$

If the stars are in thermal equilbirium (*i.e.*, in perfect energy balance, see Chap. 5), then this loss of energy must be made good by energy sources in the star. If most of the energy is produced in, say, the inner 10 per cent of the mass of the star, as is normally the case, then we should write

$$1 \lesssim \bar{\varepsilon}(\text{erg/gm/sec}) \lesssim 10{,}000$$

in the energy-producing regions of main sequence stars. In both of the above cases, the lower limit applies to M stars, the upper limit to B stars. This gives us some idea as to the rate of energy production required to maintain the radiation of stars. For comparison, we note that the rate of human metabolism is about 10^4 erg/gm/sec. The reason that people are not as bright as stars is that they are not as massive.

One possible kind of energy source can be ruled out immediately, and this is energy supplied by chemical reactions as in the case, for example, of

* The curve defined by the main sequence can be represented, roughly, by the relation $L \propto T_e^{\,6}$; also the mass-luminosity relation is, roughly, $L \propto M^4$. Thus we have $M \propto T_e^{\,3/2}$, whence $\bar{\varepsilon} \propto T_e^{\,9/2}$, with $3000\,^\circ\text{K} \lesssim T_e \lesssim 25{,}000\,^\circ\text{K}$ for main sequence stars.

some sort of burning of a chemical fuel. Let us suppose that the sun radiates by virtue of some such sort of chemical burning process and that the luminosity, $L = 4 \times 10^{33}$ erg/sec, has remained constant since the "birth" of the sun. Then we ask, "How long could the sun have radiated at its present rate if only chemical sources provided the sole energy output?" The most efficient chemical cources can liberate $\sim 10^{12}$ erg/gm. But the rate of expenditure of energy by the sun is ~ 2 erg/gm/sec; hence the sun could radiate for $t \sim (10^{12}$ erg/gm)/(2 erg/gm/sec) $\sim 5 \times 10^{11}$ sec $\sim 2 \times 10^4$ years, on the basis of chemical sources alone. Since this time is orders of magnitude shorter than the age of the sun as inferred from geologic evidence, we conclude that chemical sources are hopelessly inadequate to account for the energy release by the sun. We note also that the sun is quite literally too "hot" to burn; for burning is a chemical reaction between *atoms* (or molecules) and most atoms are nearly completely ionized throughout most of the mass of a star.

Two other important sources of stellar energy remain, and these are gravitational and nuclear sources. Although, as we shall see, only nuclear sources can account for the release of energy by the stars over the required great lengths of time, gravitational sources nevertheless play an important role in several phases of the life of a star (see Chaps. 25 and 26). Hence gravitational sources will be discussed at some length in Sects. 17.1 through 17.6; the remaining sections in this chapter will be devoted to nuclear energy sources.

The gravitational potential energy of a star will be defined in Sect. 17.1. A fairly general form of the virial theorem will be derived in Sect. 17.2; this very important theorem will be applied in a number of different contexts in several places throughout the remainder of the book. The internal and total energy of a star will be defined in Sect. 17.3. In Sects. 17.4 and 17.5 the virial theorem will be applied to the general problem of gravitational contraction, including some conditions for gravitational contraction. Expressions will be derived in Sect. 17.6 for the *local* rate of release of gravitational energy in a star. Our entire discussion of gravitational energy sources will be based on Newtonian mechanics and special relativity. Considerations based on general relativity are generally not important (except possibly in very late stages of evolution; see Chaps. 25 and 26) for stars with masses less than some 100 solar masses (which is approximately the upper mass limit for the continued existence of "ordinary" stars, see Sect. 27.6a).

In Sect. 17.7 upper limits to stellar lifetimes on the basis of nuclear energy sources will be estimated. Some basic properties of atomic nuclei will be summarized in Sect. 17.8, and the Bohr picture of a nuclear reaction will be discussed in Sect. 17.9. The Breit-Wigner one-level dispersion formula for the cross section for nuclear reactions will be "derived" and discussed in Sect. 17.10, and in Sect. 17.11 the low energy limit (of most interest for

nuclear reactions in stars) of this formula will be obtained. General formulas for the rates of thermonuclear reactions will be derived in Sect. 17.12, and in Sects. 17.13 and 17.14 these formulas will be applied to obtain, respectively, the non-resonant and the resonant contributions to the thermonuclear reaction rate. Electron screening and its effect on the thermonuclear reaction rate will be discussed in Sect. 17.15. In Sects. 17.16 and 17.17 hydrogen burning reactions (particularly the "proton chain" and the "carbon cycle") and their rates of energy production will be considered. Helium burning reactions (particularly the "triple α reaction") will be discussed in Sect. 17.18, and carbon, oxygen, and neon burning in Sect. 17.19. Finally, neutrino energy losses will be considered briefly in Sect. 17.20.

Much of the material in Sects. 17.16 through 17.20 is based on a recent comprehensive review article by Reeves [Re65]; see also Reeves [Re66].

We do not consider, except occasionally in an incidental way, the important problem of element synthesis in stars, which is obviously closely related to the material in this chapter (the basic reference on this subject is Burbidge, Burbidge, Fowler, and Hoyle [Bu57]; see also the review article by Bashkin [Ba65e] and references given there).

17.1 Gravitational Potential Energy of a Star

We consider a star as initially "dispersed" to infinity, in which state the potential energy is taken to be zero. We know that gravitational energy will be released if this matter is brought into a configuration of finite radius. Suppose that we have already "brought" in from infinity into a sphere of radius r an amount of matter $M(r)$. (See Fig. 17.1.) We shall now bring in

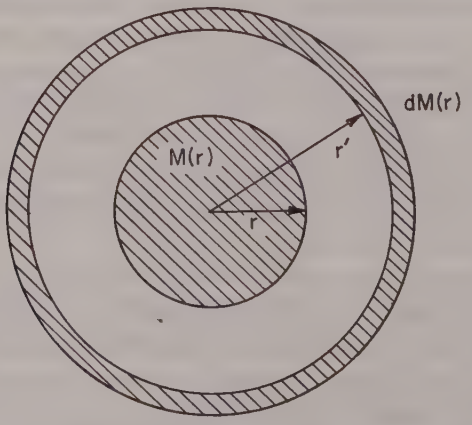


Fig. 17.1 Illustration for gravitational potential energy.

from infinity an additional amount of matter $dM(r)$ in the form of a thin spherical shell. When this shell is at distance r' from the center of the configuration, the attractive force due to the mass $M(r)$ will be $GM(r)dM(r)/r'^2$. The work required to bring $dM(r)$ from infinity to r will then be

$$d\Omega = -GM(r)dM(r)\int_r^\infty \frac{dr'}{(r')^2} = -G\frac{M(r)dM(r)}{r}. \tag{17.1}$$

Hence the amount of work required to bring the entire star in from infinity, or the *gravitational potential energy* of a spherically symmetric star of mass M, is

$$\Omega = -G\int_0^M \frac{M(r)dM(r)}{r}. \tag{17.2}$$

Equation (17.2) may also be written as

$$\Omega = -q\cdot\frac{GM^2}{R}, \tag{17.3}$$

where

$$q \equiv \int_0^1 \frac{M(r/R)/M}{r/R}\cdot\frac{dM(r/R)}{M} \tag{17.4}$$

is a dimensionless number whose value depends only on the mass concentration of the star.

A lower limit to the value of q may be found as follows: We defined the integral $I_{\sigma,\nu}$ by the relation (*cf.* (1.16))

$$I_{\sigma,\nu} = \int_0^1 \left[\frac{M(r)}{M}\right]^\sigma \left(\frac{r}{R}\right)^{-\nu} d\left[\frac{M(r)}{M}\right] \quad [3(\sigma+1)>\nu].$$

Thus we have

$$\Omega = -\frac{GM^2}{R}I_{1,1}, \tag{17.5}$$

which shows that $q = I_{1,1}$. Assuming only that hydrostatic equilibrium obtains and that $\bar{\rho}(r)$ does not increase outward, we showed that

$$I_{\sigma,\nu} \geq \frac{3}{3(\sigma+1)-\nu}.$$

Hence we have

$$q = I_{1,1} \geq \frac{3}{5} \quad \text{or} \quad \Omega \leq -\frac{3}{5}\frac{GM^2}{R};$$

the equality sign corresponds to the case of the *homogeneous* star, *i.e.*, a star of constant density throughout. For models of main sequence stars q is near unity, say $q \simeq 1.5$. We note here that for stars which are *polytropic* throughout, *i.e.*, stars in which $P = const.\ \rho^{(n+1)/n}$, with n constant throughout the star, the gravitational potential energy is (see Sect. 23.1)

$$\Omega = \frac{-3}{5-n} \cdot \frac{GM^2}{R}.$$

Note that $\Omega < 0$ for a star of finite radius; Ω is numerically equal to the amount of gravitational energy released in the process of contraction from a state of infinite dispersion.

17.2 The Virial Theorem

The virial theorem is one of the most powerful tools for obtaining an understanding of many general features of the stars and their evolution. The virial theorem may be expressed in a variety of different forms and also may be interpreted in a number of different ways. We shall derive a fairly general form of the virial theorem and then shall specialize it to deal with the various applications which we shall make of it. Our derivation will be based on a representation in which the system is regarded as a *continuous fluid*; this approach results in a fairly general form of the virial theorem. The virial theorem has been generalized to the "tensor virial theorem" by Chandrasekhar [Ch61, Chap. 13].

Assuming Newtonian mechanics to be applicable, the equation of motion of our fluid may be written in the general form (see, for example, Landau and Lifshitz [La59])

$$\ddot{\mathbf{r}} = -\frac{1}{\rho}\nabla \cdot \mathsf{P} + \mathbf{F}, \tag{17.6}$$

where the term $-\nabla \cdot \mathsf{P}/\rho$ (P = pressure tensor) is the force per unit mass due to the pressure gradient (or pressure tensor divergence) and $\mathbf{F}$ is the total "body" force acting upon unit mass, exclusive of forces due to pressure gradients. We now form the scalar product between $\mathbf{r}$ and (17.6) and integrate over the entire mass M of the fluid:

$$\int_M \mathbf{r} \cdot \ddot{\mathbf{r}} dm = -\int_V \mathbf{r} \cdot (\nabla \cdot \mathsf{P}) dV + \int_M \mathbf{r} \cdot \mathbf{F} dm, \tag{17.7}$$

where V denotes the total volume occupied by the fluid.

Consider first the left side of (17.7). We have

$$\mathbf{r}\cdot\ddot{\mathbf{r}} = \mathbf{r}\cdot\frac{\mathrm{d}}{\mathrm{d}t}(\dot{\mathbf{r}}) = \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r}\cdot\dot{\mathbf{r}}) - \dot{\mathbf{r}}^2 = \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}(\mathbf{r}\cdot\mathbf{r}) - \dot{r}^2,$$

so that

$$\int_M \mathbf{r}\cdot\ddot{\mathbf{r}}\,\mathrm{d}m = \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\int_M r^2\,\mathrm{d}m - \int_M \dot{r}^2\,\mathrm{d}m = \frac{1}{2}\ddot{I} - 2\mathscr{T}_{\text{mass}}, \tag{17.8}$$

where

$$I \equiv \int_M r^2\,\mathrm{d}m \tag{17.9}$$

is the "moment of inertia" of the fluid about the origin of coordinates and

$$\mathscr{T}_{\text{mass}} \equiv \int_M \frac{1}{2}\dot{r}^2\,\mathrm{d}m \tag{17.10}$$

is the total kinetic energy of "mass motion" (such as turbulence, rotation, pulsation, etc.) of the fluid.

Consider next the first term on the right side of (17.7). It is easy to verify the identity

$$\mathbf{r}\cdot(\nabla\cdot\mathsf{P}) = \nabla\cdot(\mathbf{r}\cdot\mathsf{P}) - \sum_i P_{ii}, \tag{17.11}$$

where $\sum_i P_{ii} = Tr\mathsf{P} = 3\bar{P}$, $\bar{P}$ being the *mean* pressure (*cf.* Sect. 10.1), averaged over all possible orientations of a surface element. Hence the appropriate term in (17.7) becomes

$$\int_V \mathbf{r}\cdot(\nabla\cdot\mathsf{P})\,\mathrm{d}V = \int_V \nabla\cdot(\mathbf{r}\cdot\mathsf{P})\,\mathrm{d}V - 3\int_V \bar{P}\,\mathrm{d}V.$$

The second term in the above equation can be transformed into a surface integral by use of the generalized divergence theorem (*cf.* Phillips [Ph33, Chap. 10]), so that we now have

$$\int_V \mathbf{r}\cdot(\nabla\cdot\mathsf{P})\,\mathrm{d}V = \oint_S (\mathbf{r}\cdot\mathsf{P}_S)\cdot\mathrm{d}\mathbf{A} - 3\int_V \bar{P}\,\mathrm{d}V, \tag{17.12}$$

where S denotes the (closed) surface bounding the fluid and P_S denotes the value of P on S. If $\mathsf{P}_S = P_S\mathsf{I}$, where I denotes the *unit* tensor (*i.e.*, if the stresses reduce to a pure hydrostatic pressure on the surface S), we have $(\mathbf{r}\cdot\mathsf{P}_S)\cdot\mathrm{d}\mathbf{A} = P_S\mathbf{r}\cdot\mathrm{d}\mathbf{A}$. If, in addition, P_S is constant on S, we have

$$\oint_S (\mathbf{r}\cdot\mathsf{P}_S)\cdot\mathrm{d}\mathbf{A} = P_S\oint_S \mathbf{r}\cdot\mathrm{d}\mathbf{A} = P_S\int_V \nabla\cdot\mathbf{r}\,\mathrm{d}V = 3P_SV. \tag{17.13}$$

We shall assume throughout the remaining discussion, merely for simplicity of writing, that (17.13) is valid. Note that this surface integral vanishes if $P_S = 0$ on S.

Finally, the second term on the right side of (17.7), termed the "virial of Clausius,"* is

$$\text{Virial} \equiv \int_M \mathbf{r} \cdot \mathbf{F} \mathrm{d}m . \tag{17.14}$$

Thus (17.7) may finally be written in the form

$$\frac{1}{2} \ddot{I} = 2\mathscr{T}_{\text{mass}} + 3 \int_V \bar{P} \mathrm{d}V - 3P_S V + \text{Virial} . \tag{17.15}$$

For a system in or nearly in a *static* state (neither expanding nor contracting secularly), we could set $\ddot{I} = 0$, in which case (17.15) would yield the relation between the various quantities therein for the static state. For a system in a *steady* state in which $\ddot{I} \neq 0$ (for example, a pulsating system or one with turbulent motions), a time average of (17.15) may be taken over a time long compared to the characteristic scale of time changes for the system. In this case, then, the left side of (17.15) may still be taken as zero if the quantities on the right side are regarded as appropriate time averages.

It is also possible to write (17.15) in a form in which the dynamical motions are treated relativistically (*i.e.*, in accordance with *special* relativity). We simply replace $\ddot{\mathbf{r}}$ (the force per unit mass) in (17.7) by the relativistically correct term $\dot{\mathbf{p}}$, where $\mathbf{p}$ is the relativistic momentum per unit rest mass (we shall always mean *rest* mass when we refer to mass). The relativistic relation between $\mathbf{p}$ and $\dot{\mathbf{r}}$ is (see, for example, Goldstein [Go50, Chap. 6]):

$$\mathbf{p} = \gamma \dot{\mathbf{r}} , \tag{17.16}$$

where

$$\gamma \equiv 1/\sqrt{1-\beta^2} , \ \beta \equiv v/c , \ v \equiv |\dot{\mathbf{r}}| ; \tag{17.17}$$

γ is also equal to the ratio of the total energy of a body to its rest energy and c is the velocity of light in vacuo. Upon forming the scalar product between $\mathbf{r}$ and (17.6), we obtain for the left side

$$\mathbf{r} \cdot \dot{\mathbf{p}} = \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r} \cdot \mathbf{p}) - \mathbf{p} \cdot \dot{\mathbf{r}}$$

$$= \frac{\mathrm{d}}{\mathrm{d}t}(\gamma \mathbf{r} \cdot \dot{\mathbf{r}}) - \gamma \dot{r}^2$$

* It is also possible to define the virial so as to include the pressure forces, but we do not do so here.

$$= \frac{\mathrm{d}}{\mathrm{d}t}\left[\gamma \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathbf{r}\cdot\mathbf{r}}{2}\right)\right] - \gamma\dot{r}^2$$

$$= \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left[\gamma\left(\frac{r^2}{2}\right)\right] - \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{r^2}{2}\frac{\mathrm{d}\gamma}{\mathrm{d}t}\right] - \gamma\dot{r}^2 . \tag{17.18}$$

We may write (17.18) in terms of the translational kinetic energy per unit mass which, according to the definition of γ, is given by the relation (in the relativistic sense kinetic energy means "total" energy minus rest energy)

$$t_{\mathrm{K}} = (\gamma - 1)c^2 \tag{17.19}$$

$$= \frac{\gamma^2 - 1}{\gamma + 1}c^2$$

$$= \frac{\gamma^2\beta^2}{\gamma + 1}c^2 \quad [\text{by (17.17)}]$$

or

$$t_{\mathrm{K}} = \frac{\gamma^2}{\gamma + 1}\dot{r}^2 . \tag{17.20}$$

Hence (17.18) becomes

$$\mathbf{r}\cdot\dot{\mathbf{p}} = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left[\gamma\left(\frac{r^2}{2}\right)\right] - \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{r^2}{2}\frac{\mathrm{d}\gamma}{\mathrm{d}t}\right] - \frac{\gamma+1}{\gamma}t_{\mathrm{K}} . \tag{17.21}$$

Integrating over the entire mass of the fluid, we have

$$\int_M \mathbf{r}\cdot\dot{\mathbf{p}}\,\mathrm{d}m = \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\int_M \gamma r^2\,\mathrm{d}m - \frac{\mathrm{d}}{\mathrm{d}t}\int_M \frac{r^2}{2}\frac{\mathrm{d}\gamma}{\mathrm{d}t}\,\mathrm{d}m - \int_M \frac{\gamma+1}{\gamma}t_{\mathrm{K}}\,\mathrm{d}m . \tag{17.22}$$

Defining now

$$I \equiv \int_M \gamma r^2\,\mathrm{d}m , \qquad \mathscr{T}_{\text{mass}} \equiv \int_M t_{\mathrm{K}}\,\mathrm{d}m \tag{17.23}$$

as the relativistic "moment of inertia" about the origin and the total relativistic "kinetic energy of mass motions," the relativistic form of the virial theorem becomes

$$\frac{1}{2}\ddot{I} - \frac{\mathrm{d}}{\mathrm{d}t}\int_M \frac{r^2}{2}\frac{\mathrm{d}\gamma}{\mathrm{d}t}\,\mathrm{d}m = \left\langle\frac{\gamma+1}{\gamma}\right\rangle \mathscr{T}_{\text{mass}} + 3\int_V \bar{P}\,\mathrm{d}V - 3P_S V + \text{Virial}, \tag{17.24}$$

where $\langle(\gamma+1)/\gamma\rangle$ is an appropriate average over the whole mass of the fluid. Again, the left side of (17.24) may be set equal to zero for a steady or stationary state. Note that in the non-relativistic limit $(\gamma \to 1)$, $\langle(\gamma+1)/\gamma\rangle \to 2$

and (17.24) reduces to the non-relativistic form (17.15). In the extreme relativistic limit ($\gamma \gg 1$), $\langle(\gamma+1)/\gamma\rangle \to 1$.

Let us now consider the expression for the virial when the only forces acting on an element of the fluid, other than pressure-gradient forces, are those due to self-gravitation of the fluid mass. If the fluid is spherically symmetric, we have

$$\mathbf{F} = -G\frac{M(r)}{r^3}\mathbf{r}, \tag{17.25}$$

which gives for the virial

$$\begin{aligned} \text{Virial} &= \int_M \mathbf{r}\cdot\mathbf{F}\mathrm{d}m \\ &= -\int_M \frac{GM(r)\mathrm{d}M(r)}{r} \\ &= \Omega \end{aligned} \tag{17.26}$$

by (17.2), where Ω is the gravitational potential energy of the fluid sphere.

More generally, we may for the moment consider the discrete-particle representation, for which the virial is given by

$$\text{Virial} = \sum_i \mathbf{r}_i \cdot \mathbf{F}_i, \tag{17.27}$$

where $\mathbf{F}_i$ is the force exerted on particle i with position vector $\mathbf{r}_i$ and the summation is extended over all the particles in the system. We suppose that the only forces entering into the virial are the mutual forces of interaction among the various particles in the system. We also assume that the forces acting between every pair of particles are of the same physical nature and that they are derivable from a potential (*i.e.*, conservative forces). Let Ω_{ij} be the potential energy of particle i due to the presence of particle j. We assume that $\Omega_{ij} = \Omega_{ji} = f(r_{ij})$, or that Ω_{ij} is a function only of the scalar distance r_{ij} between particles i and j; we also take $\Omega_{ii} = 0$.

The potential energy of particle i due to the presence of all the particles in the system is now

$$\Omega_i = \sum_j \Omega_{ij} \tag{17.28}$$

and the total potential energy of the system due to the mutual forces of interaction is obtained by summing Ω_{ij} over all *distinct pairs* of particles:

$$\Omega = \sum_{i<j} \Omega_{ij} = \frac{1}{2}\sum_{ij} \Omega_{ij} = \frac{1}{2}\sum_i \Omega_i, \tag{17.29}$$

where the factor (1/2) arises from the fact that in summing over *both* i and j, each pair is counted twice. Since the forces are assumed conservative, we may write

$$\mathbf{F}_{ij} = -\nabla_i \Omega_{ij} \tag{17.30}$$

for the force exerted on particle i by particle j, where ∇_i is the "field" gradient, taken with respect to the coordinates of particle i [$\nabla_i = \nabla_i(\partial/\partial x_i, \partial/\partial y_i, \partial/\partial z_i)$]. The total force exerted on particle i by all the particles in the system is then

$$\mathbf{F}_i = \sum_j \mathbf{F}_{ij} = -\nabla_i \Omega_i .$$

For $\Omega_{ij} = \Omega_{ij}(r_{ij})$, we have for the virial, using (17.28),

$$\text{Virial} = -\sum_{ij} \mathbf{r}_i \cdot \nabla_i \Omega_{ij} = -\sum_{ij} \frac{d\Omega_{ij}}{dr_{ij}} \mathbf{r}_i \cdot \nabla_i r_{ij} . \tag{17.31a}$$

But

$$\nabla_i r_{ij} = \nabla_i |\mathbf{r}_i - \mathbf{r}_j| = \nabla_i \sqrt{(x_i-x_j)^2+(y_i-y_j)^2+(z_i-z_j)^2} = \frac{\mathbf{r}_i - \mathbf{r}_j}{r_{ij}},$$

whence (17.31a) becomes

$$\begin{aligned} \text{Virial} &= -\sum_{ij} \frac{d\Omega_{ij}}{dr_{ij}} \frac{\mathbf{r}_i \cdot (\mathbf{r}_i - \mathbf{r}_j)}{r_{ij}} \\ &= \sum_{ij} \frac{d\Omega_{ij}}{dr_{ij}} \frac{\mathbf{r}_j \cdot (\mathbf{r}_i - \mathbf{r}_j)}{r_{ij}}, \end{aligned}$$

where in the second equality we have simply interchanged i and j (we can do this since i and j are just dummy indices). Adding the above two equations and noting that $(\mathbf{r}_i - \mathbf{r}_j) \cdot (\mathbf{r}_i - \mathbf{r}_j) = r_{ij}^2$, we obtain

$$\text{Virial} = -\frac{1}{2} \sum_{ij} r_{ij} \frac{d\Omega_{ij}}{dr_{ij}} . \tag{17.31b}$$

If Ω_{ij} is a homogeneous function of r_{ij} of degree $\nu+1$, *i.e.*, if $\Omega_{ij} = const.\ r_{ij}^{\nu+1}$, then it follows from Euler's theorem on homogeneous functions that $r_{ij} d\Omega_{ij}/dr_{ij} = (\nu+1)\Omega_{ij}$, whence we have

$$\text{Virial} = -\frac{1}{2}(\nu+1) \sum_{ij} \Omega_{ij} = -(\nu+1)\Omega \tag{17.32}$$

by (17.29). In particular, if the forces are purely gravitational, we have $\nu = -2$ and (17.32) reduces to (17.26). We shall always assume that the

forces are gravitational unless we explicitly state otherwise, so that we shall always use (17.26) as the expression for the virial.

Thus, for only gravitational and pressure forces present the non-relativistic form of the virial theorem, (17.15), becomes

$$\frac{1}{2}\ddot{I} = 2\mathscr{T}_{\text{mass}} + 3\int_V \bar{P}\,dV - 3P_S V + \Omega. \tag{17.33}$$

It should be pointed out that the virial theorem need not necessarily apply to the *entire* system, but may apply to only a *part* of the system. In the case of a spherically symmetric distribution of matter, for example, (17.33) applies as it stands to the spherical part of the system lying interior to the radial distance r ($< R$), provided 1) that the integrals in I, $\mathscr{T}_{\text{mass}}$, and Ω are taken only over $M(r)$, the mass interior to a sphere of radius r; 2) that the integral over $\bar{P}$ is taken only over $V(r)$, the volume of a sphere of radius r; 3) that P_S is taken as $P(r)$, the pressure at radial distance r; and 4) that $V \equiv V(r)$ in the P_S term. An application of the virial theorem as applied only to some interior region of a star will be found in Sect. 23.1.

If the pressure is due solely to the transfer of momentum by the particles of the system and if these particles obey Newtonian mechanics, we have (*cf.* (10.8))

$$\bar{P} = \bar{P}_{\text{kin}} = \frac{2}{3}u_{\text{kin}} \tag{17.34}$$

where u_{kin} is the translational kinetic energy per unit volume. In this case (17.33) becomes

$$\frac{1}{2}\ddot{I} = 2\mathscr{T}_{\text{mass}} + 2\mathscr{T}_{\text{th}} - 3P_S V + \Omega, \tag{17.35}$$

where

$$\mathscr{T}_{\text{th}} = \int_V u_{\text{kin}}\,dV \tag{17.36}$$

is the total energy of thermal motions in the system. The two terms $\mathscr{T}_{\text{mass}}$ and $\mathscr{T}_{\text{th}}$ of course add up to $\mathscr{T}$, the total kinetic energy in the system. Assuming for the moment that $P_S = 0$, we then obtain the form usually given for the virial theorem:

$$\frac{1}{2}\ddot{I} = 2\mathscr{T} + \Omega. \tag{17.37}$$

If there is a contribution to the pressure from sources other than the transfer of momentum by material particles (such as radiation pressure,

intermolecular forces such as are dominant in solids and liquids, magnetic stresses, etc.), then we may write for the total mean pressure

$$\begin{aligned}\bar{P} &= \bar{P}_{\text{kin}} + \bar{P}' \\ &= \frac{2}{3} u_{\text{kin}} + \bar{P}',\end{aligned} \tag{17.38}$$

assuming the material particles to obey Newtonian mechanics. In this case (17.33) becomes

$$\frac{1}{2}\ddot{I} = 2\mathscr{T}_{\text{mass}} + 2\mathscr{T}_{\text{th}} + 3\int_V \bar{P}' \mathrm{d}V - 3P_S V + \Omega, \tag{17.39}$$

where $\bar{P}'$ is the mean pressure from sources other than momentum transfer by material particles. For example, if $\bar{P}'$ is radiation pressure, then (neglecting effects of dispersion, *cf.* Sect. 2.5 and (10.9)) $\bar{P}' = (1/3)u_{\text{rad}}$, u_{rad} being the energy density of radiation, and the term containing $\bar{P}'$ becomes u_{rad}, the total radiant energy in the system.

Another example is provided by the case of a system permeated by a magnetic field. If the current density is $\mathbf{J}$ and the magnetic induction is $\mathbf{B} = \mu\mathbf{H}$, where μ is the magnetic permeability (assumed constant here) and $\mathbf{H}$ is the magnetic field strength vector, then the "ponderamotive" force per unit volume is (see, for example, Alfvén and Fälthammer [Al63, Chap. 3])

$$\mathbf{f} = \frac{\mathbf{J} \times \mathbf{B}}{c}.$$

If we neglect the displacement current in the Maxwell equations (as is conventionally done in hydromagnetics, *cf.* Alfvén and Fälthammer [Al63, Chap. 3]), we may write

$$\mathbf{J} = \frac{c}{4\pi} \nabla \times \mathbf{H},$$

whence

$$\mathbf{f} = \frac{\mu}{4\pi}(\nabla \times \mathbf{H}) \times \mathbf{H} = \frac{\mu}{4\pi}\left(\mathbf{H} \cdot \nabla \mathbf{H} - \frac{1}{2}\nabla H^2\right),$$

where we have made use of a well-known vector identity. Noting the vector identity $\nabla \cdot (\mathbf{AA}) = (\nabla \cdot \mathbf{A})\mathbf{A} + \mathbf{A} \cdot \nabla\mathbf{A}$ and recalling that $\nabla \cdot \mathbf{B} = 0$ from the Maxwell equations, we see that

$$\mathbf{f} = \frac{\mu}{4\pi}\left[\nabla \cdot (\mathbf{HH}) - \frac{1}{2}\nabla H^2\right] = \frac{\mu}{4\pi}\nabla \cdot \left(\mathbf{HH} - \frac{1}{2}H^2\mathsf{I}\right) = -\nabla \cdot \mathsf{P}_M,$$

where P_M, the magnetic pressure tensor, is given by the relation

$$\mathsf{P}_M \equiv \frac{\mu H^2}{8\pi}\mathsf{I} - \frac{\mu}{4\pi}\mathbf{HH},$$

where I is the unit tensor. If the only other source of pressure, aside from the magnetic field, is momentum transfer by Newtonian mass points, we have for the pressure tensor

$$\mathsf{P} = \left(P_{\text{kin}} + \frac{\mu H^2}{8\pi}\right)\mathsf{I} - \frac{\mu}{4\pi}\mathbf{HH}. \tag{17.40}$$

The mean pressure is then given by the relation (*cf.* (10.5))

$$\bar{P} = \frac{1}{3}\overline{Tr\mathsf{P}} = \frac{1}{3}\left[3\left(P_{\text{kin}} + \frac{\mu H^2}{8\pi}\right) - \frac{\mu H^2}{4\pi}\right] = \frac{1}{3}\left[3P_{\text{kin}} + \frac{\mu H^2}{8\pi}\right], \tag{17.41}$$

where $\mu H^2/8\pi$ is the energy density of the magnetic field.* Thus, writing $P_{\text{kin}} = (2/3)u_{\text{kin}}$, we have

$$3\int_V \bar{P}\,dV = 2\mathscr{I}_{\text{th}} + \mathscr{M} \tag{17.42}$$

where

$$\mathscr{M} \equiv \int_V \frac{\mu H^2}{8\pi}\,dV \tag{17.43}$$

is the total magnetic energy in the system.

A fairly general form of the virial theorem, then, including radiant and magnetic energy, is

$$\frac{1}{2}\ddot{I} = 2\mathscr{I}_{\text{mass}} + 2\mathscr{I}_{\text{th}} + U_{\text{rad}} + \mathscr{M} - \oint_S \mathbf{r}\cdot\mathsf{P}_S\cdot d\mathbf{A} + \Omega\,.\dagger \tag{17.44}$$

The introduction of the term $\mathscr{M}$ into the virial theorem was first effected by Chandrasekhar and Fermi [Ch53].

Finally, still another form of the virial theorem results from relating the *non-magnetic* contribution to the total pressure to the total internal energy per unit volume from all sources under consideration except magnetic energy and kinetic energy of mass motions; *i.e.*,

$$P_{\text{non-mag}} = (\gamma - 1)u, \tag{17.45}$$

* Note that the quantity $(1/3)(\mu H^2/8\pi)$ may be regarded as the "magnetic pressure."

† We have here replaced the term $-3P_S V$ by the more general surface integral, since P_S will in general not reduce to a simple hydrostatic pressure when magnetic fields are present.

where u is the total internal energy per unit volume exclusive of magnetic energy and kinetic energy of mass motions, and γ (generally a function of the state variables of the system) is defined by (17.45).* Replacing P_{kin} in (17.41) by $P_{\text{non-mag}}$, we obtain

$$\frac{1}{2}\ddot{I} = 2\mathscr{T}_{\text{mass}} + 3\langle\gamma-1\rangle U + \mathscr{M} - \oint_S \mathbf{r}\cdot\mathsf{P}_S\cdot d\mathbf{A} + \Omega, \tag{17.46}$$

where

$$U = \int_V u dV \tag{17.47}$$

is the total internal energy of the system exclusive of magnetic energy and kinetic energy of mass motions, and $\langle\gamma-1\rangle$ is an appropriate average of $(\gamma-1)$ over the entire volume of the fluid.

17.3 Internal Energy and Total Energy of a Star

We now apply (17.46) to a star in hydrostatic equilibrium (or in "quasi-hydrostatic" equilibrium) whose kinetic energy of mass motions may be neglected. We also assume that all pressures (including magnetic stresses, if any) vanish on the surface of the star.† Then (17.46) yields the result (dropping angular brackets from $\gamma-1$ and letting the symbol γ itself stand for an average of γ over the whole volume of the fluid)

$$3(\gamma-1)U + \mathscr{M} + \Omega = 0,$$

so that the total internal energy of the star is related to the gravitational potential energy and the magnetic energy by the relation

$$U = -\frac{1}{3(\gamma-1)}(\Omega+\mathscr{M}). \tag{17.48}$$

We define the *total* energy of the star in the present case as the sum of the internal, gravitational potential, and magnetic energies:

$$E = U+\Omega+\mathscr{M} \tag{17.49a}$$

$$= -(3\gamma-4)U \tag{17.49b}$$

$$= \frac{3\gamma-4}{3(\gamma-1)}(\Omega+\mathscr{M}) \tag{17.49c}$$

* Note that γ is not, in general, equal to any ratio of specific heats; $\gamma = c_p/c_V$ only in the special case of a simple perfect gas, *cf.* Sects. 9.14–9.18.

† If magnetic fields are present, one may take the surface so large that the magnetic field strength is very small over the surface.

for a star in or near hydrostatic equilibrium. We note that since $\Omega<0$ for a star of finite radius, $E<0$ for $\gamma>(4/3)$, provided that $\mathscr{M}<-\Omega$.

17.4 Gravitational Contraction

Gravitational contraction as a possible stellar energy source was originally proposed by H.L.F. Von Helmholtz and further developed by Lord Kelvin; hence the process is sometimes referred to as "Helmholtz-Kelvin," or simply "Kelvin," contraction.

Writing

$$\Omega = -q \cdot \frac{GM^2}{R} \tag{17.50}$$

(*cf.* (17.3)), where q is typically of order unity ($\simeq 1.5$ for main sequence stars), and neglecting magnetic energy for the present, we have for the total energy of a star

$$E = -\frac{3\gamma-4}{3(\gamma-1)} \cdot q \cdot \frac{GM^2}{R}. \tag{17.51}$$

We consider here only the case $\gamma>(4/3)$. We see, then, from (17.51) that $E=0$ when the star is dispersed to infinity and that $E<0$ when the star has condensed to the radius R; *i.e.*, the total energy of the star has decreased in shrinking from infinite radius to the radius R. This loss of energy has appeared in the form of radiation and represents the total energy radiated by the star in shrinking from infinite dispersion to the radius R. If the star is not deriving energy from nuclear sources (which is implicit in our assumption that $E=U+\Omega$, *i.e.*, in our not including nuclear energy in E), its total energy must be decreasing at the rate

$$-\frac{dE}{dt} = L(R) = -\frac{3\gamma-4}{3(\gamma-1)} \cdot q \cdot \frac{GM^2}{R^2}\frac{dR}{dt} \tag{17.52}$$

(assuming that q and γ do not change with time), whence the time required for the star to contract from infinite radius to the radius R is given (assuming no mass loss) by

$$t_{\text{con}} = \frac{3\gamma-4}{3(\gamma-1)} \cdot q \cdot GM^2 \int_R^\infty \frac{dR'}{R'^2 L(R')}. \tag{17.53}$$

The case $L(R')=L=$ constant yields the so-called "Kelvin" time t_{K}:

$$t_{\text{K}} = \frac{3\gamma-4}{3(\gamma-1)} \cdot q \cdot \frac{GM^2}{LR}. \tag{17.54}$$

Taking $\gamma=(5/3)$ (hence all internal energy is assumed here to be in the form of translational kinetic energy of molecular motions) and $q=1.5$ gives, with L, M, and R in solar units,

$$t_K \simeq 2 \times 10^7 \frac{M^2}{LR} \text{ years}, \tag{17.55}$$

i.e., $t_K \simeq 2 \times 10^7$ years for the sun. This time is at least two orders of magnitude shorter than the age of the sun as established on the basis of geologic evidence.

We note from (17.54) that, to order of magnitude,

$$t_K \sim \frac{|\Omega|}{L} \sim \frac{GM^2}{LR}. \tag{17.56}$$

The time t_K for the sun may also be estimated in the following way,which is of course basically equivalent to the above method. Taking $\gamma=(5/3)$, we have from (17.49) $|E| = U = -\Omega/2$. Hence, in contracting from infinite dispersion to its present radius, one-half of the gravitational energy released by the sun has gone into radiation and one-half into internal energy, thus raising the mean temperature to its present value. We have seen that the mean temperature of the sun is $\overline{T} \sim 10^7\,^\circ\mathrm{K}$; thus the average internal energy per gram is $(c_V/\mu)\overline{T} = (3\mathscr{R}/2\mu)\,\overline{T} \sim 2 \cdot 10^{15}$ erg/gm. Since the average rate of loss of energy per gram from the sun is $\bar{\varepsilon} = L/M \simeq 2$ erg/gm/sec, and since $(c_V/\mu)\overline{T}$ is the amount of energy contained in one gram, it follows that the Kelvin time for the sun is

$$t_K \simeq 2 \cdot 10^{15}/2 \simeq 10^{15} \text{ sec} = 3 \times 10^7 \text{ years},$$

in agreement with our previous results.

Another time, t_{grav}, may be defined by the relation

$$\frac{1}{t_{\text{grav}}} \equiv -\frac{\mathrm{d} \ln R}{\mathrm{d}t}, \tag{17.57}$$

so that t_{grav} is the time required for a star to change its radius by an appreciable fraction of itself as a result of gravitational contraction (if t_{grav} is constant, then it is the "e-folding" time for radius changes during gravitational contraction). From (17.52) we see that

$$t_{\text{grav}} = \frac{3\gamma - 4}{3(\gamma - 1)} \cdot q \cdot \frac{GM^2}{L(R)R}, \tag{17.58}$$

so that

$$t_{\text{grav}} = t_K(R). \tag{17.59}$$

Thus the time required for a star to change its present radius appreciably by means of gravitational contraction is comparable to the time which would have been required for the star to shrink from infinite dispersion to its present radius if its luminosity had remained constant during the contraction.

It is important to note that in actual stars $L(R)$ may decrease rapidly during at least the initial phases of gravitational contraction (*cf.* Chap. 26 and Hayashi, Hōshi, and Sugimoto [Ha62a]), so that t_{con} may actually be smaller than t_{K} by an appreciable factor. Moreover, recent studies (for example, Gausted [Ga63]; Gould [Go64]; Hayashi and Nakano [Ha65b]; Hayashi [Ha66a]) suggest that actual star formation occurs on a much shorter time scale than t_{K}. See Sect. 26.2 for further discussion of these and related points.

17.5 Some Conditions for Gravitational Contraction

In the present considerations we ignore mass motions, mass loss, and magnetic energy; we then have from (17.48) and (17.49) for a star in or near hydrostatic equilibrium

$$U = -\frac{1}{3(\gamma-1)}\Omega, \quad \Delta U = -\frac{1}{3(\gamma-1)}\Delta\Omega, \tag{17.60a}$$

$$E = \frac{3\gamma-4}{3(\gamma-1)}\Omega, \quad \Delta E = \frac{3\gamma-4}{3(\gamma-1)}\Delta\Omega, \tag{17.60b}$$

where we have assumed γ to remain constant during any changes in the condition of the star. It should be emphasized that (17.60) are valid *only* if the star is in or near hydrostatic equilibrium ($\ddot{I} = 0$, $\mathscr{I}_{\text{mass}} = 0$).

We consider three cases:

(1) $\gamma > (4/3)$: If the star is to radiate, and if the only sources of energy are thermal and gravitational, then $\Delta E < 0$, whence $\Delta\Omega < 0$; this implies a secular contraction, with the fraction $1/(3(\gamma-1))$ of the gravitational energy $|\Delta\Omega|$ released going into internal energy and the rest being radiated away. For $\gamma = (5/3)$ (perfect monatomic gas), $(1/2)|\Delta\Omega|$ goes into internal energy and an equal amount of energy is lost by radiation. Thus a star having $\gamma > (4/3)$ can furnish the energy radiated away by a slow, secular contraction.

(2) $\gamma = (4/3)$: Here we see from (17.60b) that $E = 0$ and $\Delta E = 0$ regardless of the values of Ω and $\Delta\Omega$. Thus a star with $\gamma = (4/3)$ could go from one equilibrium radius to another without any change in energy. This case represents a "transition" case, corresponding to a state of "neutral" equilibrium with zero total energy. Here $\Delta U = -\Delta\Omega$, so that all the gravitational energy released goes into internal energy, and none is left over for radiation.

If, however, a star with no nuclear energy sources has in the past (when γ was greater than (4/3)) lost energy by radiation (as would be the case for a real star), then its total energy E must be negative. If, now, for some reason (see a few paragraphs further on) γ abruptly decreases to the value (4/3) (*i.e.*, in so short a time that E cannot have changed appreciably), then the first equation of (17.60b) can no longer be valid and the star cannot, therefore, be in hydrostatic equilibrium. The star must, in fact, be in a state of dynamical collapse, as may be seen by considering the more general form of the virial theorem (17.46) (still neglecting magnetic energy and taking the surface integral equal to zero):

$$\frac{1}{2}\ddot{I} = 3(\gamma-1)U+\Omega+2\mathscr{I}_{\text{mass}}. \tag{17.61}$$

Taking now $E = U+\Omega+\mathscr{I}_{\text{mass}}$ for the total energy of the system, eliminating U between this equation and (17.61), and solving for E, we obtain

$$E = \frac{3\gamma-4}{3(\gamma-1)}\Omega + \frac{3\gamma-5}{3(\gamma-1)}\mathscr{I}_{\text{mass}} + \frac{1}{3(\gamma-1)}\frac{1}{2}\ddot{I}. \tag{17.62}$$

If the first term on the right side of (17.62) abruptly increases from a large negative value of order Ω to zero (because γ becomes equal to (4/3)) and if E remains negative, then the sum of the last two terms in (17.62) must abruptly become negative. If the star were assumed initially (before the abrupt decrease in γ) in approximately a steady state in which $\mathscr{I}_{\text{mass}}$ was negligible, then $\ddot{I} = (\mathrm{d}^2/\mathrm{d}t^2)\int_M r^2 \mathrm{d}m$ would have to be negative, *i.e.*, the star would have to be in a state of dynamical collapse.

That the collapse would be dynamical rather than secular follows from the following considerations. Suppose that γ had a value not greatly different from (5/3) prior to the abrupt decrease. Then we would have, according to the first equation of (17.60b) (which is valid prior to the abrupt decrease in γ), $E \sim (1/2)\Omega$. Since E is supposed not to change significantly during the abrupt decrease in γ, then E still has about this same value when $\gamma = (4/3)$, and (17.62) (with $\mathscr{I}_{\text{mass}}$ neglected, see preceding paragraph) then requires that $\ddot{I} \sim \Omega$. We make the order-of-magnitude approximation $\ddot{I} \sim I/\tau^2$, where τ is the time scale for the collapse, and write $I = MR_g^2$, where R_g (defined by this equation) is the "radius of gyration" of the star, of the order of the stellar radius R. We then obtain

$$\tau \sim [I/(-\Omega)]^{1/2} \sim (R^3/GM)^{1/2}, \tag{17.62'}$$

which is, to order of magnitude, the "free-fall" time (see Chaps. 0, 1, and 27). The collapse under discussion is therefore indeed a dynamical one.

We conclude that a star with no nuclear energy sources and having $\gamma=(4/3)$ would either be a non-radiating object in neutral equilibrium with zero total energy at any radius, or else possibly an object undergoing dynamical collapse. The more rigorous discussion in Sect. 27.5b (not based on the virial theorem) shows that dynamical expansion is also a possibility for a spherical object with $\gamma = (4/3)$.

On the other hand, a cloud of gas initially "dispersed to infinity" but not capable of dynamical collapse (for example, one which is stable against gravitational collapse), would simply remain dispersed to infinity and would not contract to form a star.

(3) $\gamma<(4/3)$: Here we see from (17.60b) that $\Delta E>0$ if $\Delta\Omega<0$; *i.e.*, a secular contraction would cause the total energy of the star to *increase*. Since this is clearly impossible for a star with no nuclear or external energy sources, we conclude that such a star would tend to remain dispersed to infinity. In this case we see from (17.60a) that $\Delta U>-\Delta\Omega$, which means that such a star in hydrostatic equilibrium demands a larger increase in internal energy than can be supplied by the gravitational energy released. Thus some source of energy other than thermal and gravitational is required for contraction.

Just as a dynamical collapse is required in the above case of a real star of negative total energy whose value of γ abruptly decreases to (4/3), a dynamical collapse would also result if γ abruptly decreased to a value less than (4/3). In this case, however, the collapse would be more violent than before, because now the first term on the right side of (17.62) would be *positive* for $\gamma<(4/3)$. Again, the work in Sect. 27.5a shows that dynamical expansion is also a possibility for a spherical object with $\gamma<(4/3)$.

The case (3) may apply to the contraction of a protostar to a true star, when the internal temperature of the protostar has risen to $\sim 1800\,^{\circ}\mathrm{K}$, at which temperature the hydrogen molecule H_2 begins to dissociate into atomic hydrogen. In this case the average γ for the whole protostar is likely to fall to a value less than (4/3), whereupon the protostar would undergo a rapid collapse. Cameron [Ca62a] has shown that the dynamical collapse will begin, for a star of solar mass, when its radius $R\sim 100$ astronomical units (A.U.), and will not stop until essentially all the hydrogen and helium in the protostar are completely ionized, when $R\sim 0.2$ A.U. This and related problems connected with star formation are discussed in Sect. 26.2 (see also the end of Sect. 17.4).

Such a dynamical collapse may also occur in the post-main sequence stages in the evolution of certain types of stars, as a result of an abrupt transformation of elements in the "iron group" (*cf.* Sect. 17.8) into helium under equilibrium conditions (*cf.* Chap. 26 and Hoyle and Fowler [Ho60]).

We conclude that secular gravitational contraction can supply the energy radiated by a star only for $\gamma > (4/3)$; for $\gamma \leq (4/3)$, the star would be dynamically unstable and could not contract slowly with release of gravitational energy. Thus stars whose material has six or more effective degrees of freedom per molecule (*cf.* (10.84)), or one in which gas pressure is negligible compared to radiation pressure, would probably be dynamically unstable. Also, if the gas particles are moving with velocities close to that of light, γ will approach (4/3), and again dynamical instability could result.

Finally, we note that dynamical instability could also result if a star possessed a sufficiently strong magnetic field in its interior. This may be seen from (17.49) giving the total energy (including magnetic energy) of a star in hydrostatic equilibrium:

$$E = \frac{3\gamma - 4}{3(\gamma - 1)}(\Omega + \mathcal{M}), \tag{17.63}$$

where $\mathcal{M}$ denotes the total magnetic energy of the star. Since E must be negative for a star in gravitational equilibrium, we must have (assuming that $\gamma > (4/3)$)

$$\mathcal{M} < -\Omega, \tag{17.64}$$

i.e., the total magnetic energy of the star must be less than the magnitude of the gravitational potential energy. Equation (17.64) provides an estimate of the order of magnitude of the maximum magnetic field strength which could exist in a gravitationally stable star. Writing $|\Omega| \sim GM^2/R$ and $\mathcal{M} \sim (4/3)\pi R^3 \cdot \bar{H}^2/8\pi$, we obtain

$$\bar{H} < \sim (6G)^{1/2} M/R^2 . \tag{17.65}$$

With M and R in solar units, we have

$$\bar{H} < \sim 3 \times 10^8 M/R^2 \text{ gauss} . \tag{17.66}$$

Hence, a star similar to the sun could be gravitationally stable only if the average magnetic field strength therein were less than some 10^8 gauss.

Further discussion of certain types of stellar instabilities may be found in Sect. 25.5 and in Chaps. 26 and 27.

17.6 Local Energy Release from Gravitational Contraction

We wish to consider now the energy release resulting from gravitational contraction, not from the star *as a whole*, but *locally*; *i.e.*, we wish to know the

rate of release of gravitational energy per unit mass *at each point* in a star which has no nuclear energy sources and which is contracting gravitationally.

We start with the first law of thermodynamics which, for unit mass of material, is

$$\frac{dQ}{dt} = \frac{\partial \mathscr{E}}{\partial t} + P\frac{\partial}{\partial t}\left(\frac{1}{\rho}\right), \tag{17.67}$$

where dQ/dt is the rate of gain of heat per unit mass, $\mathscr{E}$ is the internal energy per unit mass, and the time derivatives in (17.67) are "Stokes derivatives", taken following the motion of a particular element of fluid.* We now regard $\mathscr{E} = \mathscr{E}(P, \rho)$ as a function of P and ρ, so that (17.67) becomes

$$\begin{aligned}\frac{dQ}{dt} &= P\left(\frac{\partial \mathscr{E}}{\partial P}\right)_\rho \frac{\partial \ln P}{\partial t} - \left[\frac{P}{\rho} - \rho\left(\frac{\partial \mathscr{E}}{\partial \rho}\right)_P\right]\cdot\frac{\partial \ln \rho}{\partial t} \\ &= P\left(\frac{\partial \mathscr{E}}{\partial P}\right)_\rho \left\{\frac{\partial \ln P}{\partial t} - \frac{[P/\rho - \rho(\partial \mathscr{E}/\partial \rho)_P]}{P(\partial \mathscr{E}/\partial P)_\rho}\frac{\partial \ln \rho}{\partial t}\right\}.\end{aligned} \tag{17.68}$$

(We are here neglecting any change in chemical composition which may result from nuclear transmutations.) We note, however, that, since $\partial \ln P/\partial t \equiv \Gamma_1 \partial \ln \rho/\partial t$ for *adiabatic* motion ($dQ/dt = 0$), then we have

$$\Gamma_1 \equiv \frac{[P/\rho - \rho(\partial \mathscr{E}/\partial \rho)_P]}{P(\partial \mathscr{E}/\partial P)_\rho}. \tag{17.69}$$

We also note that we can write

$$P\left(\frac{\partial \mathscr{E}}{\partial P}\right)_\rho = P\cdot\frac{(\partial \mathscr{E}/\partial T)_\rho}{(\partial P/\partial T)_\rho} = T\frac{(\partial \mathscr{E}/\partial T)_\rho}{(\partial \ln P/\partial \ln T)_\rho} = \frac{c_V T}{\chi_T}, \tag{17.70}$$

where

$$c_V \equiv (\partial \mathscr{E}/\partial T)_\rho \tag{17.71}$$

is the specific heat per unit mass at constant volume and

$$\chi_T \equiv (\partial \ln P/\partial \ln T)_\rho. \tag{17.72}$$

Now, the net rate of gain of heat per unit mass, dQ/dt, is in general made up of energy released from nuclear sources at the rate ε_N per unit mass and of energy flowing out of the matter at the rate $(1/\rho)\nabla \cdot \mathbf{F} = \partial L_r/\partial M_r$ per unit mass, where $\mathbf{F}$ is the net flux and L_r is the net rate at which energy escapes via

* We use the notation dQ/dt, rather than $\partial Q/\partial t$, because dQ is not an exact differential (*cf.* Sect. 9.8), and use of the partial derivative seems to imply (incorrectly) that Q is a function in the same sense as are the thermodynamic variables.

radiation, convection, or conduction from a sphere of radius r (clearly, if $L_{r+dr} > L_r$, then more energy is flowing out of a shell of mass dM_r and thickness dr than is flowing in). There may also be energy losses at the rate ε_ν per unit mass due to the production of neutrinos (*cf.* Sect. 17.20). (We ignore other losses such as dissipation through viscous forces.) Thus $dQ/dt = \varepsilon_N - \varepsilon_\nu - \partial L_r/\partial M_r$, so that (17.68) may finally be written in the form

$$\varepsilon_N - \varepsilon_\nu - \frac{\partial L_r}{\partial M_r} = \frac{c_V T}{\chi_T}\left\{\frac{\partial}{\partial t}\left(\ln\frac{P}{\rho^{\Gamma_1}}\right) + (\ln\rho)\frac{\partial \Gamma_1}{\partial t}\right\}. \tag{17.73}$$

We note that we may write (17.73) in the form

$$\varepsilon_N + \varepsilon_{\text{grav}} - \varepsilon_\nu - \frac{\partial L_r}{\partial M_r} = 0, \tag{17.74}$$

where $\varepsilon_{\text{grav}}$, which denotes the local rate of gravitational energy release per unit mass, is given by

$$\varepsilon_{\text{grav}} = -\frac{c_V T}{\chi_T}\left[\frac{\partial}{\partial t}\left(\ln\frac{P}{\rho^{\Gamma_1}}\right) + (\ln\rho)\frac{\partial \Gamma_1}{\partial t}\right]. \tag{17.75}$$

If $\mathscr{E} = \mathscr{E}(\rho, T)$ were to be regarded as a function of ρ and T, the following alternative expression for $\varepsilon_{\text{grav}}$ would result:

$$\varepsilon_{\text{grav}} = -c_V T\left[\frac{\partial}{\partial t}\left(\ln\frac{T}{\rho^{\Gamma_3 - 1}}\right) + (\ln\rho)\frac{\partial \Gamma_3}{\partial t}\right]. \tag{17.75'}$$

It is clear from the definition of entropy that, if the changes under consideration in the system are reversible in the thermodynamic sense (*cf.* Chap. 9), we have

$$\varepsilon_{\text{grav}} = -T\frac{\partial S}{\partial t}, \tag{17.75''}$$

where S is the entropy per unit mass (however, see next paragraph). Hence, gravitational energy release by slow (reversible) contraction is always accompanied by a diminution of the entropy.

It should be noted that (17.75″) is *incorrect* if nuclear reactions or neutrino emission are occurring at the point under consideration (*i.e.*, if $\varepsilon_N \neq 0$ or $\varepsilon_\nu \neq 0$), unless chemical equilibrium (*cf.* Sect. 9.12) obtains among the nuclei, neutrinos, and other relevant particles (this condition may not obtain anywhere in the real universe because it is questionable whether neutrinos, with their negligible interaction with matter,* can ever come to

* At very high energies, however, the interaction of neutrinos with matter may not be negligible. The interaction of high energy neutrinos with matter has, in fact, been postulated as an important element in a recent theory of supernovae (see Colgate and White [Co66]).

equilibrium with matter). The correct expression in this case is (see Sect. 9.12)

$$\varepsilon_{\text{grav}}'' = -T\frac{\partial S}{\partial t} - \sum_i \mu_i \frac{\partial}{\partial t}\left(\frac{n_i}{\rho}\right), \tag{17.75'''}$$

where μ_i and (n_i/ρ) are, respectively, the chemical potential and the number per unit mass of the i^{th} kind of particle, and the summation is extended over all relevant kinds of particle. According to the second law of thermodynamics (*cf.* Chap. 9), this second term on the right always gives a *positive* (or zero) contribution to the entropy. Consequently, the summation in (17.75‴) must always be intrinsically *negative* (or zero).

In case $\varepsilon_N = \varepsilon_\nu = 0$, $\partial L_r/\partial M_r$ is seen to be determined entirely by $\varepsilon_{\text{grav}}$. In the special case of a simple perfect gas we have $\chi_T = 1$, $\Gamma_1 = \gamma$, $\partial\Gamma_1/\partial t = 0$, and $\mathscr{E} = c_V T$, so that we have

$$\varepsilon_{\text{grav}} = -\mathscr{E}\frac{\partial}{\partial t}\left(\ln\frac{P}{\rho^\gamma}\right) \tag{17.76}$$

$$= -\mathscr{E}\frac{\partial}{\partial t}\left(\ln\frac{T}{\rho^{\gamma-1}}\right). \tag{17.76'}$$

We note from (17.76) and (17.76′) that the local energy release is the result of departures from adiabaticity during the motion, since $P \propto \rho^\gamma$ and $T \propto \rho^{\gamma-1}$ along an adiabat. We note also that the assumption of gravitational contraction has not entered into our derivation of (17.75), (17.75′), (17.76) and (17.76′); indeed, these are merely alternative ways of writing the first law of thermodynamics. We see from (17.76) and (17.76′) that the local rate of energy release per unit mass is determined by the local internal energy per unit mass (*i.e.*, the temperature in the case of a perfect gas) and the local time rate of change of the quantity (P/ρ^γ) or of the quantity $(T/\rho^{\gamma-1})$, *i.e.*, of the entropy per unit mass.

Let us now apply (17.76) to a star that contracts *homologously* (*cf.* Chap. 22 for further details). This means that the star contracts in such a way that the *relative mass distribution* remains unchanged during the contraction, and the star always remains in almost perfect hydrostatic equilibrium. It can be shown that in this case the *shapes* of the curves of P and ρ as functions of $x = r/R$ are unaltered during the contraction, but the *values* of P and ρ at a given value of x do change during the contraction. If we consider an element of matter a fraction $x = r/R$ of the way from the center to

the surface, then, as we shall show in Sect. 22.1, we will have for the pressure and density of the mass element under consideration during a homologous contraction:

$$P(x) \propto R^{-4}, \tag{17.77}$$

$$\rho(x) \propto R^{-3}. \tag{17.78}$$

Thus

$$\frac{P(x)}{\rho^{\gamma}(x)} = f(x) R^{3\gamma-4} \tag{17.79}$$

and

$$\ln \frac{P}{\rho^{\gamma}} = \ln f(x) + (3\gamma-4) \ln R. \tag{17.80}$$

We now note that x does not change with time during a homologous contraction, since each element of matter always remains at the same *fractional* distance x from the center of the star. Thus we have

$$\frac{\partial}{\partial t}\left(\ln \frac{P}{\rho^{\gamma}}\right) = (3\gamma-4) \frac{\mathrm{d} \ln R}{\mathrm{d}t},$$

whence (17.76) becomes

$$\varepsilon_{\mathrm{grav}} = -(3\gamma-4) \mathscr{E} \frac{\mathrm{d} \ln R}{\mathrm{d}t}, \tag{17.81}$$

$$= -\frac{3\gamma-4}{(\gamma-1)} \frac{\mathscr{R}T}{\mu} \frac{\mathrm{d} \ln R}{\mathrm{d}t}, \tag{17.82}$$

where we have used the relations $\mathscr{E} = u/\rho$, $P = \mathscr{R}\rho T/\mu = (\gamma-1)u$ (u = internal energy per unit volume), valid for a simple perfect gas. Thus, for a star contracting homologously the rate of local gravitational energy release per unit mass is determined by the local internal energy per unit mass and the fractional rate of shrinkage of the star. Note that at each instant of time $\varepsilon_{\mathrm{grav}} \propto T$ for homologous contraction. Note also that $\varepsilon_{\mathrm{grav}} = 0$ for $\gamma = (4/3)$, which is in agreement with our previous conclusion that a star with $\gamma = (4/3)$ would be in neutral equilibrium at any radius. Equation (17.81) provides the direct mathematical justification for our crude calculation of the time scale for gravitational contraction in Sect. 17.4 since, to order of magnitude,

$$t_{\mathrm{grav}} \sim \frac{\bar{\mathscr{E}}}{\bar{\varepsilon}}.$$

Finally, note that in the present case of a simple perfect gas, homologous gravitational contraction leaves the relative entropy distribution in the star unaltered.

17.7 Nuclear Energy Production

The basic principle underlying all theories of nuclear energy production in stars is the conversion of matter into energy. The energy ΔE released due to a mass loss Δm is given by the famous Einstein relation

$$\Delta E = \Delta m c^2 .$$

This relation, along with certain nuclear data,* permits an estimate to be made of an upper limit to the lifetime of the sun on the basis of conversion of matter into energy. If hydrogen (H^1) is converted into helium (He^4), the mass of hydrogen consumed per helium nucleus formed is

$$\Delta m = 4 \times 1.0081 - 4.0039 = 0.0285 \text{ AMU},$$

or $\Delta m \sim 0.007$ AMU/nucleon. Since $M_{\text{nucleon}} \simeq 1$ AMU, then only about 0.7 per cent of the mass of a given quantity of hydrogen disappears when this hydrogen is converted into helium. If hydrogen were converted into iron, say, then about 0.85 per cent of the mass would be lost.

Thus the energy available per gram for the conversion of H^1 into He^4 is $\sim 0.007\ c^2 \simeq 0.007 \times 9 \times 10^{20} \simeq 6 \times 10^{18}$ erg/gm. If, now, the sun were composed entirely of hydrogen and if all this hydrogen were converted to helium, the time t required for this, assuming the sun to continue radiating at its present rate, would be

$$t \sim (6 \times 10^{18} \text{erg/gm})/(2 \text{erg/gm/sec}) \simeq 3 \times 10^{18} \text{sec} \sim 10^{11} \text{ years}.$$

This is the upper limit to the lifetime of the sun on the basis of conversion of hydrogen into helium. If, say, only 10 per cent of the hydrogen in the sun is available for conversion into helium, before its observable properties change significantly, then this lifetime would be $\sim 10^{10}$ years, which is certainly of the right order of magnitude to be in agreement with geologic evidence.

For a star composed initially of pure hydrogen we would then have for its "nuclear" time scale t_{nuc}

$$t_{\text{nuc}} = \frac{1}{10} \times 6 \times 10^{18} \frac{M}{L} \tag{17.83a}$$

$$\simeq 10^{10} M/L \text{ years}, \tag{17.83b}$$

* The following relations will find frequent use in the following sections:

$$1 \text{ electron volt (ev)} = 1.602 \times 10^{-12} \text{erg} = 11.605 \times 10^3 \,^\circ\text{K}$$

Atomic mass unit rest energy (physical scale):	1 AMU	= 931.1 Mev (million electron volts)
Electron rest energy:	$m_e c^2$	= 0.5110 Mev
Mass of hydrogen atom:	H^1	= 1.0081 AMU (atomic mass unit)
Mass of helium atom:	He^4	= 4.0039 AMU

where M and L are in solar units in (17.83b), and where again we have assumed that only ~ 10 per cent of the initial hydrogen is available for conversion into helium before the star's properties change significantly.

17.8 Basic Properties of Atomic Nuclei

We define the following quantities, which will be used extensively in the following work:

Z = atomic number = number of protons in a nucleus.

A = atomic mass number (integer) = number of nucleons (protons + neutrons) in a nucleus.*

$N = A - Z$ = number of neutrons in a nucleus.

Symbolically, a nucleus of atomic number Z and mass number A is denoted by $(Z)^A$. (Examples: C^{12}, N^{13}, H^2, O^{16}, ...).

Nuclei with common Z but different A are called *isotopes*;
nuclei with common A but different Z are called *isobars*;
nuclei with common N are called *isotones*.

Most of our brief survey of properties of atomic nuclei is based on such reference works as Blatt and Weisskopf [Bl52], Bethe and Morrison [Be56], and Preston [Pr62].

We define the *binding energy* of a nucleus as

$$\begin{aligned} \text{B.E.} &= [M_p Z + (A-Z)M_n - M_{\text{nuc}}]c^2 \\ &= [M_{\text{H}^1} Z + (A-Z)M_n - M_{\text{at}}]c^2, \end{aligned} \tag{17.84}$$

where M_{H^1} and M_{at} are the masses of the *neutral* atoms, and the binding energy of the atomic electrons has been neglected (being of the order of some tens of ev, as compared to some tens of Mev for B.E.). The binding energy is the amount of energy that would be required to separate the particles in a nucleus to infinity; conversely, it is the amount of energy that would be released in bringing the nuclear particles in from infinity to form a nucleus. The B.E. of a stable nucleus is always positive because the mass of such a nucleus is always less than the sum of the masses of its constituent parts.

* In other chapters in this book we have used the symbol A to denote *atomic mass* (in units of AMU), and not *atomic mass number* (an integer). We change our notation in this chapter to comply with the convention used in nuclear physics. In this chapter the symbol $\mathscr{A}$ will be used to denote atomic mass, and the symbol A to denote mass number.

We now define the *average binding energy per nucleon*, or the *binding fraction*, f:

$$f = (\text{B.E.})/A \sim 7 \text{ Mev/nucleon}. \tag{17.85}$$

A plot of f as a function of A would appear, schematically, as shown in Fig. 17.2, where we have drawn the curve as if it were smooth (actually, it has irregular "wiggles" of magnitude about $\pm$ 0.1 Mev). The approximate constancy of f with respect to A over large ranges in A suggests a "saturation" of nuclear forces; each nucleon is roughly equally tightly bound, as is the case of atoms or molecules in liquids and solids. This property provides, in part, a justification for the "liquid drop" model of the nucleus. Note that f reaches a maximum of about 8.5 Mev in the region $40 \lesssim A \lesssim 80$, the so-called "iron

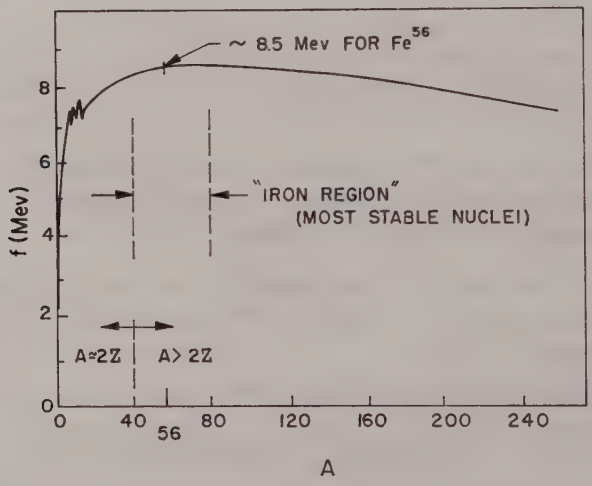


Fig. 17.2 Binding fraction as a function of mass number (schematic).

region." This region contains the most tightly bound, and therefore the most stable, nuclei. For $A \lesssim 40$, $A \simeq 2Z$ for stable isotopes, while for $A > 40$, $A > 2Z$ for stable isotopes. The decrease in f in the region to the right of the iron region is due to the Coulomb repulsions of the protons in the nucleus on one another. The decrease in f in the region to the left of the iron region is a surface effect; nucleons on the surface of a nucleus are less tightly bound than those inside, and, since surface area per unit volume increases as A decreases, this results in a lowering of the average binding energy per nucleon for the smaller nuclei. The irregularities in Fig. 17.2 are particularly pronounced in the region of small A ($A \lesssim 20$); for example, a high peak occurs for He^4, indicating that this is an especially stable nucleus.

Nuclei with even Z are more stable than those with odd Z, and nuclei with even A are more stable than those with odd A. Stable nuclei with even A have even Z, with the exceptions of H^2, Li^6, B^{10}, and N^{14}; "even-even" nuclei with $A = 2Z$ are especially stable; examples are He^4, C^{12}, O^{16}, Ne^{20}, Mg^{24}. (*Note:* By "stable" nuclei we mean those whose mean lives are greater than, say, 10^{12}–10^{14} years).

Nuclei for which either Z or N (or both) have the "magic number" values 2, 8, 20, 28, 50, 82, and 126 are especially stable and therefore relatively abundant in nature. These "magic numbers" have been accounted for on the basis of the "shell model" of the nucleus. In this model (sometimes also referred to as the "independent-particle" model) each nucleon moves in a "smoothed-out" potential well whose exact shape is not known but which is usually assumed to be approximately rectangular (see Fig. 17.3). The

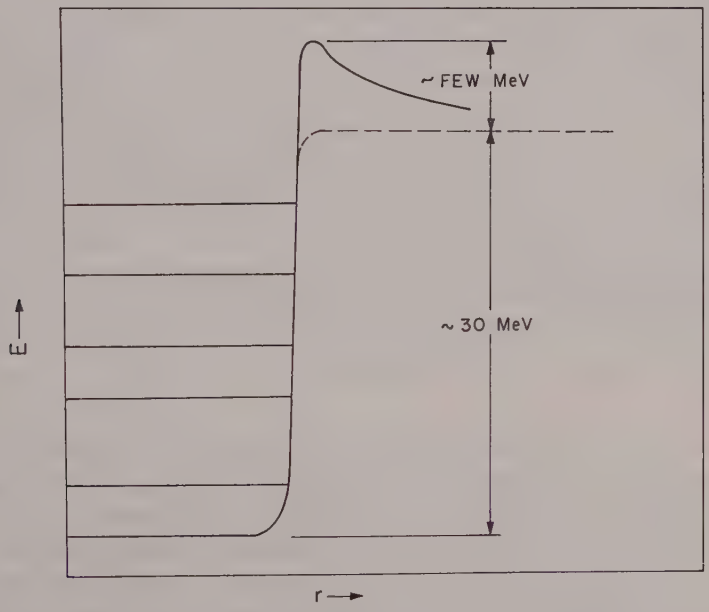


Fig. 17.3 Schematic energy level diagram for a nucleus. The dashed curve is for neutrons (assuming no "centrifugal barrier") and the Coulomb potential barrier is for positively charged particles.

nucleus therefore possesses discrete energy levels, or "shells," just as do atoms. Since nucleons obey the Pauli exclusion principle, each shell can accommodate only so many nucleons. The "filled shells" should then make for enhanced stability and presumably correspond to the "magic number" nuclei (analogous to closed-shell atoms such as He, Ne, A).

It is the Pauli principle, in fact, which explains why there are approximately as many protons as neutrons in a nucleus, and why neutrons in a nucleus do not decay into protons by β-emission, as neutrons outside the nucleus normally do (because the rest mass of a neutron is greater than that of a proton). If all the nucleons in a given shell were all neutrons or all protons (*i.e.*, identical particles), then the shell could accommodate, because of the Pauli principle, only half as many nucleons as in the case when half are protons and half are neutrons. The half of the (assumed identical) nucleons that could not be accommodated by the shell under consideration would have to move to higher, more energetic shells. Since the lower lying shells are separated in energy by, say, a few Mev or so (to order of magnitude) on the average, then a nucleus having a given total number of nucleons A would have a much larger energy if the nucleons were all identical than if they consisted of approximately equal numbers of neutrons and protons (the Coulomb energy in case all the nucleons are protons is usually considerably smaller than the energies under discussion here; also, the energy difference between the neutron and proton rest masses ($\sim$0.8 Mev per nucleon) is usually smaller than the energies we are considering here). Since nuclei always tend to seek the state of lowest energy, a nucleus of given A and arbitrary $Z(\leq A)$ will adjust its Z value by electron or positron emission until the Z/A value is such (roughly, one-half) that the total energy of the nucleus is near its minimum value (with respect to Z).

Only nuclei whose masses are less than the sum of the masses of all their possible constituents are stable; *i.e.*, the condition for stability is that

$$M_n < \sum_i M_i . \tag{17.86}$$

Thus He^4 is stable because $M_{He^4} < 4M_{H^1}$. However, Li^5 is unstable against breakup into He^4 and H^1; He^5 is unstable against breakup into He^4 and n; and Be^8 is unstable against breakup into $2He^4$. This instability, incidentally, explains why Li^5, He^5, and Be^8 are not ordinarily found in nature. This mass defect of stable nuclei appears as binding energy by the Einstein relation $E = mc^2$ and is one of the most elementary and convincing demonstrations of the equivalence of mass and energy.

The nuclear radius can be expressed, on the basis of empirical evidence, in terms of mass number A by means of the following expression:

$$R \simeq 1.4 \times 10^{-13} A^{1/3} \text{ cm} . \tag{17.87}$$

Thus each particle in the nucleus occupies approximately the same volume, regardless of the size of the nucleus; the nucleons are evidently rather tightly "packed" within the nucleus, somewhat as oranges in a crate.

Fundamental particles in nuclei are *protons* and *neutrons*, not protons and electrons, as was believed to be the case until 1932. One reason why electrons cannot exist in the nucleus is as follows: The de Broglie wavelength λ of a particle of momentum p is given by

$$p = h/\lambda,$$

where h = Planck's constant. An electron within a nucleus would be likely to be moving with relativistic energies; the kinetic energy of such an electron would be

$$E \simeq pc = hc/\lambda.$$

For an electron to be "in" the nucleus, however, its de Broglie wavelength must be smaller than the nucleus: $\lambda < R \sim 10^{-12}$ cm. Hence the kinetic energy of the electron would have to be $E \gtrsim 7 \times 10^{-27} \times 3 \times 10^{10}/10^{-12} \times 200 \times 10^{-6}$ ergs ~ 100 Mev. But the observed energies of β-particles from radioactive nuclei are only of the order of 1 to 2 Mev; hence the small sizes of nuclei preclude the existence of electrons in nuclei. The assumption that nuclei are composed of protons and electrons also fails to explain the spin properties of nuclei (*cf.* Bethe and Morrison [Be56, Chap. 5]).

The above argument does not exclude the presence of π mesons (rest mass ~ 0.1 times the nucleon rest mass) in nuclei. These π mesons are thought to be the "glue" that holds the nucleons together within a nucleus; the π mesons are pictured as being continually emitted and reabsorbed by the nucleons. For separations $\lesssim 10^{-13}$ cm, very powerful short-range attractive forces exist between nucleons; these forces are independent of the charges on the nucleons. The range of nuclear forces is given approximately by the *Compton wavelength* of a π meson:

$$\lambda\!\!\!^{-} \equiv \lambda/2\pi = \hbar/m_\pi c \simeq 2 \times 10^{-13} \text{ cm},$$

where m_π is the rest mass of the π meson ($\simeq 200 m_e$) and $\hbar \equiv h/2\pi$. Thus a nucleon is attracted effectively only by its immediate neighbors in the nucleus.

It is important to note that the kinetic energies of the particles in stellar interiors are quite low as judged by terrestrial nuclear physics laboratory standards. Let $f(E)\mathrm{d}E$ denote the fraction of particles whose kinetic energies lie between E and $E+\mathrm{d}E$. If a Maxwellian velocity distribution is assumed, then $f(E)$ is given by the relation (*cf.* (3.19))

$$f(E) = \frac{2}{\pi^{1/2}} \frac{1}{(kT)^{3/2}} e^{-E/kT} E^{1/2}. \tag{17.88}$$

The maximum of $f(E)$ occurs at $E = (1/2)kT$, and thus most particles at

temperature T have about this energy. For $T\sim(10^7$–$10^9)\,°\mathrm{K}$, $kT\sim$(1–100) Kev. Thus most particles in stellar interiors have energies of the order of a few Kev, or quite low by nuclear physics laboratory standards of several Mev to a few Bev. This is why low energy nuclear reactions are of particular importance for work in stellar interiors.

We consider now some relevant energy relations. Consider the general nuclear reaction

$$a+X\rightarrow Y+b, \tag{17.89}$$

where X and Y represent the nucleus before and after collision, respectively, and a and b are the incident and emitted particles, respectively (b may also represent a γ-ray). Such a reaction is also frequently written as

$$X(a,b)Y.$$

(Examples: $\mathrm{H}^1+\mathrm{N}^{15}\rightarrow\mathrm{C}^{12}+\mathrm{He}^4$ or $\mathrm{N}^{15}(p,\alpha)\mathrm{C}^{12}$;

$\mathrm{He}^4+\mathrm{C}^{13}\rightarrow\mathrm{O}^{16}+n$ or $\mathrm{C}^{13}(\alpha,n)\mathrm{O}^{16}$.)

We shall usually consider the reaction in the center of mass system, *i.e.*, in a coordinate system in which the center of mass of the reacting particles is at rest. Then the relative kinetic energy of motion E between a and X is

$$E=\frac{1}{2}mv^2, \tag{17.90}$$

where m is the "reduced mass"

$$m=M_aM_X/(M_a+M_X), \tag{17.91}$$

and v is the relative velocity between a and X when they are far apart. The total kinetic energy E in the laboratory system, E_{lab}, can easily be shown to be

$$E_{\mathrm{lab}}=\frac{M_a+M_X}{M_X}E\simeq E \text{ for } M_X\gg M_a, \tag{17.92}$$

where in a laboratory situation it is understood that the target particle, X, is initially at rest (or at least has negligible velocity compared to that of the incident particle, a).

In order for the reaction (17.89) to proceed for $E\rightarrow 0$, it must be *exothermic*; *i.e.*, some energy Q (>0) must be released from the reaction. This energy Q given off in the reaction is related to the masses of the reacting particles as follows:*

$$Q=[M_a+M_X-M_Y-M_b]c^2. \tag{17.93}$$

* In most astrophysical applications Q is defined slightly differently than in (17.93) in cases of electron or positron emission; see Sect. 17.16.

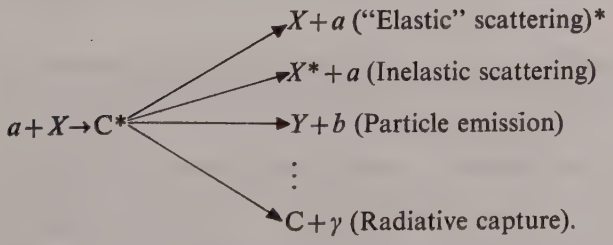
For an exothermic reaction $Q>0$, as will be the case for most of the reactions we shall consider. The form in which this energy Q is released will depend on the specific kind of reaction that is under consideration.

17.9 Bohr Picture of a Nuclear Reaction

For low-energy nuclear reactions (say $E \lesssim 50$ Mev) involving a nucleon and a nucleus or else two nuclei, the "Bohr" picture is generally assumed to be applicable. While results of modern experimental nuclear physics show that this picture is a considerably over-simplified representation of an actual nucleus, such a picture is nevertheless adequate for our purposes and will be adopted in the following developments.

In the Bohr picture the nuclear reaction $a+X \rightarrow Y+b$, where a denotes a nucleon or nucleus, X and Y are nuclei, and b may be either a particle or a photon, proceeds in two separate steps: (1) formation of a "compound nucleus" C*, which is always formed in an excited state (since C* has the kinetic energy of the incident particle and also the binding energy of this particle); (2) disintegration of C* into the particular reaction product or "channel" of interest. (The asterisk on C indicates that C is in an excited state.) We are here using the term "channel "in a loose sense to denote simply a specific reaction product such as $Y+b$ (as opposed to, say, $Z+c$). Actually, each such reaction product can be further subdivided according to the resultant quantum states in which the residual nucleus and emerging particles may find themselves. These subdivisions must of course obey the conservation laws for energy, angular momentum, and parity to be capable of realization. Each such "allowed" subdivision really constitutes a "channel" in the correct sense of the word. However, in our work we shall not distinguish between these various subdivisions.

Schematically, the Bohr picture may be represented as follows:



* Here the term "elastic" scattering represents the combination of a and X to form C*, followed by the exact time-reversed process, and not elastic scattering without penetration of a into X to form C*.

The compound nucleus, being unstable, exists for a short time, then decays into one of the "channels" indicated. The lifetime of the compound nucleus is, however, long compared to the nuclear time scale.* The Bohr assumption states that the processes (1) and (2) are independent of each other, in the sense that the mode of disintegration of C* depends only on its energy, angular momentum, and parity, but not on the specific way in which C* was formed. In other words, the compound nucleus "forgets" how it acquired its energy, angular momentum, and parity. Thus the relative probability of decay of C* into some specific channel depends only on the excitation energy of C* and on its other properties.

There are two general ways in which C* may decay:

(1) *Particle Emission*: This includes elastic and inelastic scattering as special cases (although the Bohr theory is not to be applied to elastic scattering, *cf.* Bethe and Morrison [Be56, pp. 192]). If the excitation energy of C* is great enough, *fission*, or breakage of C*into two particles of roughly comparable mass, is most likely to occur. Otherwise, C* will emit a neutron, proton, or alpha particle, generally in that order of likelihood, the energy Q appearing as kinetic energy of motion of the residual nucleus and emitted particle. Strictly speaking, β-emission, in which C* retains the incident particle and emits a positron or electron and a neutrino or anti-neutrino, is also a kind of particle emission.† We do not consider this process in detail in this section, however, primarily because of its relative improbability as compared to other kinds of processes. The lifetimes for β-emission typically range from fractions of a second to some thousands of years, *i.e.*, long compared to the nuclear time scale. In addition, the physical mechanism of β-emission (being a manifestation of the "weak interaction") is intrinsically different from that for other kinds of particle emissions (which are manifestations of the "strong interaction"). Electrons, in fact, are not even "in" the nucleus before the emission takes place. We also exclude inverse β decay (capture of a positron or electron by the nucleus) from the present discussion, although such inverse β decays are important under some stellar

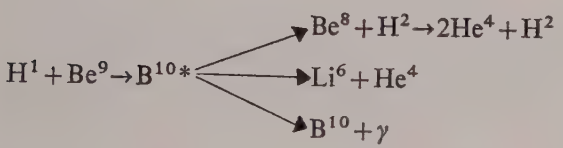
* The nuclear time scale is defined as the order of the time required for a nucleon moving with speed $\sim 0.1c$ to traverse a distance of the order of a nuclear dimension; thus the nuclear time scale is $\sim 10^{-12}/10^{9} \sim 10^{-21}$ sec.

† A useful rule for remembering whether a neutrino or anti-neutrino is emitted in a β-decay process is the principle of "lepton number conservation." Assign the number $(+1)$ to each *lepton* (electron, μ meson, neutrino) and the number (-1) to each *anti-lepton* (the corresponding anti-particle) participating in a reaction. The principle of lepton number conservation then states that the lepton number (the algebraic sum of the numbers introduced above for the leptons participating in a reaction) must be the same after a reaction as before.

conditions, particularly at high densities (see, for example, Sects. 25.3a, 26.5, and near the end of Sect. 17.16). A recent discussion of electron capture in stellar interiors has been given by Bahcall [Ba64]. Photodisintegration, in which "particle" a is a photon, may also be included in this category (1) (although then the Bohr theory would not be applicable).

(2) *Radiative Capture*: If the excitation energy of C* is not great enough for particle emission, then C* may retain the incident particle and get rid of its excess energy by emission of a γ-ray.* This process is generally relatively improbable if the possibility of particle emission exists.

An example of such a situation is the following: In the combination of H^1 and Be^9, the only exothermic possibilities are:



	Q(Mev)
$Be^8+H^2\rightarrow 2He^4+H^2$	0.64
Li^6+He^4	2.12
$B^{10}+\gamma$	6.49

The possible modes of decay of the compound nucleus B^{10}* are given here in their relative orders of likelihood.

The validity of the Bohr picture rests on the assumption that the entering particle quickly shares its energy with the other nucleons in the nucleus, as a result of the strong short-range forces of attraction between the nucleons. Thus, it takes a long time for enough energy to become concentrated on any one particle to cause its emission, and by this time the nucleons in the nucleus have become thoroughly "scrambled", so that the compound nucleus has effectively "forgotten" how it was formed. This is why we said that the lifetime of C* is very long compared with the nuclear time scale of $\sim 10^{-21}$ sec.

In order that the Bohr picture be valid, two conditions would appear to be necessary: (1) The mean free path Λ of a nucleon in a nucleus must be small compared to the nuclear radius:

$$\Lambda \ll R. \tag{17.94}$$

Experiments on (p,n) scattering imply that $\Lambda \lesssim 1 \times 10^{-13}$ cm for $E \lesssim 50$ Mev (see Blatt and Weisskopf [Bl52, pp. 341–342]), so that (17.94) is satisfied for such energies. (Recent evidence, however, suggests that Λ may be larger than these estimated values by factors up to ~ 20; this fact, however, appears not to invalidate the general features of the Bohr picture.) (2) The average energy per

* If emission of a γ-ray is forbidden by selection rules, C* may get rid of its excess energy by creation of a positron-electron pair or by "K-capture"; see, for example, Preston [Pr62, Chap. 15].

nucleon gained from the incoming particle must be much less than the minimum separation energy S of a particle from the nucleus (*i.e.*, S is the minimum amount of energy ($\sim$7–8 Mev) which would be required to remove a particle from the nucleus to infinity). If $\mathscr{E}$ denotes the excitation energy of C* and E denotes the relative kinetic energy of a and X before collision, we have

$$\mathscr{E} = E + S_a, \tag{17.95}$$

where S_a is the separation energy of particle a. We must then require that

$$\frac{\mathscr{E}}{A} = \frac{E+S_a}{A} \ll S.$$

Since $S_a \simeq S$, we have that

$$E \ll (A-1)S. \tag{17.96}$$

Since $S \sim 8$ Mev, it follows that $E < 50$ Mev for $A \gtrsim 10$. Note that the upper allowable limit for E (for validity of the Bohr picture) increases with increasing A.

17.9a Occurrence of Resonances

One of the striking facts about low-energy nuclear reactions is the occurrence of very high and sharp peaks, or "resonances", in the cross section $\sigma(\mathscr{E})$ for formation of C*, these peaks occurring at certain "resonant"

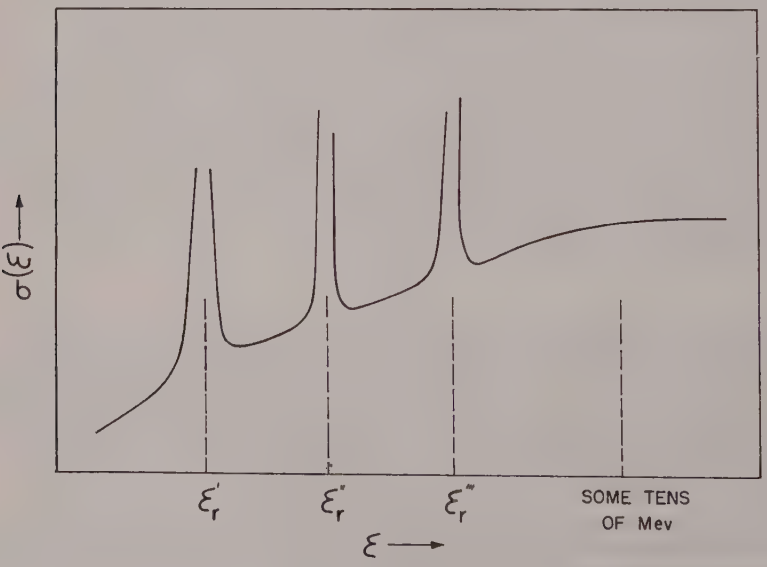


Fig. 17.4 Schematic appearance of resonances.

excitation energies $\mathscr{E}_r$. At these energies $\mathscr{E}_r$, the probability of formation of C* is very greatly enhanced. This resonance phenomenon is observed to exist only below some excitation energy which depends on the specific nuclear reaction under consideration but which is generally of the order of some tens of Mev. Above this energy $\sigma(\mathscr{E})$ is a relatively smooth function of the excitation energy $\mathscr{E}$. The schematic appearance of $\sigma(\mathscr{E})$ is shown in Fig. 17.4.

To facilitate our discussion of the resonance phenomenon, we state three additional assumptions about the nucleus, following Blatt and Weisskopf [Bl52] (hereafter, "*BW*"):

(1) The nucleus has a well-defined surface which is a sphere of radius R, where R is the "nuclear interaction radius" (to which we shall refer loosely as the "nuclear radius"), given by Burbidge, Burbidge, Fowler, and Hoyle [Bu57] (hereafter, "B^2FH") as (*cf.* (17.87))

$$R \simeq 1.44 \times 10^{-13}(A_a^{1/3} + A_X^{1/3}) \text{ cm}$$
$$= R_a + R_X (\simeq R_{\text{nucleus}}) . \tag{17.97}$$

The interaction radius is the distance beyond which the nuclear forces no longer operate on particle a.

(2) If particle a penetrates into the nucleus, it moves with an average kinetic energy T_{in} which is much larger than the relative kinetic energy E between a and X outside the nucleus (because of the very strong attraction of intranuclear forces):

$$T_{\text{in}} = E + T_0 , \tag{17.98}$$

where T_0 is the average intranuclear kinetic energy per nucleon ($\sim$20 Mev for neutrons and protons, somewhat less for alpha particles, according to *BW*).

(3) Particle a is subject to very strong interactions within the nucleus, so that it interchanges its energy very rapidly with the other nucleons. (This is actually not a new assumption.)

These assumptions are equivalent to supposing that a sharp change in the potential at the radius R occurs such that, for $r < R$, a deep potential well exists (see Fig. 17.3). The large kinetic energy of particle a for $r < R$ arises from the sharp drop in potential at the surface (*i.e.*, from the strong nuclear forces in the nucleus).

With these assumptions it is possible to understand the occurrence of resonances at low interaction energies. In keeping with the spirit of quantum mechanics, one must represent particles by probability waves, the square of the absolute value of which gives the probability of finding a particle at the space-time point of interest. Consider now the interaction of particle a with

the nucleus X. Because of the possibility of elastic reflection from the nuclear surface (this possibility is a result of the abrupt change in potential at the nuclear surface, in accordance with assumption (1) above), the wave function for particle a outside the nucleus ($r > R$) must be written as the sum of two terms, the first representing the incident wave, the second the reflected wave. If the reflection is elastic, then both incident and reflected waves have the same wave number $k = 2\pi/\lambda$, where λ is the de Broglie wavelength corresponding to the relative kinetic energy E of a and X when far apart. We have

$$k = (2mE)^{1/2}/\hbar \tag{17.99a}$$

$$\simeq 2.18 \times 10^{12} (\mathscr{A} E_{\text{Mev}})^{1/2} \text{ cm}^{-1}, \tag{17.99b}$$

where $\hbar \equiv h/2\pi$, m is the reduced mass of a and X, $\mathscr{A}$ is the "reduced atomic weight" of a and X:

$$\mathscr{A} = \frac{\mathscr{A}_a \mathscr{A}_X}{\mathscr{A}_a + \mathscr{A}_X} = N_0 m, \tag{17.100}$$

N_0 is Avogadro's number, and E is in Mev in (17.99b). We have assumed the classical approximation $E = p^2/2m$ in (17.99).

Inside the nucleus ($r < R$), now, we must in general also allow for both an incoming wave, with wave number $K \gg k$,* representing particle a going toward the nuclear interior, and an outgoing wave, with the same wave number, representing particle a going toward the nuclear surface after having interacted strongly with the particles within the nucleus. The phase of the returning wave will be determined by the complicated interactions that a has undergone while within the nucleus, and is therefore a function of the relative kinetic energy E.

Proper matching of the inside and outside waves at the nuclear surface $r = R$ then requires that the magnitudes and slopes of these waves be equal there.

In our very much oversimplified picture, then, the problem is essentially one of fitting a wave of very long wavelength to one of very short wavelength at the nuclear surface. The matching problem is illustrated schematically in Fig. 17.5. This figure shows that the amplitude of the "inside" wave will be much smaller (perhaps by a factor of order $k/K \ll 1$) than that of the "outside" wave unless the phase of the inside wave (which depends on the relative kinetic energy E of a and X when far apart) has certain particular values. The values of E corresponding to these particular phases of the inside wave for which strong penetration occurs are the "resonance" energies referred to

* $K \sim 1 \times 10^{13}$ cm^{-1}, corresponding to an intranucleon kinetic energy $T_0 \sim$ 20–30 Mev.

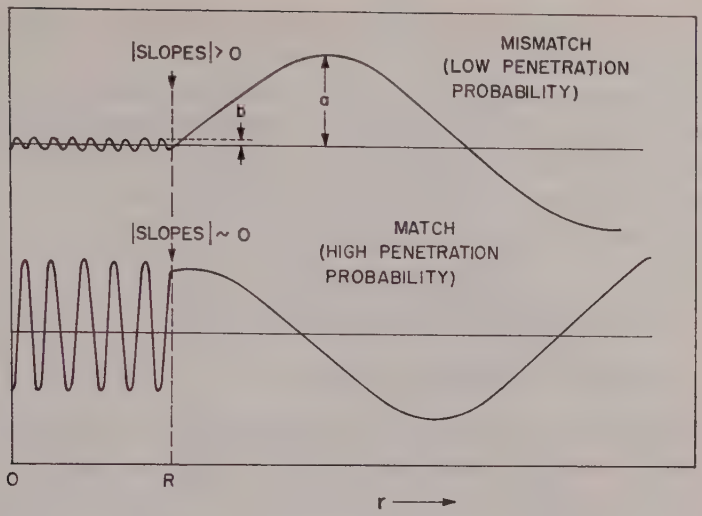


Fig. 17.5 Illustration of matching conditions for inside and outside waves. The amplitude ratio b/a in the case of mismatch is of the order of the ratio k/K, where k and K are the wave numbers corresponding to the relative motion of a and X outside and inside, respectively, the nucleus.

earlier. Thus we see that according to our simple picture these resonances occur because the inside and outside wave functions are matched at a *sharp* boundary and because the phase of the inside wave depends on the energy E.

In our treatment of the wave (inside the nucleus) returning from the interior, we have essentially assumed that the excitation energy of C* is sufficiently small that the number of "channels" through which C* may possibly decay will be small enough that decay of C* through the "entrance channel" in which C* was formed has a relatively high probability. If, however, the excitation energy of C* is *very high*, there will be a great many possible "channels" through which C* may decay, so that the probability of a being re-emitted through the "entrance channel" in which C* was formed is very small. In this case the amplitude of the "outgoing" wave *inside* the nucleus is very small and this wave may, in fact, be neglected for sufficiently high excitation energies. In the absence of this returning wave, the "inside" wave is now a purely "ingoing" wave whose phase no longer depends on the excitation energy of C*. The problem in this case is then just one of reflection from a discontinuity in potential at the nuclear surface, and resonance phenomena should not occur. The disappearance of resonances at high energies is thus qualitatively explained. *These resonance energies, $\mathscr{E}_r$, are just*

the excitation energies of the quasi-stationary states in which the compound nucleus C* *may find itself* (*cf.* later in this section).

It is convenient at this point to define a *transmission factor* for penetration of a into X. For generality, we suppose that particle a has the orbital angular momentum $\sqrt{\ell(\ell+1)}\hbar$ relative to particle X (although cases with $\ell = 0$, *i.e.*, zero orbital angular momentum, are of most interest to us, as we shall see later) and define the transmission factor as follows:

$$T_\ell(a) \equiv 1 - \frac{|\text{Amplitude of outside reflected wave}|^2}{|\text{Amplitude of outside incident wave}|^2} \qquad (17.105)$$

$$= \text{probability of penetration of } a \text{ into } X.$$

When the possibility of resonances is not allowed for, a calculation by *BW* (p. 361) shows that, if $k \ll K$ (*i.e.*, $E \ll T_{\text{in}}$, or $E \lesssim$ several Mev),

$$T_\ell(a) \sim \frac{4k}{K} \cdot P_\ell(a) \quad (k \ll K), \qquad (17.106)$$

where $P_\ell(a)$ is the *barrier penetration factor*, equal to unity if a is a neutron with $\ell = 0$ and very small compared to unity if a is a positively charged particle. $P_\ell(a)$ decreases with increasing ℓ for both neutrons and positively charged particles. Values of $P_\ell(a)$ will be considered in Sect. 17.11. We see, then, from (17.106) that penetration of particles into a nucleus is impeded by two different effects: (1) an "impedance mismatch" (represented by the first factor in (17.106)), resulting from the sharp change in potential at the nuclear surface, and (2) if particle a is charged, a barrier penetration factor resulting primarily from the strong Coulomb repulsion of the particle by the nucleus.

The correction for the occurrence of resonances will be made in Sect. 17.10.

17.9b Compound Nucleus

The compound nucleus C*, consisting of the incident particle a and the nucleus X, is always created in an excited state because a brings into C* not only its kinetic energy E but also its binding energy S_a.

If the excitation energy $\mathscr{E}$ (*i.e.*, energy above the ground state) of C* is less than S_{min}, where S_{min} is the minimum amount of energy required to remove some particle from the compound nucleus, then the energy levels of C* will be stationary aside from the possibility of radiative transitions to lower levels (just as in an ordinary atom). For $\mathscr{E} > S_{\text{min}}$, however, the energy levels of C* form, strictly speaking, a continuum, since the possibility now

exists for the emission of a particle with any final kinetic energy up to $\mathscr{E} - S_{\min}$ ($S_{\min}$ plays a role in nuclei analogous to the ionization potential in atoms). However, because of the large potential jump at the nuclear surface, a particle will not be emitted for a long time even if $\mathscr{E} > S_{\min}$. Hence the

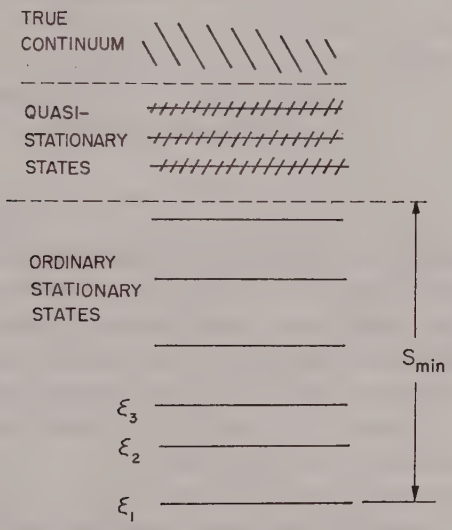


Fig. 17.6 Schematic energy level diagram of a compound nucleus.

states with $\mathscr{E} > S_{\min}$ are "quasi-stationary" or "metastable" if $\mathscr{E}$ is not too large, and these states do not differ qualitatively from those having $\mathscr{E} < S_{\min}$ (see Fig. 17.6). The lifetime of the states for which $\mathscr{E} > S_{\min}$ is less than that of the states for which $\mathscr{E} < S_{\min}$, simply because, in the former case, particle emission is possible in addition to radiative transitions.

The mean life, τ, of a given state of the compound nucleus, *i.e.*, the reciprocal of the probability per unit time of transition to some lower state, is related to the "width," Γ, of the given energy level by the relation

$$\Gamma = \hbar/\tau . \tag{17.107a}$$

If τ is in seconds, we have

$$\Gamma = 6.6 \times 10^{-16}/\tau \text{ ev}. \tag{17.107b}$$

(This relation is consistent with the Heisenberg uncertainty relation: $\Delta E \Delta t \sim \hbar$, where ΔE and Δt are the uncertainties in energy and time, respectively). If the possibility of decay into several specific "channels" exists, we may write

$$\Gamma = \Gamma_\gamma + \sum_i \Gamma_i , \tag{17.108}$$

where Γ_γ is the "radiation width" and the Γ_i are the respective "partial particle emission widths." That is, Γ_γ is associated with τ_γ, the lifetime of a level against emission of a γ-ray, and Γ_i are associated with τ_i, the respective lifetimes of a level against emission of particle i. Clearly, $\Gamma_i/\hbar$ is the probability per unit time for decay of the particular state of C* through the "channel" i, assuming all other "channels" to be closed by some artifice.

In order for a discrete spectrum of levels with $\mathscr{E} > S_{\min}$ to exist, it is clear that the level width Γ of the excited state of the compound nucleus must be smaller than the energy spacing D between neighboring states:

$$\Gamma < D . \tag{17.109}$$

The level width Γ increases with increasing excitation energy $\mathscr{E}$, primarily because the number of "channels" through which the compound nucleus can decay increases, thus resulting in a larger number of terms on the right side of (17.108). In addition, each Γ_i generally increases with increasing $\mathscr{E}$. Thus, at some excitation energy $\mathscr{E} > S_{\min}$, Γ becomes larger than D and the energy spectrum of C* becomes truly continuous. Above this critical energy resonances no longer occur (see Fig. 17.6).

It can be shown on the basis of general principles of quantum mechanics (*cf. BW*, p. 387, for example) that for a system with equally spaced energy levels, the "period" P of the motion of the constituents of the system is related to D, the level spacing, by the relation

$$P = 2\pi\hbar/D . \tag{17.110}$$

If the levels are not equally spaced, then D must be interpreted as the average level spacing, and we still expect to have, to order of magnitude,

$$P \sim 2\pi\hbar/D . \tag{17.111}$$

For example, (17.110) can be verified directly on the basis of the Bohr theory of hydrogen-like atoms for n (principal quantum number) $\gg 1$, where P in this case is the period of the electron in the orbit n (this is equivalent to applying the Bohr correspondence principle). In the case of a compound nucleus C* which has been formed by capture of particle a through some definite "channel", P may be interpreted as the time after which a again assumes its original "position" and is ready to leave through the same "channel" in which it entered.

Because of the abrupt potential jump at the nuclear surface, however, the particle a will in general not leave the nucleus within the time P after the formation of C*, but will be reflected at the nuclear surface back into the

interior and so will repeat the motion a number of times. It is clear that a large number of such repetitions of this motion is, in fact, necessary for the existence of a well-defined quasi-stationary state. For, if particle a had left the nucleus after the time P, the lifetime of the excited state would be

$$\tau = \hbar/\Gamma = P \sim \frac{2\pi\hbar}{D},$$

or

$$\Gamma \sim D/2\pi,$$

i.e., Γ would be of order D, and the states would not be discrete. Hence we must have $\tau \gg P$ for the existence of discrete levels.

It is clear from symmetry or from arguments based on detailed balancing that the number of times that particle a is reflected back into the nucleus, after attempting to escape through the channel in which it entered to form C*, before finally escaping, is just $1/T_\ell(a)$, where $T_\ell(a)$ is the "transmission factor" (*cf.* (17.105)) for penetration of a *into* X through the particular channel of interest. Thus we have

$$\tau_a = \hbar/\Gamma_a = P/T_\ell(a) \sim \frac{2\pi\hbar}{T_\ell(a)D},$$

where τ_a is the lifetime of the excited state against emission of particle a through the "channel" in which it entered. We then obtain the important relation

$$\Gamma_a \sim T_\ell(a) \cdot \frac{D}{2\pi}, \tag{17.112}$$

which relates the partial width for *decay* of C* through a particular "channel" to the transmission factor for *formation* of C* through this same "channel." (See Bethe and Morrison [Be56, pp. 195ff] for a derivation of (17.112) directly from arguments based on detailed balancing.) We note that $\Gamma_a \ll D$, as required, since $T_\ell(a) \ll 1$ in general (at least for low E). If we use the approximation (17.106) for $T_\ell(a)$, valid for E not close to resonance energies, we may also write

$$\Gamma_a \sim \frac{4k}{K} \cdot P_\ell(a) \cdot \frac{D}{2\pi} \qquad (k \ll K), \tag{17.113a}$$

where it will be recalled that $k = 1/\lambda\!\!\!^{-}_{\text{out}}$ is the wave number corresponding to the relative motion of a and X when they are far apart and $K = 1/\lambda\!\!\!^{-}_{\text{in}}$ is the wave number corresponding to the average motion of a *inside* the compound nucleus C*.

If we take $K \simeq 1.5 \times 10^{13}$ cm^{-1} and use (17.99), we may also write (17.113a) as

$$\Gamma_a/D \simeq 3 \times 10^{-3} \cdot P_\ell(a) \cdot (\mathscr{A} E_{\text{Kev}})^{1/2}, \tag{17.113b}$$

where E is in Kev and $\mathscr{A}$ is the "reduced atomic weight":

$$\mathscr{A} \equiv \frac{\mathscr{A}_a \mathscr{A}_x}{\mathscr{A}_a + \mathscr{A}_x}.$$

We note that the width Γ_a is expressed in terms of a "reduced width" γ_a by means of the following relation:

$$\Gamma_a \equiv (2kRP_\ell)\gamma_a, \tag{17.114}$$

where the factor in parentheses depends only on the "channel energy" E and other conditions *outside* the nucleus (which are presumably known), and γ_a depends on conditions *inside* the nucleus. Comparing with (17.113a), we see that, to order of magnitude,

$$\gamma_a \sim \frac{D}{\pi KR} \tag{17.115}$$

which shows that γ_a does not depend on the "channel energy" E in this approximation. Note that for charged particles the widths Γ_i contain the barrier penetration factors P_ℓ.

For "light" and "medium" nuclei (say $A \lesssim 60$) and for excitation energies $\mathscr{E}$ in the range of, say, 1–10 Mev, the mean energy spacing D between levels ranges from a few Kev to a few Mev; D decreases strongly both with increasing E and increasing A.

(According to *BW*, p. 391, the mean level spacing D in the relation $P \sim 2\pi\hbar/D$ must be the mean spacing between levels of equal angular momentum and parity.)

17.10 Cross Section for Nuclear Reactions

We shall write the expression for the cross section $\sigma(a,b)$ for the reaction $a+X \rightarrow C^* \rightarrow Y+b$ as a product of a number of factors, and shall then explain the significance and purpose of each factor in turn. Our treatment is therefore not a true derivation of this expression but, rather, a series of semi-quantitative plausibility arguments which result in the correct final expression. More rigorous treatments may be found in *BW* and in Preston [Pr62]. The relative kinetic energy between particles a and X when they are

far apart is E, and the orbital angular momentum of a relative to X is $L = \sqrt{\ell(\ell+1)}\hbar$. We have

$$\sigma(a,b) = (2\ell+1)\pi\lambda\!\!\!^{-2} \cdot \omega \cdot T_\ell(a) \cdot y(E) \cdot \mathscr{S} \cdot G(b). \tag{17.116}$$

In the first factor, $(2\ell+1)\pi\lambda\!\!\!^{-2}$,

$$\lambda\!\!\!^{-} = \lambda/2\pi = \frac{1}{k} = \hbar/p = \hbar/\sqrt{2mE} = 4.59 \times 10^{-13}(\mathscr{A} E_{\text{Mev}})^{-1/2}\text{cm} \tag{17.117}$$

is the (wavelength/2π) corresponding to the relative motion between a and X when far apart (m = reduced mass of a and X, $\mathscr{A}$ = reduced atomic weight of a and X (*cf.* (17.100)). This first factor is the maximum possible cross section for the ℓ^{th} "partial wave" (see Fig. 17.7), *i.e.*, for particles whose

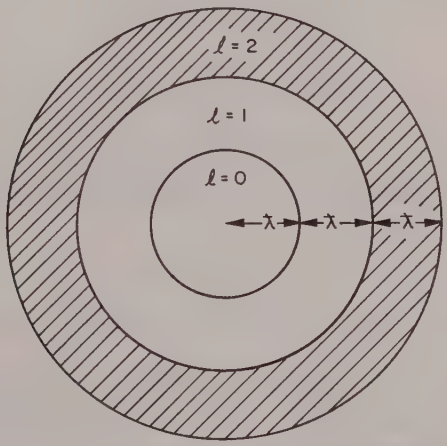


Fig. 17.7 Illustration for the maximum possible cross section of the nucleus. For example, the shaded area corresponds to the maximum cross section for $\ell = 2$.

impact parameters lie between $\ell\lambda\!\!\!^{-}$ and $(\ell+1)\lambda\!\!\!^{-}$. This factor takes account of the fact that particles a and X cannot "see" each other more distinctly than within a distance of the order of $\lambda\!\!\!^{-}$; this is consistent with the Heisenberg uncertainty principle. This factor represents the maximum possible cross section only if E is small enough that $\lambda\!\!\!^{-} > R$, R being the "nuclear interaction radius" (*cf.* (17.97)) or, loosely speaking, the nuclear radius. It can easily be shown from (17.97) and (17.117) that, if $A_X \gg A_a$:

$$E \lesssim \frac{10}{A_a A_X^{2/3}} \text{ Mev} \tag{17.118}$$

for $\lambda\!\!\!^{-} \geq R$, where we have taken

$$\mathscr{A} \simeq \frac{A_a A_X}{A_a + A_X}.$$

Hence this factor is applicable for $E \lesssim$ a few Mev for light nuclei (which is the range of energy of most interest to us here).

The second factor, ω, is a statistical factor of order unity which accounts for the "spins" or "internal" angular momenta of a, X, and the compound nucleus C*. This factor is

$$\omega = \frac{2J+1}{(2I_a+1)(2I_X+1)}, \tag{17.119}$$

where the "spin" angular momenta of a, X, and the excited level of C* are given, respectively, by

$$\left.\begin{aligned} |\mathbf{I}_a| &= \sqrt{I_a(I_a+1)}\,\hbar\,, \\ |\mathbf{I}_X| &= \sqrt{I_X(I_X+1)}\,\hbar\,, \\ |\mathbf{J}| &= \sqrt{J(J+1)}\,\hbar\,, \end{aligned}\right\} \tag{17.120}$$

where I_a and I_X may have integral or half-integral values. Furthermore, $\mathbf{J}$ is the vector sum of $\mathbf{I}_a$ and $\mathbf{I}_X$:

$$\mathbf{J} = \mathbf{I}_a + \mathbf{I}_X, \tag{17.121}$$

and the total angular momentum quantum number J may take the values

$$J = |I_a - I_X|, |I_a - I_X| + 1, \ldots, I_a + I_X. \tag{17.122}$$

Since $\Sigma_{\text{all } J}\, \omega = 1$, it follows that ω is the relative probability that C* has the total angular momentum $\mathbf{J}$.

The third factor, $T_\ell(a)$, is the non-resonant "transmission factor" (*cf.* (17.106)), *i.e.*, the probability of penetration of a into X in the absence of the possibility of resonance. This factor is related to the partial width, Γ_a, for *emission* of particle a by C* by the relation (*cf.* (17.112))

$$T_\ell(a) \sim \frac{2\pi}{D} \cdot \Gamma_a. \tag{17.123}$$

The fourth factor, $y(E)$, is a dimensionless profile factor which corrects for the occurrence of resonances in the "low-energy" region. In the "high-energy" region where resonances do not occur, $y(E)$ is unity. In the region where there are resonances, it is clear that $y(E)$ should have a sharp peak at $E = E_r$, where E_r is a resonance energy, and should decrease rapidly as $|E - E_r|$ increases. The form of $y(E)$ can be obtained from the following con-

siderations. Since a resonance corresponds to an excited state of C* and since an excited state always has a finite width Γ, then the excited state may be regarded as a "decaying" state; *i.e.*, the probability of finding C* in this particular state will decrease exponentially with time, the "e-folding" time for the probability being $\hbar/\Gamma$. The time-dependent part of the wave function for this state can then be written in the form

$$\psi(t) = \text{const.} \cdot \exp\left(-\frac{i}{\hbar} E_r t\right) \cdot \exp\left(-\frac{\Gamma}{2\hbar} t\right), \tag{17.123a}$$

where the factor (1/2) in the second exponent arises from the fact that the probability is proportional to $|\psi(t)|^2$. To obtain the frequency spectrum of $|\psi(t)|^2$, we Fourier analyze $\psi(t)$ by means of the usual Fourier integrals:

$$\psi(t) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} A(\omega) e^{i\omega t} \mathrm{d}\omega, \tag{17.123b}$$

$$A(\omega) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \psi(t) e^{-i\omega t} \mathrm{d}t. \tag{17.123c}$$

We substitute (17.123a) into (17.123c) and assume that $\psi(t) \equiv 0$ for $t < 0$ to avoid a divergence of the integral; we obtain

$$A(\omega) = \text{const.} \cdot \frac{1}{i(E_r/\hbar + \omega) + \dfrac{\Gamma}{2\hbar}}.$$

Now, the absolute square of $A(\omega)$ gives the frequency distribution of $|\psi(t)|^2$:

$$|A(\omega)|^2 = \text{const.} \frac{1}{\left(\dfrac{E_r}{\hbar} + \omega\right)^2 + \left(\dfrac{\Gamma}{2\hbar}\right)^2}. \tag{17.123d}$$

This function clearly has a single maximum at $\omega = -E_r/\hbar$; we must therefore associate $-\omega$ with $E/\hbar$,* so that we have

$$|A(E)|^2 \equiv y(E) = \text{const.} \frac{1}{(E - E_r)^2 + (\Gamma/2)^2}, \tag{17.124}$$

* The usual association made in quantum mechanics is $+\omega = E/\hbar$. We must use the negative sign here, however, if we adopt the conventional forms (17.123b,c) for the Fourier integrals, because of the presence of the minus sign in the first exponent in (17.123a). We could have used the usual association between ω and E if we had defined the Fourier integrals with the signs of the exponents reversed or if we had used a plus sign in the first exponential in (17.123a). This point clearly has no physical significance.

which is the familiar "resonance" form. Note that

$$\Gamma = \Gamma_\gamma + \sum_i \Gamma_i \tag{17.125}$$

is the total width at half-maximum of the excited level of C*, as can be seen by setting $|E - E_r| = (1/2)\Gamma$ in (17.124).

The constant in (17.124) is chosen in such a way that

$$\int_{\substack{\text{Over one} \\ \text{resonance peak}}} y(E)\mathrm{d}E = D, \tag{17.126}$$

where D is the average energy spacing between levels in the vicinity of E_r, so that the average of $y(E)$ over several resonances is unity. If $\Gamma \ll D$, then the limits of the integral in (17.126) can be extended to $E - E_r = \pm\infty$ with negligible error, so that the constant in (17.124) can be evaluated. The expression for $y(E)$ is then, finally,

$$y(E) = \frac{D}{2\pi} \cdot \frac{\Gamma}{(E - E_r)^2 + \left(\frac{1}{2}\Gamma\right)^2}. \tag{17.127}$$

The next-to-last factor, $\mathscr{S}$, is a "symmetry" factor, such that $\mathscr{S} = 1$ if a and X are different particles and $\mathscr{S} = 2$ if a and X are identical and spinless (because *two* particles of the same kind then "disappear" in each reaction; see Mott and Massey [Mo65a, Chap. 11]; Paquette and Reeves Pa64b] for further discussion of this symmetry factor).

These first five factors together give the cross section for formation of the compound nucleus in an excited level. The last factor, $G(b)$, is the "branching ratio," *i.e.*, the relative probability of decay of C* into the specific "channel" corresponding to the emission of "particle" b. Hence

$$G(b) = \Gamma_b/\Gamma, \tag{17.128}$$

where Γ_b is the associated partial width for emission of "particle" b by C*, since, according to the Bohr picture, $G(b)$ depends only on the properties of C* and not on the way the compound nucleus was formed.

Putting these six factors together, we obtain, finally, the well-known *Breit-Wigner one-level dispersion formula*:

$$\sigma(a,b) = (2\ell + 1)\pi\lambda\!\!\!^{-2} \cdot \omega \cdot \mathscr{S} \cdot \frac{\Gamma_a \Gamma_b}{(E - E_r)^2 + \left(\frac{1}{2}\Gamma\right)^2} \tag{17.129}$$

(see Fig. 17.8) which fits the experimental data best in the vicinity of E_r. Note that the barrier penetration factors are contained in the Γ's.

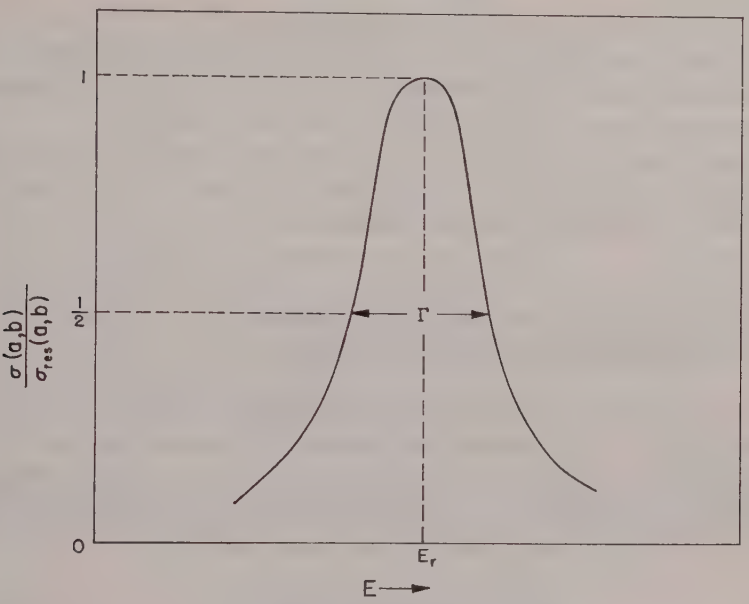


Fig. 17.8 The Breit-Wigner resonance form (schematic).

At the *peak* of the resonance ($E = E_r$) we have

$$\sigma_{\text{res}}(a,b) = (2\ell + 1)\pi\lambda\!\!\!^{-2} \cdot \omega \cdot \mathcal{P} \cdot \frac{\Gamma_a \Gamma_b}{\left(\frac{1}{2}\Gamma\right)^2}. \tag{17.130}$$

For low excitation energies Γ will contain only a few terms and hence may be of the same order of magnitude as Γ_a or Γ_b. If Γ_a and Γ_b are also of the same order of magnitude, then the factor in (17.130) involving the Γ's may not differ from unity by many orders of magnitude. Further, if either I_a or I_X is small ((1/2), say), then we see from (17.119) that $\omega \sim 1$. Hence it is possible in some cases that $\sigma_{\text{res}}(a,b) \sim (2\ell+1)\pi\lambda\!\!\!^{-2}\mathcal{P}$, *i.e.*, that $\sigma_{\text{res}}(a,b)$ may approach in order of magnitude the maximum possible cross section.

We note that Γ_a and Γ_b both depend on E, the relative kinetic energy of a and X when they are far apart; Γ_b depends only indirectly on E, since E_b, the relative kinetic energy of b and Y when they are far apart, is related to E and Q by conservation of total energy:

$$(M_a + M_X)c^2 + E = E_b + (M_b + M_Y)c^2$$

or

$$E_b = E + Q \tag{17.131}$$

from (17.93).

The ratio Γ_a/D for particle emission is given by (17.113); for light elements the Γ_a may be as high as a few tenths of an Mev if a is a neutron and much less than this if a is a low-energy proton. Radiation widths Γ_γ for the light elements are typically in the range of ~ 0.1 ev to ~ 100 ev.

If more than one resonance contribute appreciably to $\sigma(a,b)$, then (ignoring the possibility of interference) the right side of (17.129) must be replaced by a sum over the relevant resonances.

17.11 Cross Section for Low Energy Exothermic Nuclear Reactions

Clearly, $\sigma(a,b)$ approaches zero as E approaches zero unless the general reaction $X(a,b)Y$ is exothermic; we assume this general reaction to be exothermic here and in most of the following. We may then lump all quantities in the Breit-Wigner dispersion formula (17.129) which do not vanish as $E \to 0$ into a single constant. We obtain, as $E \to 0$,

$$\sigma(a,b) = \text{const.}\, \lambda\!\!\!^{-2} \Gamma_a \quad (E \to 0). \tag{17.132}$$

However, we had (see (17.113a)) for $k \ll K$ and for E far from any resonances

$$\Gamma_a \simeq \frac{4k}{K} P_\ell(E) \cdot \frac{D}{2\pi} \propto k \cdot P_\ell(E),$$

where $k = 1/\lambda\!\!\!^{-}$ is, as usual, the wave number corresponding to the relative kinetic energy E, and $P_\ell(E)$ is the barrier penetration factor. We then have, using the relation (17.99) between k and E,

$$\sigma(a,b) \simeq \text{const.}\, E^{-1/2} P_\ell(E) \quad (E \to 0). \tag{17.133}$$

If a is a neutron with zero orbital angular momentum ($\ell = 0$), then $P_\ell(E) = 1$ and we obtain

$$\sigma(a,b) \propto E^{-1/2} \propto v^{-1}, \tag{17.134}$$

which is the well-known "$1/v$ law" for absorption of low-energy neutrons.

If a is a charged particle, we must evaluate the penetration factor $P_\ell(E)$, which gives the probability of penetration of the potential barrier surrounding the nucleus. We do this by making use of the "W.K.B. approximation" of quantum mechanics (*cf.*, for example, Schiff [Sc55, Chap. 7]), although more

accurate methods are available. According to *BW* (pp. 358ff), for orbital angular momentum $\sqrt{\ell(\ell+1)}\hbar$ of a relative to X the penetration factor is

$$P_\ell \simeq \frac{K_\ell(R)}{k} \exp[-2\int_R^{r_0} K_\ell(r)\mathrm{d}r], \tag{17.135}$$

where

$$K_\ell(r) \equiv \sqrt{\frac{2m}{\hbar^2} U_\ell(r) - k^2} \tag{17.136}$$

is in general the wave number corresponding to the difference between the barrier energy at r and the relative kinetic energy at $r = \infty$; k is the wave number when $r = \infty$; m is as usual the reduced mass of a and X; and $U_\ell(r)$ is the potential barrier itself. The "boundaries" of the potential barrier are taken at R, the "nuclear interaction radius," and at r_0, the "classical distance of closest approach," defined by the relation

$$U_\ell(r_0) = E, \tag{17.137}$$

where E is the relative kinetic energy of a and X when far apart. If the potential barrier arises only from Coulomb and centrifugal forces, we have

$$U_\ell(r) = V(r) + \frac{\ell(\ell+1)\hbar^2}{2mr^2}, \tag{17.138}$$

where

$$V(r) = \frac{Z_a Z_X e^2}{r} (r \geq R) \tag{17.139}$$

is the ordinary Coulomb potential energy and the last term in (17.138) is the "centrifugal potential energy." For $r<R$, we assume that the potential energy becomes large and negative, corresponding to the deep potential well inside the nucleus.

Because of the very small cross sections for charged-particle reactions at low energies, particles with $\ell = 0$ (zero relative orbital angular momentum) make by far the greatest contribution to $\sigma(a,b)$; consequently, we henceforth consider only such particles (thus we consider only "head-on" collisions). The case $\ell \neq 0$ is treated in Frank-Kamenetskii [Fr62, pp. 172–174]. For the case $\ell = 0$ we have

$$U_0(r) = V(r) = \frac{Z_a Z_X e^2}{r} (r \geq R) \tag{17.140}$$

and

$$K_0(r) = \sqrt{\frac{2mZ_a Z_X e^2}{\hbar^2 r} - k^2}. \tag{17.141}$$

We note that $\hbar^2 K_0^2(r)/2m$ is negative for $r > r_0$. Thus, classically, we may interpret $-\hbar^2 K_0^2(r)/2m$ as the relative kinetic energy between a and X at distance $r(>r_0)$. However, for $r < r_0$ this interpretation is not valid, as the relative kinetic energy is zero, classically, for $r < r_0$. The situation is depicted in Fig. 17.9. The nuclear barrier height B, *i.e.*, $V(R)$, is given by

$$B \equiv V(R) = \frac{Z_a Z_X e^2}{R}. \tag{17.142}$$

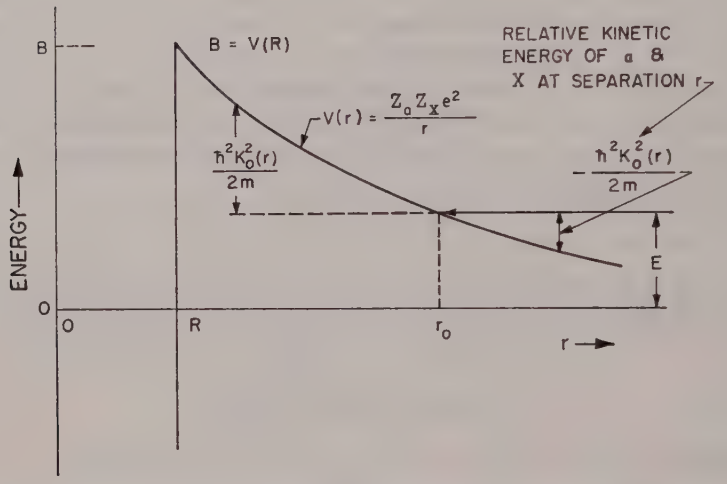


Fig. 17.9 The Coulomb potential barrier about a nucleus (schematic).

If $\mathscr{A}_X \gg \mathscr{A}_a$, we may write $R \simeq (3/2) \times 10^{-13} \mathscr{A}_X^{1/3}$ cm; also we may let $Z_X \simeq \mathscr{A}_X/2$ if $\mathscr{A}_X > 1$, so that we have

$$B \simeq 0.76 Z_a Z_X^{2/3} \text{ Mev}. \tag{17.143}$$

We see, then, that nuclear potential barriers are of the order of a few Mev in height. Since energies E of the order of a few Kev are of most interest in astrophysical applications, we see that $E \ll B$ for these energies.

The classical distance of closest approach, r_0, is given by (17.137) and (17.140) as

$$r_0 = \frac{Z_a Z_X e^2}{E}. \tag{17.144a}$$

Comparing this with (17.142), we then have that

$$r_0/R = B/E \sim 10^3 \tag{17.144b}$$

if E is of the order of a few Kev. Thus, classically, very few particles in a stellar interior would ever come closer to a nucleus than $\sim 10^3 R$, so that vanishingly few nuclear reactions would ever occur. The "tunnelling" through the potential barrier (strictly a quantum-mechanical effect), however, renders a non-negligible probability for a particle's penetrating the barrier even if $E \ll B$. We now calculate this probability of penetration.

Taking $\ell = 0$, we have from (17.135) and (17.141)

$$P_0 = \frac{1}{k}\sqrt{\frac{2mZ_aZ_Xe^2}{\hbar^2 R} - k^2}\cdot e^{-\mathcal{J}}$$

$$= \sqrt{\frac{B}{E} - 1}\cdot e^{-\mathcal{J}}, \tag{17.145}$$

since $E = \hbar^2 k^2/2m$, where

$$\mathcal{J} \equiv 2\int_R^{r_0} \sqrt{\frac{2mZ_aZ_Xe^2}{\hbar^2 r} - k^2}\, \mathrm{d}r. \tag{17.146}$$

Using (17.144), we write this as

$$\mathcal{J} = 2\frac{r_0}{\lambda}\int_R^{r_0} \sqrt{\frac{r_0}{r} - 1}\,\frac{\mathrm{d}r}{r_0}. \tag{17.147}$$

Letting $\xi^2 \equiv r/r_0$, we have

$$\mathcal{J} = 4\frac{r_0}{\lambda}\int_{\sqrt{\frac{R}{r_0}}}^{1} \sqrt{1-\xi^2}\,\mathrm{d}\xi \quad = 4\frac{r_0}{\lambda}\int_{\sqrt{\frac{E}{B}}}^{1} \sqrt{1-\xi^2}\,\mathrm{d}\xi. \tag{17.148a}$$

The integral in (17.148a) is elementary, and we obtain

$$\mathcal{J} = 4\frac{r_0}{\lambda}\left\{\frac{1}{2}\cos^{-1}\left(\frac{E}{B}\right)^{1/2} - \frac{1}{2}\left[\frac{E}{B}\left(1-\frac{E}{B}\right)\right]^{1/2}\right\} \tag{17.148b}$$

for all $E/B \leq 1$. A more useful result, however, is obtained if we expand the integral in (17.148a) in powers of $(E/B)^{1/2}$ about $\xi = 0$; we obtain

$$\mathcal{J} = 4\frac{r_0}{\lambda}\left[\int_0^1 \sqrt{1-\xi^2}\,\mathrm{d}\xi - \left(\frac{E}{B}\right)^{1/2} + \frac{1}{6}\left(\frac{E}{B}\right)^{3/2} + \right.$$

$$\left. + \frac{1}{40}\left(\frac{E}{B}\right)^{5/2} + \frac{1}{112}\left(\frac{E}{B}\right)^{7/2} + \cdots\right]. \tag{17.149}$$

The integral in (17.149) is just the area of one quadrant of a circle of unit radius, *i.e.*, $\pi/4$. We then obtain

$$\mathscr{J} = 4\frac{r_0}{\lambda}\left[\frac{\pi}{4} - \left(\frac{E}{B}\right)^{1/2} + \frac{1}{6}\left(\frac{E}{B}\right)^{3/2} + \frac{1}{40}\left(\frac{E}{B}\right)^{5/2} + \right.$$
$$\left. + \frac{1}{112}\left(\frac{E}{B}\right)^{7/2} + \cdots\right], \qquad (17.150)$$

where we note that all terms involving E inside the square brackets will be small compared to the first term if E is considerably smaller than, say, one Mev. For most purposes we indeed have $E/B \ll 1$, in which case it is usually sufficient to retain only the first two terms in square brackets in (17.150). Inclusion of the third term will give $\mathscr{J}$ to an accuracy of a few per cent or better for $E/B \lesssim 0.5$ or 0.6. Using (17.144), (17.142), and the relation $1/\lambda = p/\hbar = \sqrt{2mE}/\hbar$, we obtain, keeping the first three terms in (17.150),

$$\mathscr{J} \simeq \pi\frac{(2m)^{1/2}Z_aZ_Xe^2}{\hbar}E^{-1/2} - 8\frac{e}{\hbar}\sqrt{\frac{1}{2}mZ_aZ_XR} +$$
$$+ \frac{2}{3}\frac{(2m)^{1/2}R^{3/2}}{\hbar(Z_aZ_Xe^2)^{1/2}}\cdot E\,. \qquad (17.151)$$

If we express R in terms of $\mathscr{A}_a$ ($\simeq A_a$) and $\mathscr{A}_X$ ($\simeq A_X$) (*cf.* (17.97)), we may write (17.151) in the form

$$\mathscr{J} \equiv 31.28\,Z_aZ_X\mathscr{A}^{1/2}/E_{\text{Kev}}^{1/2} - 1.26\sqrt{\mathscr{A}Z_aZ_X(\mathscr{A}_a^{1/3}+\mathscr{A}_X^{1/3})}$$
$$+2.11\times10^{-4}\sqrt{\mathscr{A}(\mathscr{A}_a^{1/3}+\mathscr{A}_X^{1/3})^3/Z_aZ_X}\cdot E_{\text{Kev}}, \qquad (17.151a)$$

where $\mathscr{A}$ is the "reduced atomic weight" (*cf.* (17.100)) and E is in Kev. Because of the smallness of the third term on the right side of (17.151a) for values of E less than about an Mev, we can neglect this term to good accuracy and shall do so in the following.

It is customary to define a dimensionless quantity η as follows:

$$\eta \equiv \left(\frac{m}{2}\right)^{1/2}\frac{Z_aZ_Xe^2}{\hbar E^{1/2}} = \frac{Z_aZ_Xe^2}{\hbar v}. \qquad (17.152)$$

(For an interesting interpretation of η, see Frank-Kamenetskii [Fr62, p. 175].) Since we are assuming that $E/B \ll 1$, we can replace the square root factor in (17.145) by $(B/E)^{1/2}$. We then have, retaining only the first two terms in (17.151),

$$P_0 \simeq \left[B^{1/2}\exp\left(8\frac{e}{\hbar}\sqrt{\frac{1}{2}mZ_aZ_XR}\right)\right]\cdot E^{-1/2}e^{-2\pi\eta} \quad (E/B \ll 1), \qquad (17.153)$$

where the factor $e^{-2\pi\eta}$ is often called the "Gamow penetration factor." We note that the factor in square brackets in (17.153) does not depend on the relative energy E but does depend on the charges Z_a and Z_X and on the nuclear interaction radius R. Using (17.142), (17.97), and (17.151a), we may write (17.153) in the form

$$P_0 \simeq \left\{31.6\left[\frac{Z_a Z_X}{(\mathscr{A}_a^{1/3}+\mathscr{A}_X^{1/3})}\right]^{1/2}\exp\left[1.27\sqrt{\mathscr{A}Z_aZ_X(\mathscr{A}_a^{1/3}+\mathscr{A}_X^{1/3})}\right]\right\}\times$$

$$\times E_{\text{Kev}}^{-1/2}\exp\left[-31.28\,Z_aZ_X\mathscr{A}^{1/2}/E_{\text{Kev}}^{1/2}\right] \quad (E/B \ll 1). \tag{17.154}$$

Consider, for example, the penetration of a proton into a carbon nucleus, H^1+C^{12}. We have $\mathscr{A}_a \simeq 1$, $\mathscr{A}_X \simeq 12$; $Z_a = 1$, $Z_X = 6$; and we take $E = 10$ Kev. The probability of penetration of the potential barrier is then given by (17.154) as $P_0 \sim 10^{-21}$, which is a small number indeed. The presence of a Coulomb barrier can thus lower the cross section of a nuclear reaction by many orders of magnitude.

We recall from (17.133) that $\sigma(E) \equiv \sigma(a,b) \propto E^{-1/2}P_\ell(E)$ for $E \to 0$ and far from any resonances. Using (17.153) for $P_0(E)$, we then have

$$\sigma(E) \simeq \text{const.}\, E^{-1}e^{-2\pi\eta} \quad (\ell = 0).$$

We shall henceforth assume that $\ell = 0$ unless we explicitly state otherwise. It is customary to define the "astrophysical cross section factor" S as follows:

$$\sigma(E) = (S/E)\,e^{-2\pi\eta} \quad \left(\frac{E}{B} \ll 1\right), \tag{17.155}$$

where $\sigma(E)$ is the actual cross section for the reaction, and S is a slowly varying function of E for $E \to 0$ and far from any resonances. Although S can be computed in some cases from theory, the calculations are not reliable except for the very simplest nuclei, and the usual procedure is to use experimentally determined values of S. If S is in Kev-barns (1 barn = 10^{-24} cm^2) and E is in Kev, we have

$$\sigma(E) = \left(\frac{S_{\text{Kev-barns}}}{E_{\text{Kev}}}\right)\exp\left[-31.28\,Z_aZ_X\mathscr{A}^{1/2}/E_{\text{Kev}}^{1/2}\right]\text{barns}\left(\frac{E}{B} \ll 1\right). \tag{17.156}$$

Note that since S is normally determined experimentally, there is no need to introduce a "symmetry factor" (see Sect. 17.10) into (17.155) or (17.156); S, so determined, automatically includes the symmetry factor.

Finally, we discuss briefly the experimental determination of S. Because of the presence of Coulomb barriers, cross sections for proton-induced reactions with light nuclei become too small ($\lesssim 10^{-11}-10^{-10}$ barns) to be

observable in the laboratory at energies E below about 100 Kev. On the other hand, the energies of most astrophysical interest are in the range of, say, (5–50) Kev (*cf.* Sect. 17.8) for $T \sim 10^7$–10^8 °K. A considerable extrapolation is therefore required to reach the "astrophysical" energy range, at which S should be evaluated.

The usual procedure for extrapolating to values of S in the astrophysical range is as follows. The cross section $\sigma(E)$ is measured at several energies in the range of, say (100–200) Kev. The cross section factor S is then computed from the measured values of $\sigma(E)$, using (17.155), so that one obtains $S(E)$ empirically as a function of energy in this range of (100–200) Kev. If there are no resonances in or near this energy range, then $S(E)$ will be a slowly varying function of E and extrapolation of $S(E)$ down to the astrophysical range can be done reasonably reliably, on the basis of the behavior of $S(E)$ in the experimentally accessible range. The existence of an unsuspected resonance in the range $E < 100$ Kev could, of course, lead to a completely wrong value of S in the astrophysical range.

For proton-induced reactions with the C-N-O group of elements, values of S are typically in the range of a few Kev-barns.

17.12 Thermonuclear Reaction Rate

Here we are concerned with the rate at which the reaction

$$a + X \rightarrow Y + b \tag{17.157}$$

proceeds as a result of thermally induced collisions among the particles in a stellar interior, as a function of density and temperature. Once this rate is known, the rate of energy production can be obtained immediately since the Q-value (*cf.* (17.93)) for the reaction is assumed known.

We use subscripts 1 and 2 for the present to distinguish between the two types (a and X) of interacting particles, assumed non-identical. It is clear that the probability per unit time for *one* particle of type 1 or 2 to "interact" with *unit* number density of particles of the other type is $\sigma(v)v$, where v is the *relative* velocity between the two particles. If $\mathrm{d}N_1(\mathbf{v}_1)$ and $\mathrm{d}N_2(\mathbf{v}_2)$ denote the respective numbers of particles of the two types per unit volume having velocity vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ (relative to some fixed frame) lying within the "volume" elements $\mathrm{d}^3\mathbf{v}_1$ and $\mathrm{d}^3\mathbf{v}_2$ in the velocity spaces of the two types of particles, then it is clear that the total number of reactions (17.157) per unit volume and time is given by

$$r = \int_{-\infty}^{\infty}\int \mathrm{d}N_1(\mathbf{v}_1)\mathrm{d}N_2(\mathbf{v}_2)v\sigma(v), \tag{17.158}$$

where the integrations are to be carried out over the entire velocity spaces of the two types of particle. (We note that if (17.157) is an *endothermic* reaction, then $\sigma(v)$ would be zero if v were below the threshhold for the reaction.) Since nucleons and nuclei do not become degenerate except at extremely high densities (say $\rho \gtrsim 10^{14}\text{gm/cm}^3$, or of the order of nuclear densities), we may assume that both types of particle have Maxwellian velocity distributions. Thus we have for particles of type i (see Sect. 3.4b)

$$dN_i(\mathbf{v}_i) = N_i\left(\frac{m_i}{2\pi kT}\right)^{3/2} e^{-m_i v_i^2/2kT}\, d^3v_i \quad (i = 1,2), \tag{17.159}$$

where N_i is the total number density for particles of type i and m_i is the mass per particle. We then have from (17.158)

$$r = N_1 N_2 \left(\frac{m_1}{2\pi kT}\right)^{3/2}\left(\frac{m_2}{2\pi kT}\right)^{3/2} \int_{-\infty}^{\infty}\int \exp$$

$$\left(-\frac{m_1 v_1^2}{2kT} - \frac{m_2 v_2^2}{2kT}\right) \cdot v\sigma(v) d^3v_1 d^3v_2 . \tag{17.160}$$

We now transform to relative velocity $\mathbf{v}$ and center of mass velocity $\mathbf{V}$:

$$\mathbf{v} = \mathbf{v}_1 - \mathbf{v}_2 , \tag{17.161}$$

$$(m_1 + m_2)\mathbf{V} = m_1\mathbf{v}_1 + m_2\mathbf{v}_2 . \tag{17.162}$$

From these equations we obtain

$$\mathbf{v}_1 = \mathbf{V} + \frac{m_2}{m_1 + m_2}\mathbf{v}, \tag{17.163a}$$

$$\mathbf{v}_2 = \mathbf{V} - \frac{m_1}{m_1 + m_2}\mathbf{v}. \tag{17.163b}$$

The total kinetic energy then becomes

$$\frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 = \frac{1}{2}(m_1 + m_2)V^2 + \frac{1}{2}mv^2 , \tag{17.164}$$

where

$$m = \frac{m_1 m_2}{m_1 + m_2} \tag{17.165}$$

is the reduced mass of the two types of particle. It is also clear that, since the particle velocities are according to (17.159) isotropically distributed and since

the integration in (17.160) is over the entire velocity space, we may formally replace the element of "volume" $\mathrm{d}^3v_1\mathrm{d}^3v_2$ in velocity space by the "volume" element $\mathrm{d}^3V\mathrm{d}^3v = 4\pi V^2\mathrm{d}V \cdot 4\pi v^2\mathrm{d}v$. We then obtain for (17.160)

$$r = N_1N_2\left(\frac{m_1}{2\pi kT}\right)^{3/2}\left(\frac{m_2}{2\pi kT}\right)^{3/2}\int_0^\infty\int \exp\left[-\frac{(m_1+m_2)V^2}{2kT} - \frac{mv^2}{2kT}\right]\cdot v\sigma(v)\cdot 4\pi V^2\mathrm{d}V\cdot 4\pi v^2\mathrm{d}v\,. \tag{17.166}$$

Carrying out the integration over V, we obtain

$$\int_0^\infty \exp\left[-\frac{(m_1+m_2)V^2}{2kT}\right]\cdot 4\pi V^2\mathrm{d}V = \left(\frac{2\pi kT}{m_1+m_2}\right)^{3/2},$$

whence (17.166) becomes

$$r = 4\pi N_1N_2\left(\frac{m}{2\pi kT}\right)^{3/2}\int_0^\infty e^{-mv^2/2kT}\cdot v\sigma(v)\cdot v^2\mathrm{d}v\,. \tag{17.167}$$

In terms of the relative kinetic energy

$$E = \frac{1}{2}mv^2 \tag{17.168}$$

between particles 1 and 2 when far apart, we have

$$r = N_1N_2\langle\sigma v\rangle, \tag{17.169a}$$

where

$$\langle\sigma v\rangle \equiv \frac{2}{\pi^{1/2}}\frac{1}{(kT)^{3/2}}\int_0^\infty e^{-E/kT}\cdot v\sigma(v)\cdot E^{1/2}\mathrm{d}E \tag{17.170a}$$

is the average over a Maxwellian velocity distribution of the quantity σv.

A general, physical interpretation of $\langle\sigma v\rangle$ is the following: $\langle\sigma v\rangle$ is the probability per unit time that *two* particles, confined to a *unit* volume, will react with each other. Note that $\langle\sigma v\rangle$ does not depend on density ρ (except for possible indirect effects on the cross section σ, see Sect. 17.15) and is a function only of temperature T for a given type of reaction.

We may also write (17.170a) in the form

$$\langle\sigma v\rangle = \int_0^\infty n(E)\cdot v\sigma(v)\cdot\mathrm{d}E\,, \tag{17.170b}$$

where $n(E)\mathrm{d}E$ is the *fraction* of all particles of type 1 or 2 whose kinetic energy relative to *one* particle of the other type lies between E and $E+\mathrm{d}E$. Comparison with (17.170a) yields the result

$$n(E) = \frac{2}{\pi^{1/2}} \frac{1}{(kT)^{3/2}} \cdot e^{-E/kT} \cdot E^{1/2} . \tag{17.170c}$$

In the case where the reacting particles are identical, we may deduce the reaction rate by appealing to the physical interpretation of $\langle \sigma v \rangle$ presented above: If $\langle \sigma v \rangle$ is the probability per unit time that a pair of particles, confined to a unit volume, will react, then the probability per unit time that a pair of particles will react anywhere within a system of volume V is $\langle \sigma v \rangle / V$. Now, the total number of distinct pairs of identical particles in a system consisting of $\mathcal{N}$ identical particles is $\mathcal{N}(\mathcal{N}-1)/2$. Thus, the total reaction rate (number of reactions per unit time) in the system, rV, is $\mathcal{N}(\mathcal{N}-1)\langle \sigma v \rangle / 2V$, where r is the total reaction rate per unit volume. Thus, for reactions between like particles the total reaction rate per unit volume is

$$r = \frac{1}{V^2} \frac{\mathcal{N}(\mathcal{N}-1)}{2} \langle \sigma v \rangle = \frac{N\left(N - \frac{1}{V}\right)}{2} \langle \sigma v \rangle \simeq \frac{N^2}{2} \langle \sigma v \rangle , \tag{17.169b}$$

where N is the average number density of particles. The approximate equality in (17.169b) is extremely accurate in all astrophysical applications when one considers a large number of particles, $\mathcal{N}$.

We consider now the *mean life* of a nucleus. If we consider reactions between two types of *non-identical* particles, X and a, say, it is convenient to return to our original interpretation of $\langle \sigma v \rangle$ (beginning of this section): $\langle \sigma v \rangle$ is the probability per unit time that *one* particle of *either* type will react with *unit number density* of the *other* type. It is then clear that the probability of reaction per unit time per particle of type X is

$$p_X = N_a \langle \sigma v \rangle \equiv 1/t_X , \tag{17.171}$$

where t_x is the *mean life* of the nucleus of type X against reaction with particles of type a, and where N_a is the total number density of type a particles. Similarly, for the probability p_a of reaction per unit time per particle of type a, we have

$$p_a = N_X \langle \sigma v \rangle \equiv 1/t_a . \tag{17.172}$$

If the relative mass abundance (*i.e.*, mass fraction) of type a particles, for example, is x_a, then we have $\rho x_a = m_a N_a$, where ρ denotes the mass density of the stellar material. Writing $m_a = \mathscr{A}_a / N_0$ for the mass per particle of type

a, where N_0 is Avogadro's number and $\mathscr{A}_a$ is the atomic mass (mass per mole) of a-type particles, we have

$$N_a = N_0 \rho x_a / \mathscr{A}_a . \tag{17.173}$$

Thus we may also write (17.171) in the form

$$p_X = 1/t_X = N_0 \rho (x_a/\mathscr{A}_a) \cdot \langle \sigma v \rangle . \tag{17.174}$$

If the particles of type a are protons (as is often the case), we have $x_a = X$, $\mathscr{A}_a \simeq 1$, so that

$$p_X = 1/t_X = N_0 \rho X \langle \sigma v \rangle . \tag{17.175}$$

If the reacting particles are identical, then the probability p of reaction per unit time per particle is seen from (17.169b) to be just twice the total reaction rate in the system, rV, divided by the total number of particles, $\mathscr{N}$, where the factor of two appears because *two* particles are involved in *each* reaction:

$$p = 1/t = 2(rV)/\mathscr{N} = \left(N - \frac{1}{V}\right) \langle \sigma v \rangle \simeq N \langle \sigma v \rangle ,$$

where t is the mean life of a particle in the system.

We consider now the rate of energy production. If Q denotes the energy given off in a single reaction of the type $X(a,b)Y$, then the rate of release of thermonuclear energy per unit mass is clearly $r \cdot Q/\rho$, where r is the number of reactions per unit volume and time. Using (17.169a) for r, we then obtain for the rate of release of thermonuclear energy per unit mass from a given type of reaction (in which the reacting particles are not identical)

$$\varepsilon = N_a N_X \langle \sigma v \rangle \cdot Q/\rho . \tag{17.176a}$$

Alternatively, noting (17.171) and (17.172), we may write (17.176a) as

$$\varepsilon = \frac{N_x}{t_x} \frac{Q}{\rho} = \frac{N_a}{t_a} \frac{Q}{\rho} . \tag{17.176b}$$

Using expressions analogous to (17.173) for the N's, we may also write (17.176a) as

$$\varepsilon = N_0^2 \rho \cdot \frac{x_a}{\mathscr{A}_a} \cdot \frac{x_X}{\mathscr{A}_X} \cdot \langle \sigma v \rangle \cdot Q . \tag{17.177a}$$

If the particles participating in the reaction are identical, we must use (17.169b) for the reaction rate r, and we obtain

$$\varepsilon = \frac{1}{2} N_0^2 \rho \left(\frac{x}{\mathscr{A}}\right)^2 \langle \sigma v \rangle Q , \tag{17.177b}$$

where x and $\mathscr{A}$ are, respectively, the relative mass abundance and atomic mass of the (identical) particles participating in the reaction.

For approximate work it is sometimes found convenient to approximate the quantity $\langle \sigma v \rangle$ in the vicinity of some reference temperature T_0 by the following interpolation formula (recall that $\langle \sigma v \rangle$ does not depend on density except for effects of screening of the nuclear charge by electrons, see Sect. 17.15):

$$\langle \sigma v \rangle_T \simeq \langle \sigma v \rangle_{T_0} (T/T_0)^\nu , \tag{17.178}$$

where

$$\nu \equiv \mathrm{d} \ln \langle \sigma v \rangle / \mathrm{d} \ln T \big|_{T=T_0} . \tag{17.179}$$

In this case we have in the vicinity of T_0

$$\varepsilon \simeq \varepsilon_0 \rho T^\nu , \tag{17.180}$$

where

$$\varepsilon_0 = N_0^2 \cdot \frac{x_a}{\mathscr{A}_a} \cdot \frac{x_X}{\mathscr{A}_X} \cdot Q \cdot \frac{\langle \sigma v \rangle_{T_0}}{T_0{}^\nu} \tag{17.181}$$

if particles a and X are not identical. Explicit expressions for ν will be given in later sections.

We see, then, that the crucial quantity which enters into all our equations is $\langle \sigma v \rangle$, which we now proceed to evaluate for two important and common situations.

17.13 Non-Resonant Contribution

Here we shall compute the quantity $\langle \sigma v \rangle$ in the case of low energies far from any resonances. In this case we saw (Sect. 17.11) that the cross section for the reaction $X(a,b)Y$ could be written in the form

$$\sigma(E) = (S/E) e^{-2\pi\eta} , \tag{17.182}$$

where S is the cross section factor, E is the relative kinetic energy of a and X when they are far apart, and η is given by the relation

$$\eta = \left(\frac{m}{2} \right)^{1/2} \frac{Z_a Z_X e^2}{\hbar E^{1/2}} , \tag{17.183}$$

where m is the reduced mass of particles a and X. The cross section factor S is found experimentally to be a slowly varying function of E in most cases, so we shall treat it as constant here (in case the variation of S with E is non-negligible, the usual procedure is to estimate the value of S at the energy of the "Gamow peak" (discussed later in this section) by extrapolation from the experimentally accessible energy range).

Expressing v in terms of E ($v = \sqrt{2E/m}$) and using (17.182) for $\sigma(E)$, we have from (17.170)

$$\langle \sigma v \rangle = \left(\frac{2}{m}\right)^{1/2} \frac{2}{\pi^{1/2}} \frac{1}{(kT)^{3/2}} \cdot S \cdot \int_0^\infty e^{-E/kT - BE^{-1/2}} dE, \tag{17.184}$$

where*

$$B \equiv \pi(2m)^{1/2} \frac{Z_a Z_X e^2}{\hbar}. \tag{17.185}$$

Now let

$$y \equiv E/kT; \tag{17.186}$$

then we have

$$\langle \sigma v \rangle = \left(\frac{2}{m}\right)^{1/2} \frac{2}{\pi^{1/2}} \frac{1}{(kT)^{1/2}} \cdot S \cdot \int_0^\infty e^{-y - Cy^{-1/2}} dy, \tag{17.187}$$

where

$$C \equiv \frac{B}{(kT)^{1/2}} = \pi(2m)^{1/2} \frac{Z_a Z_X e^2}{\hbar (kT)^{1/2}}. \tag{17.188}$$

We write (17.187) in the form

$$\langle \sigma v \rangle = \left(\frac{2}{m}\right)^{1/2} \frac{2}{\pi^{1/2}} \frac{1}{(kT)^{1/2}} \cdot S \cdot \mathcal{I}, \tag{17.189}$$

where the dimensionless integral $\mathcal{I}$ is given by

$$\mathcal{I} = \int_0^\infty e^{-y - Cy^{-1/2}} dy. \tag{17.190}$$

Note that the integrand of $\mathcal{I}$ consists of the product of a rapidly decreasing function, e^{-y}, and a rapidly increasing function, $\exp(-Cy^{-1/2})$; hence this product will be a sharply-peaked function having a maximum at, say, $y = y_0$ (see Fig. 17.10).

This peak is sometimes called the "Gamow peak." We shall write the integrand as

$$f(y) = e^{-y - Cy^{-1/2}}. \tag{17.190a}$$

To locate the position of the maximum, we equate to zero the derivate of $f(y)$ and obtain

$$y_0 = \left(\frac{C}{2}\right)^{2/3} = \frac{E_{\max}}{kT}, \tag{17.190b}$$

* The symbol B used here sould not be confused with the symbol B defined in (17.142).

where y_0 and E_{max} denote the values of y and E at the maximum. We then have from (17.188),

$$E_{max} = \left[\left(\frac{m}{2}\right)^{1/2} \frac{\pi Z_a Z_X e^2}{\hbar} (kT)\right]^{2/3} \propto T^{2/3}. \tag{17.191}$$

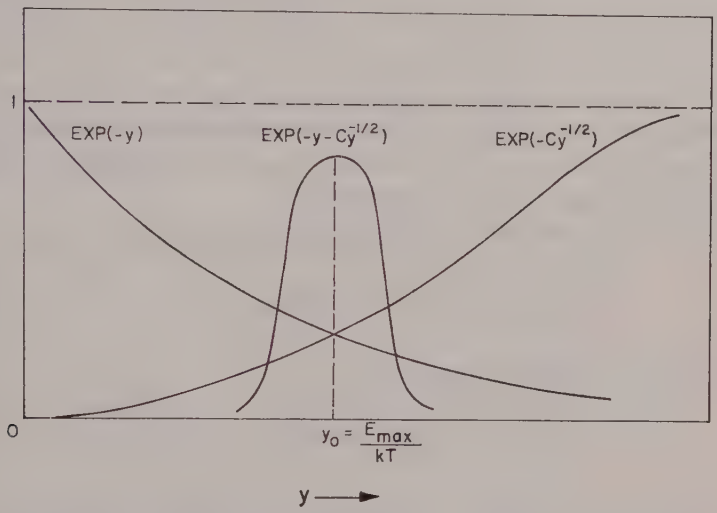


Fig. 17.10 The integrand of (17.190) showing the Gamow peak (schematic).

If we use the relations

$$m = \frac{m_a m_X}{m_a + m_X} = \frac{1}{N_0} \frac{\mathscr{A}_a \mathscr{A}_X}{\mathscr{A}_a + \mathscr{A}_X} \equiv \frac{\mathscr{A}}{N_0}, \tag{17.192}$$

where $\mathscr{A}$ is the "reduced atomic mass," we may also write (17.191) in the form

$$E_{max} = \left[\frac{1}{(2N_0)^{1/2}} \frac{\pi e^2 k 10^6}{\hbar}\right]^{2/3} [Z_a^2 Z_X^2 \mathscr{A}]^{1/3} \left(\frac{T}{10^6}\right)^{2/3}. \tag{17.193}$$

Evaluating the constants gives the result

$$E_{max} = 1.220 [Z_a^2 Z_X^2 \mathscr{A}]^{1/3} T_6^{2/3} \text{ Kev}, \tag{17.194}$$

where $T_6 \equiv T(°\text{K})/10^6$ is temperature T in units of 10^6 °K.

For example, for $Z_a = Z_X = \mathscr{A}_a = \mathscr{A}_X \simeq 1, T = 10^7$ °K, we have $E_{max} \simeq 4.5$ Kev; for $Z_a = \mathscr{A}_a \simeq 1, Z_X = 7, \mathscr{A}_X \simeq 15, T = 10^7$ °K, we have $E_{max} \simeq 20.3$ Kev.

Thus, for most thermonuclear reactions of interest and for $10^7 \lesssim T(°K) \lesssim 10^8$, E_{max} falls within the "stellar energy region," defined roughly by the limits

$$5 \lesssim E_{max}(\text{Kev}) \lesssim 100. \tag{17.195}$$

The upper limit would be about 500 Kev for temperatures near 10^9 °K.

The energy E_{max} is the energy at which practically all the thermonuclear reactions occur at the temperature T. Since most particles in a stellar interior have energies of the order of $(1/2)kT \simeq (1/2-5)$Kev for $T = 10^7$–10^8 °K, then it follows that most of the thermonuclear reactions occur at energies falling well out into the "tail" of the Maxwellian distribution; hence only a very small fraction of the particles have the proper energies ($E \simeq E_{max}$) to contribute to the thermonuclear reactions.

We are now concerned with the evaluation of the integral

$$\mathscr{I} = \int_0^\infty e^{-y-Cy^{-1/2}} dy\,;$$

since $C = 2y_0^{3/2}$, we have, alternatively,

$$\mathscr{I} = \int_0^\infty e^{-y-2y_0^{3/2}y^{-1/2}} dy\,. \tag{17.196}$$

Since the integrand here is a sharply-peaked function centered about $y = y_0$, we may expand the exponent in a Taylor series about the point $y = y_0$. Let $g(y)$ be the exponent:

$$-g(y) = y + 2y_0^{3/2}y^{-1/2}.$$

Then we have

$$g(y) = -3y_0 - \frac{3}{4}y_0^{-1}(y-y_0)^2 + \frac{5}{8}y_0^{-2}(y-y_0)^3 - \\ - \frac{35}{64}y_0^{-3}(y-y_0)^4 + \cdots \tag{17.197a}$$

$$= -3y_0 - \frac{3}{4}y_0^{-1}(y-y_0)^2\left[1 - \frac{5}{6}y_0^{-1}(y-y_0) + \right. \\ \left. + \frac{35}{48}y_0^{-2}(y-y_0)^2 - \cdots\right]. \tag{17.197b}$$

Since y_0 is a fairly large number (say ~ 5 to 100) and since the "width" of the Gamow peak is usually small compared to y_0 (see later in this section), then the quantity $(y-y_0)/y_0$ will generally be small compared to unity in those regions near the Gamow peak, where most of the contribution to the integral

$\mathscr{I}$ arises. We should therefore commit only a small error if we neglect all terms in the square brackets in (17.197b) except the first (the correction to this approximation will be given later in this section). This is the approximation which is usually made, and we see that it is equivalent to replacing the true Gamow peak by a Gaussian of the appropriate height and width. We then have

$$g(y) \simeq -3y_0 - \frac{3}{4}y_0\left(\frac{y}{y_0} - 1\right)^2. \tag{17.198}$$

It is customary to express the results in terms of a new dimensionless variable τ, defined by

$$\tau \equiv 3y_0 = 3E_{\max}/kT \tag{17.199}$$

by (17.190b). Using (17.191), we have

$$\tau = 3\left[\left(\frac{m}{2}\right)\cdot\frac{\pi^2 Z_a^2 Z_X^2 e^4}{\hbar^2(kT)}\right]^{1/3} \propto \left(\frac{\mathscr{A} Z_a^2 Z_X^2}{T}\right)^{1/3}. \tag{17.200}$$

Using the relation $m = \mathscr{A}/N_0$, $\mathscr{A}$ being the "reduced atomic mass" (*cf.* (17.192)) and N_0 being Avogadro's number, we also have

$$\begin{aligned}\tau &= 3\left[\frac{\pi^2 e^4}{2N_0\hbar^2 k 10^6}\right]^{1/3}(Z_a^2 Z_X^2 \mathscr{A})^{1/3}\left(\frac{10^6}{T}\right)^{1/3} \\ &= 42.48\,(Z_a^2 Z_X^2 \mathscr{A})^{1/3}\,T_6^{-1/3},\end{aligned} \tag{17.201}$$

where $T_6 = T(^\circ\mathrm{K})/10^6$. Since $y = E/kT$, (17.198) then becomes

$$g(y) = -\tau - \frac{\tau}{4}\left(\frac{E}{E_{\max}} - 1\right)^2. \tag{17.202}$$

We may estimate the "width" of the Gamow peak by representing this peak by the Gaussian $e^{g(y)}$, with $g(y)$ given by (17.202). We obtain for the *full* width at half maximum

$$\frac{2\Delta E}{E_{\max}} \equiv \frac{2\Delta y}{y_0} = 4\sqrt{\frac{\ln 2}{\tau}} = 4\sqrt{\frac{\ln 2}{3}\cdot\frac{kT}{E_{\max}}} = 4\sqrt{\frac{\ln 2}{3}\frac{(2N_0)^{1/2}\hbar}{\pi e^2}}\left(\frac{E_{\max}}{Z_a^2 Z_X^2 \mathscr{A}}\right)^{1/4}, \tag{17.203}$$

using (17.193) for the last equality. Expressing $E_{\max}$ in Kev, we have

$$\frac{2\Delta E}{E_{\max}} = 0.565\sqrt{\frac{T_6}{E_{\max}^{(\mathrm{Kev})}}} = 0.486\left(\frac{E_{\max}^{(\mathrm{Kev})}}{Z_a^2 Z_X^2 \mathscr{A}}\right)^{1/4}, \tag{17.204}$$

using (17.194) for the last equality. Thus $2\Delta E/E_{\max} \simeq 0.33$ for $E_{\max} = 30$ Kev and $T = 10^7\,^\circ\mathrm{K}$; hence $2\Delta E/E_{\max}$ is generally less than one but not by a large

factor. Using (17.193), we see that $2\Delta E/E_{\max} \propto (T/Z_a^2 Z_X^2 \mathscr{A})^{1/3}$, so that the width of the Gamow peak does not depend sensitively on temperature.

Using (17.202), we then have for the integral $\mathscr{I}$

$$\mathscr{I} = \int_0^\infty e^{-\tau - \frac{\tau}{4}\left(\frac{E}{E_{\max}}-1\right)^2} \cdot \frac{E_{\max}}{kT} \mathrm{d}\left(\frac{E}{E_{\max}}\right)$$

$$= \frac{1}{3}\tau e^{-\tau} \int_{-1}^{\infty} e^{-\frac{\tau}{4}\left(\frac{E}{E_{\max}}-1\right)^2} \mathrm{d}\left(\frac{E}{E_{\max}} - 1\right). \tag{17.205}$$

Since the peak of our Gaussian is centered about $(E/E_{\max} - 1) = 0$ and since the *half*-width at half maximum is generally several times smaller than $E_{\max}$, we may with only slight error extend the lower limit of integration in (17.205) to $(-\infty)$. We then obtain

$$\mathscr{I} \simeq \frac{2}{3}\pi^{1/2}\tau^{1/2}e^{-\tau}. \tag{17.206}$$

A correction factor which corrects $\mathscr{I}$ as given in (17.206) for replacing the Gamow peak by a Gaussian is given by Salpeter [Sa52b] as

$$C_1(\tau) = \left[1 + \frac{5}{12\tau} - \frac{35}{388\tau^2} + \cdots\right]. \tag{17.206a}$$

Since τ is usually greater than, say, 10 (*cf.* Tables 17.1 and 17.2 below), it follows that representing the Gamow peak by a Gaussian introduces errors into the reaction rate of no more than a few per cent.

A correction factor which accounts, in addition, for a dependence of the cross section factor $S(E)$ on energy E has been given by Bahcall [Ba66], and is

$$C_2(\tau) = 1 + \frac{5}{12\tau} + \frac{S'(0)E_{\max}}{S(0)}\left(1 + \frac{105}{36\tau}\right)$$

$$+ \frac{1}{2}\frac{S''(0)E_{\max}^2}{S(0)}\left(1 + \frac{267}{36\tau}\right) + \cdots, \tag{17.206b}$$

where $S(0)$ is the value of $S(E)$ at $E = 0$ and a prime denotes differentiation with respect to E.

Now, we had for the non-resonant contribution (*cf.* (17.189))

$$\langle \sigma v \rangle = \left(\frac{2}{m}\right)^{1/2} \frac{2}{\pi^{1/2}} \frac{1}{(kT)^{1/2}} \cdot S \cdot \mathscr{I}. \tag{17.207}$$

We use (17.200) to express kT in terms of τ, and write $m = \mathscr{A}/N_0$ ($\mathscr{A}$ = "reduced atomic mass"), to obtain, finally, the expression

$$\langle \sigma v \rangle = \left[\frac{8}{3^{5/2}} \cdot N_0 \cdot \frac{\hbar}{\pi e^2} \right] \cdot \frac{1}{Z_a Z_X \mathscr{A}} \cdot S \cdot \tau^2 e^{-\tau} \tag{17.208}$$

for the non-resonant contribution to $\langle \sigma v \rangle$. If we express S in Kev-barns, we have

$$\langle \sigma v \rangle = 7.207 \times 10^{-19} \cdot \frac{1}{Z_a Z_X \mathscr{A}} \cdot S \cdot \tau^2 e^{-\tau} \mathrm{cm}^3/\mathrm{sec}. \tag{17.209}$$

We note from (17.199) and (17.194) that

$$\langle \sigma v \rangle \propto \frac{(Z_a Z_X/\mathscr{A})^{1/3}}{T^{2/3}} \exp\left[-3\left(\frac{\pi^2 e^4}{2 N_0 \hbar^2 k} \right)^{1/3} (Z_a^2 Z_X^2 \mathscr{A}/T)^{1/3} \right].$$

Because of the exponential factor $(Z_a^2 Z_X^2 \mathscr{A})^{1/3}$, we see that charged-particle reactions in which one of the particles is a proton are vastly more likely to occur than those in which neither particle is a proton. The reaction rate goes down very rapidly as the charges on both particles increase.

For temperatures near some temperature T_0 we can use the interpolation formula

$$\langle \sigma v \rangle_T \simeq \langle \sigma v \rangle_{T_0} \left(\frac{T}{T_0} \right)^{\nu}, \tag{17.210}$$

where

$$\nu = \left. \frac{\mathrm{d} \ln \langle \sigma v \rangle}{\mathrm{d} \ln T} \right|_{T = T_0}. \tag{17.211}$$

From (17.209) and (17.201) we have

$$\ln \langle \sigma v \rangle = \text{const.} + 2 \ln \tau - \tau,$$

$$\ln T = \text{const.} - 3 \ln \tau. \quad (T \propto 1/\tau^3),$$

whence

$$\nu = \left. \frac{\mathrm{d} \ln \langle \sigma v \rangle}{\mathrm{d} \ln T} \right|_{T = T_0} = \frac{1}{3}(\tau_0 - 2). \tag{17.212}$$

Thus, as long as $\tau_0 \gg 2$, which is usually the case ($T \sim 10^7$–10^8 °K, *cf.* (17.201)), we have, approximately,

$$\nu \propto T^{-1/3}. \tag{17.213}$$

Because the rate of energy production per unit mass is $\varepsilon \propto \rho \langle \sigma v \rangle \cdot Q$, it is clear that ν is also the temperature exponent in the expression for ε in the vicinity of $T = T_0$ (*cf.* (17.180)).

We cite two examples, to illustrate the magnitudes of some of the quantities we have introduced:

(1) $Z_a = Z_X = 1$; $\mathscr{A}_a = \mathscr{A}_X \simeq 1 (H^1 + H^1)$. Table 17.1 gives values for τ and ν for three reference temperatures T_0.

(2) $Z_a \simeq \mathscr{A}_a \simeq 1$, $Z_X = 7$, $\mathscr{A}_X \simeq 14$ $(H^1 + N^{14})$. Results are given in Table 17.2.

Table 17.1

VALUES OF τ AND ν FOR THE REACTION $H^1 + H^1$

T_{06}	τ	ν
15	13.6	3.87
20	12.4	3.47
25	11.5	3.18

Table 17.2

VALUES OF τ AND ν FOR THE REACTION $H^1 + N^{14}$

T_{06}	τ	ν
20	55.9	18.0
25	51.8	16.6
30	48.7	15.6

We note that ν, the temperature exponent, does not involve the often poorly known cross section factor S, and hence is quite accurately known for a given reaction.

17.14 Resonant Contribution

We had, in general (see (17.170b)),

$$\langle \sigma v \rangle = \int_0^\infty n(E)\sigma(E)v\mathrm{d}E\,, \tag{17.214}$$

where E and v are the relative kinetic energy and velocity, respectively, of particles a and X when far apart and $n(E)\mathrm{d}E$ is the fraction of particles of

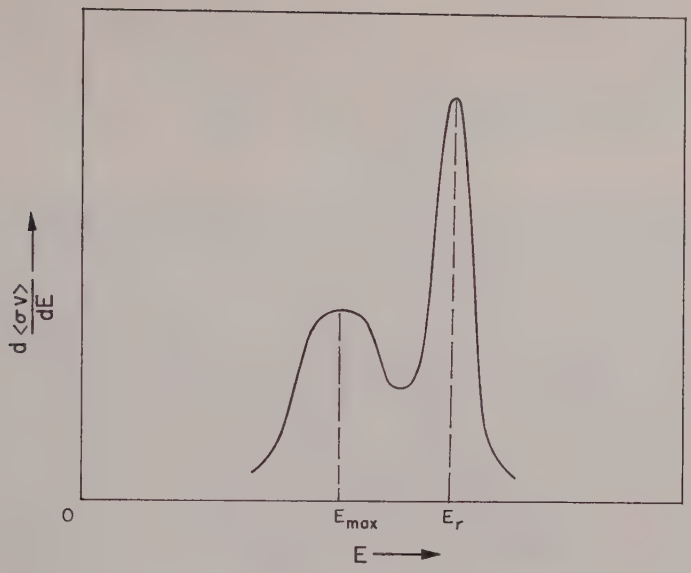


Fig. 17.11 A resonance in the vicinity of the Gamow peak (schematic).

either type having kinetic energies relative to one of the other type lying between E and $E+\mathrm{d}E$. If a strong resonance exists in or near the "stellar energy region" (*i.e.*, if the resonance energy E_r is somewhere in the vicinity of $E_{\max}$, *cf.* (17.195) and Fig. 17.11), then most of the contribution to $\langle \sigma v \rangle$ may arise from this resonance rather than from the non-resonant Gamow peak. (It is clear from the Gaussian nature of the Gamow peak that a resonance energy E_r not in the rather near vicinity of this peak would have a negligible effect on $\langle \sigma v \rangle$.) In addition, the width of such a resonance is usually smaller than the width of the Gamow peak, especially in the case where the emitted particle b is a photon or a charged particle.

Thus, if a resonance peak exists near $E_{\max}$, and if the resonance width, Γ, is very small, then one may with adequate accuracy evaluate $n(E)$ and v at the resonance energy E_r and take them outside the integral sign in (17.214):

$$\langle \sigma v \rangle \simeq n(E_r) v_r \int \sigma(E) \mathrm{d}E, \tag{17.215}$$

where the integration is now carried out over the resonant peak.

For values of E close to E_r we can use the one-level Breit-Wigner dispersion formula for $\sigma(E)$ (setting $\ell = 0$ in (17.129)):

$$\sigma(E) = \pi \lambda\!\!\!^{-2} \cdot \omega \cdot \mathscr{S} \cdot \frac{\Gamma_a \Gamma_b}{(E-E_r)^2 + \left(\frac{1}{2}\Gamma\right)^2}, \tag{17.216}$$

where ω is the statistical factor (*cf.* (17.119)), $\mathscr{S}$ is the "symmetry factor" (*cf.* (17.116)), Γ_a and Γ_b are the respective partial widths of the entrance channel a and the exit channel b, and

$$\Gamma = \sum_i \Gamma_i + \Gamma_\gamma \tag{17.217}$$

is the total width over all channels into which the compound nucleus C* may decay with the particular energy of excitation which C* has. It will be recalled that for charged-particle reactions, Γ_a contains the Gamow penetration factor, and is thus a function of E. If the resonance is narrow compared with the Gaussian peak, the Gamow factor will vary only slightly over the width of the resonance; thus we can set $\Gamma_a(E) = \Gamma_a(E_r)$ in (17.216) when we integrate over the resonance peak.

Integrating over the resonance peak gives the result

$$\int_{-\infty}^{\infty} \sigma(E)\mathrm{d}E = \pi\lambda\!\!\!^{-2}_r \cdot \omega \cdot \mathscr{S} \cdot \frac{\pi\Gamma_a(E_r)\Gamma_b}{\left(\frac{1}{2}\Gamma\right)}, \tag{17.218}$$

where for a narrow resonance peak it is realized that the limits of integration can be extended to $\pm\infty$ with negligible error. Thus, (17.215) becomes

$$\begin{aligned}\langle\sigma v\rangle &= n(E_r)v_r \cdot \pi\lambda\!\!\!^{-2}_r \cdot \omega \cdot \mathscr{S} \cdot \frac{\pi\Gamma_a(E_r)\Gamma_b}{\left(\frac{1}{2}\Gamma\right)} \\ &= n(E_r) \cdot 2^{1/2} \cdot \frac{\pi^2\hbar^2}{m^{3/2}E_r^{1/2}} \cdot \omega \cdot \mathscr{S} \cdot \frac{\Gamma_a(E_r)\Gamma_b}{\Gamma},\end{aligned} \tag{17.219}$$

where we have used the relations

$$\lambda\!\!\!^{-2}_r = \frac{\hbar^2}{2mE_r}, \tag{17.220}$$

$$v_r = \left(\frac{2E_r}{m}\right)^{1/2}, \tag{17.221}$$

m being the reduced mass of particles a and X.

For a Maxwellian distribution of particle energies we have (*cf.* (17.170c))

$$n(E) = \frac{2}{\pi^{1/2}} \frac{1}{(kT)^{3/2}} e^{-E/kT} E^{1/2}, \tag{17.222}$$

so that (17.219) becomes

$$\langle \sigma v \rangle = \left(\frac{2\pi}{m}\right)^{3/2} \hbar^2 \cdot \omega \cdot \mathcal{S} \cdot \frac{\Gamma_a(E_r)\Gamma_b}{\Gamma} \cdot \frac{1}{(kT)^{3/2}} e^{-E_r/kT}. \tag{17.223}$$

For low energy reactions it is frequently true that

$$\Gamma \simeq \Gamma_a + \Gamma_b ; \tag{17.224}$$

furthermore, usually, one of the Γ's is much larger than the other. Hence the factor containing the Γ's in (17.223) frequently reduces simply to the smaller of the two Γ's. It is clear that the enhancement of the reaction rate produced by a resonance occurring within the "stellar energy region" will be especially pronounced if E_r happens to be very close to $E_{\max}$.

If $\langle \sigma v \rangle$ is given by (17.223), then the temperature exponent is given by (*cf.* (17.211))

$$\nu = \frac{\mathrm{d} \ln \langle \sigma v \rangle}{\mathrm{d} \ln T}\Big|_{T=T_0} = \frac{E_r}{kT_0} - \frac{3}{2}. \tag{17.225}$$

17.15 Electron Screening

In the foregoing sections we have calculated the electrostatic energy between the two nuclei a and X as if they were perfectly bare nuclei. This treatment is valid if the density is sufficiently low. At higher densities, however, each nucleus tends to attract neighboring electrons and so to form a negative "charge cloud" around it; this charge cloud then partially shields the nuclear charge, thus facilitating the penetration of the Coulomb barrier and hence increasing the rate of thermonuclear reactions. Here we shall only give an approximate treatment of this effect, referred to as "electron screening," following in part the work of Salpeter [Sa54], whose results are often quoted in the literature.

We shall first calculate the electrostatic potential of a positive point charge $+Ze$ and its negative charge cloud. This potential is given by the solution of Poisson's equation

$$\nabla^2 \phi = -4\pi \rho_E , \tag{17.226}$$

where ρ_E is the charge density at the point under consideration. To calculate ρ_E, we make use of the statistical mechanical result that the number density of particles in a velocity independent potential ϕ is different from the number

density in the absence of such a potential. Specifically, we have for the number density of particles of type i, each having charge $+Z_i e$ (*cf.* Sect. 10.5),

$$N_i = N_{0i} e^{-Z_i e\phi/kT}, \tag{17.227}$$

where ϕ is the electrostatic potential and where N_{0i} is the particle density when $\phi = 0$. Similarly, for the electron density N_e we have (assuming the electron gas to be non-degenerate)

$$N_e = N_{0e} e^{+e\phi/kT}, \tag{17.228}$$

taking $Z = -1$, where N_{0e} has an interpretation analogous to that of N_{0i}. Note that $N_e > N_{0e}$ and $N_i < N_{0i}$ if $\phi > 0$. We assume that the particle densities are almost uniform, *i.e.*, that the magnitudes of the exponents in (17.227) and (17.228) are small compared with unity. Expanding the exponents and retaining only linear terms, we have

$$N_i = N_{0i}\left(1 - \frac{Z_i e\phi}{kT}\right), \tag{17.229}$$

$$N_e = N_{0e}\left(1 + \frac{e\phi}{kT}\right). \tag{17.230}$$

The total electric charge density is then given by

$$\rho_E = \sum_i N_i Z_i e - N_e e, \tag{17.231}$$

where the summation is carried out over all types of particle (other than electrons) present. Using (17.229) and (17.230), we have

$$\begin{aligned}\rho_E &= \sum_i N_{0i} Z_i e\left(1 - \frac{Z_i e\phi}{kT}\right) - N_{0e} e\left(1 + \frac{e\phi}{kT}\right) \\ &= \sum_i N_{0i} Z_i e - N_{0e} e - \sum_i N_{0i} \frac{Z_i^2 e^2 \phi}{kT} - N_{0e} \frac{e^2\phi}{kT}.\end{aligned} \tag{17.232}$$

The first two terms on the right side of (17.232), however, must cancel each other because we require the plasma to be electrically neutral when $\phi = 0$:

$$N_{0e} = \sum_i N_{0i} Z_i. \tag{17.233}$$

We then write (17.232) in the form

$$\rho_E = -\frac{\phi e^2}{kT}\chi N, \tag{17.234}$$

where

$$N \equiv \sum_i N_{0i} + N_{0e} \tag{17.235}$$

is the total particle density when $\phi = 0$, and

$$\chi \equiv \sum_i \frac{N_{0i}}{N} Z_i^2 + \frac{N_{0e}}{N}. \tag{17.236}$$

Using the condition of electrical neutrality, (17.233), we may also write

$$\chi = \sum_i \frac{N_{0i}}{N} Z_i(Z_i+1). \tag{17.237}$$

Using the relations $N_{0i} = N_0\rho(x_i/\mathscr{A}_i)$ and $N = N_0\rho/\mu$, where N_0 is Avogadro's number, x_i is the relative mass fraction of particles of type i, $\mathscr{A}_i$ is the atomic mass of a nucleus of type i, and μ is the mean molecular weight per free particle (including electrons, *cf.* Sect. 15.1), we may also write (17.237) in the form

$$\begin{aligned}\chi &= \mu \sum_i \frac{x_i}{\mathscr{A}_i} Z_i(Z_i+1) \\ &\equiv \mu\zeta,\end{aligned} \tag{17.238}$$

where ζ is defined by (17.238) and is seen to be an average (with respect to mass abundance) of the quantity $Z_i(Z_i+1)/\mathscr{A}_i$.

Introducing (17.234) for ρ_E into Poisson's equation, (17.226), and assuming spherical symmetry, we obtain

$$\frac{1}{r}\frac{\mathrm{d}^2(r\phi)}{\mathrm{d}r^2} = \frac{4\pi e^2}{kT} N\chi\phi = \frac{\phi}{r_D^2}, \tag{17.239}$$

where r_D is the "Debye length":*

$$r_D \equiv \sqrt{\frac{kT}{4\pi e^2 N\chi}}. \tag{17.240}$$

* Our definition of r_D is not universal; indeed, there does not appear to be general agreement about the precise definition of r_D. Our definition agrees with Debye's original definition (*cf.* Spitzer [Sp62, p. 22]) and with what Salpeter [Sa54] has called the "radius of the charge cloud," but not with the definitions given by Spitzer [Sp62, Eq. (2–3)], Delcroix [De60, p. 92], and others.

(An expression for r_D in which the electrons may be partially degenerate is (15.69).) The interpretation of r_D will follow presently. It is clear that (17.239) may be written as

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}(r\phi)}\left[\frac{\mathrm{d}(r\phi)}{\mathrm{d}r}\right]^2 = \frac{r\phi}{r_D{}^2},$$

which, when integrated, yields the result

$$\frac{\mathrm{d}(r\phi)}{\mathrm{d}r} = \pm\frac{r\phi}{r_D} + \text{const.} \tag{17.241}$$

We choose the minus sign and set the constant of integration equal to zero so that ϕ vanishes as $r\to\infty$ in the absence of any net charge density ($\rho_E = 0$). Integrating (17.241), we obtain

$$\phi = \text{const.}\frac{e^{-r/r_D}}{r}.$$

The constant of integration, however, must be chosen to be Ze, so that ϕ will become the potential of an isolated point charge $+Ze$ as $r\to 0$. The potential of our shielded point charge is then given by

$$\phi = \frac{Ze}{r}\cdot e^{-r/r_D}. \tag{17.242}$$

The significance of the Debye length, r_D (see (17.240)), is now clear: it is the distance beyond which the positive charge $+Ze$ is essentially "shielded" by the surrounding negative charge cloud; *i.e.*, r_D is approximately the "radius" of the charge cloud. Inserting numbers into (17.240), we have

$$r_D = 0.89\times 10^{-8}(T_6/\zeta\rho)^{1/2}\,\mathrm{cm}, \tag{17.243}$$

where ρ is mass density in gm/cm^3, $T_6 \equiv T(^\circ\mathrm{K})/10^6$, and ζ is defined by (17.238).

In order to see the effect of this shielding on the thermonuclear reaction rate, we recall from (17.144) that the "classical distance of closest approach" between two nuclei of charges Z_1e and Z_2e is given (neglecting for the moment the effects of shielding) by

$$r_0 = \frac{Z_1Z_2e^2}{E}, \tag{17.244}$$

where E is the relative kinetic energy between the two particles when far apart. We also recall that the greatest contribution to the thermonuclear reaction rate is made by particles whose relative kinetic energy is $E_{\max}$; thus the "effective" classical distance of closest approach is given by replacing E in (17.244) by $E_{\max}$:

$$r_0 = Z_1 Z_2 e^2 / E_{\max}. \tag{17.245}$$

Forming the ratio of r_D to r_0, we have

$$\frac{r_D}{r_0} \simeq 62 \frac{E_{\max}^{(\mathrm{Kev})}}{Z_1 Z_2} \left(\frac{T_6}{\zeta \rho}\right)^{1/2}, \tag{17.246}$$

where $E_{\max}$ is in Kev and ρ is in gm/cm^3. For $E_{\max} = 30$ Kev, $Z_1 Z_2 = 6$, $T_6 = 10$, $\zeta \simeq 1$, and $\rho = 100$ gm/cm^3, we have $r_D/r_0 \simeq 100$. Hence we expect to have $r_D/r_0 \gg 1$ under all ordinary stellar conditions (*i.e.*, except for exceptionally high densities) and we henceforth assume this to be the case. We also recall that the calculation of the barrier penetration factor $P(E)$ involved only values of r between R (the nuclear interaction radius) and r_0. Hence we have $r/r_D \ll 1$ for $R \leq r \leq r_0$.

The most important factor entering into the calculation of $P(E)_{\ell=0}$ in Sect. 17.11 was the quantity $V(r) - E (R \leq r \leq r_0)$, where $V(r)$ is the Coulomb potential energy between the two nuclei. Taking shielding into account (*cf.* (17.242)), we have

$$V(r) = \frac{Z_1 Z_2 e^2}{r} \cdot e^{-r/r_D}. \tag{17.247}$$

Since we have just shown that $r/r_D \ll 1$ for the values of r of interest here, we can expand the exponential in (17.247) to first order to obtain

$$V(r) \simeq \frac{Z_1 Z_2 e^2}{r} - \frac{Z_1 Z_2 e^2}{r_D}, \tag{17.248}$$

where the last term is small compared to the next-to-last and does not depend either on r or E. We then have

$$V(r) - E = \frac{Z_1 Z_2 e^2}{r} - \left(E + \frac{Z_1 Z_2 e^2}{r_D}\right). \tag{17.249}$$

It then follows from (17.249) that the effect of shielding on the penetration factor $P(E)$ is essentially to make the particles act as if their relative kinetic energy when far apart had been increased from E to $E + U_0$, where

$$U_0 \equiv \frac{Z_1 Z_2 e^2}{r_D} \tag{17.250}$$

is the correction of the Coulomb interaction energy for the effects of shielding. Using (17.250) and the approximate relation (17.245) for r_0, we see that

$$\frac{U_0}{E_{\max}} \simeq \frac{r_0}{r_D} \ll 1, \tag{17.251}$$

so that U_0 is small compared to the values of E ($\simeq E_{\max}$) of interest.

We conclude that the effects of screening on the reaction cross section can be approximately taken into account by using a slightly enhanced energy $E+U_0$ in the cross section computed without screening corrections, *i.e.*,

$$\sigma_s(E) \simeq \sigma_{n-s}(E+U_0), \tag{17.252}$$

where the subscripts s and $n-s$ stand for, respectively, "corrected for screening" and "not corrected for screening."

The thermonuclear reaction rate, now, was proportional to $\langle \sigma v \rangle$ (see Sect. 17.12). Hence the reaction rate, corrected for screening, is proportional to

$$\langle \sigma v \rangle_s = \int_0^\infty n(E)\sigma_s(E) \cdot v(E) \mathrm{d}E, \tag{17.253}$$

where

$$n(E) \propto E^{1/2} e^{-E/kT} \tag{17.254}$$

is the Maxwell-Boltzmann factor and $\sigma_s(E)$ is the cross section properly corrected for screening. Making use of (17.252), we have

$$\langle \sigma v \rangle_s = \int_0^\infty n(E)\sigma_{n-s}(E+U_0)v(E)\mathrm{d}E. \tag{17.255}$$

Changing variables by replacing E by $E'-U_0$, we have

$$\langle \sigma v \rangle_s = \int_{U_0}^\infty n(E'-U_0)\sigma_{n-s}(E')v(E'-U_0)\mathrm{d}E'. \tag{17.256}$$

Since $v(E) \propto E^{1/2}$, we have

$$n(E'-U_0)v(E'-U_0) \propto (E'-U_0)e^{-E'/kT} \cdot e^{U_0/kT} \simeq E' e^{-E'/kT} \cdot e^{U_0/kT}, \tag{17.257}$$

since (*cf.* (17.251)) $U_0 \ll E_{\max}$, which is the region of E' which contributes most to the integral (17.256). We may also to good approximation replace the lower limit of the integral in (17.256) by zero since most of the contribution to the integral comes at values of E' large compared to U_0. We see,

then, that electron screening can be taken into account approximately by simply multiplying the reaction rate computed without screening effects by the "screening factor"

$$f \equiv e^{U_0/kT}; \tag{17.258}$$

thus

$$\langle \sigma v \rangle_s = \langle \sigma v \rangle_{n-s} \cdot f. \tag{17.259}$$

The value for U_0 which we have calculated,

$$U_0 = Z_1 Z_2 e^2 / r_D, \tag{17.260}$$

is actually valid only in the case which Salpeter [Sa54] calls "weak screening." Specifically, this case requires that

$$\frac{U_0}{kT} = \frac{Z_1 Z_2 e^2}{r_D kT} \ll 1 \tag{17.261}$$

(as may be seen from our initial linearization of (17.227) and (17.228)), which means that the electrostatic interaction energy between the two nuclei when a distance r_D apart is small compared to the mean thermal energy per particle. Other requirements for validity of (17.260) are discussed by Salpeter [Sa54]. Inserting numbers into (17.261), we obtain

$$\frac{U_0}{kT} = 0.188\, Z_1 Z_2 (\zeta \rho)^{1/2} / T_6^{3/2}, \tag{17.262}$$

where ρ is in gm/cm^3, for the case of weak screening. In practice, this case obtains except at very high densities (say $\rho \lesssim 10^4$–10^6 gm/cm^3). (We have not included here effects of partial electron degeneracy; see Salpeter [Sa54] for this correction.)

The other limiting case, $U_0/kT \gg 1$, is called "strong screening" by Salpeter [Sa54], and it applies to the case of very high densities (say $\rho \gtrsim 10^4$–10^8 gm/cm^3). In this case the screening of each nucleus is almost complete; *i.e.*, r_D is considerably less than the average interparticle separation.* The linearization in (17.229) and (17.230) is then clearly not justified and another approach must be employed. The derivation for this case is more involved than for the weak screening case and we shall only quote Salpeter's result, which is

$$\frac{U_0}{kT} \simeq 0.205 \left[(Z_1 + Z_2)^{5/3} - Z_1^{5/3} - Z_2^{5/3} \right] (\xi \rho)^{1/3} / T_6, \tag{17.263}$$

where

$$\xi \equiv \sum_i x_i Z_i / \mathscr{A}_i, \tag{17.264}$$

* See footnote p. 474.

x_i being the relative mass abundance of particles of type i and ρ being in gm/cm^3.

We note from (17.262) and (17.263) that the screening factor f is always greater than unity and increases with increasing density. For normal main sequence stars having masses greater than ~ 0.06 solar masses, these screening corrections are never far from unity.

Nuclear reactions occurring under conditions for which $r_D/r_0 \ll 1$ (which is the case at extremely high densities, say $\rho \sim 10^8$–10^{10} gm/cm^3, and very low temperatures, say $T < \sim 10^6$–10^7 °K) have been called by Cameron [Ca59] "pycnonuclear reactions." The strong screening increases the barrier penetration probabilities by large factors, and the reaction rate is found to depend very sensitively on density but not at all (or only very weakly) on temperature. Pycnonuclear reactions can occur at a significant rate even at zero temperature.

At these very high densities and low temperatures (conditions which obtain in many white dwarf stars, *cf.* Chap. 25) the electrostatic energy of a nucleus in the field of other nuclei is much larger than the thermal kinetic energy $\sim kT$ and, as was pointed out in Sect. 15.5b (see also Salpeter [Sa61]), the nuclei tend to arrange themselves into a regular lattice structure, as in a crystalline solid. Under these conditions the nuclei are considerably less mobile than if they formed a gas, and one would expect the rate of nuclear reactions to be considerably smaller than if the lattice structure were not taken into account. Wolf [Wo65] has developed a theory of nuclear reactions in solid-like stars and has applied the theory to the calculation of the nuclear reaction rate for some particular nuclear reactions. He finds that the rates for

* This may be seen by forming the ratio

$$\frac{r_D}{a} = \frac{Z_1 Z_2 e^2}{akT} \cdot \frac{1}{(U_0/kT)},$$

where a is the average interparticle separation. Defining a by the relation $(4/3)\pi a^3 \sum_i N_i = 1$ ($\sum_i N_i$ = total ion number density), *i.e.*, $a = (3H/4\pi\rho)^{1/3} \langle 1/\mathscr{A} \rangle^{-1/3}$, where the second factor is a mass fraction average (*cf.* (15.40)) of $1/\mathscr{A}$, $\mathscr{A}$ denoting atomic mass, we have

$$\frac{r_D}{a} \approx \frac{0.2 Z_1 Z_2 (\rho^{1/3}/T_6)\langle 1/\mathscr{A} \rangle^{1/3}}{U_0/kT},$$

where $T_6 \equiv T(°\mathrm{K})/10^6$ and ρ is in gm/cm^3. Since the numerator will generally not be very many orders of magnitude larger than unity, even at very large densities, it is seen that $U_0/kT \gg 1$ normally implies that $r_D < a$. (*Cf.* also (15.54).)

the reactions $H^1 + H^1$ and $C^{12} + C^{12}$ are typically 1 to 10 orders of magnitude smaller than those calculated from Cameron's theory of pycnonuclear reactions (which does not take into account the lattice structure).

17.16 Hydrogen Burning Reactions

Considerations based on the observed mass and radius of the sun permit estimates of the conditions of temperature and density in the central regions to be made (see Chap. 1); such estimates give T_6 ($\equiv T(°K)/10^6$) $\sim 15-25$ and $\rho \sim 100$ gm/cm^3. It is under these conditions of temperature and density that the nuclear reactions in the sun, whatever may be their detailed nature, must generate energy at a rate sufficient to balance the energy radiated away from the surface (the time scales involved require that the sun be in or near thermal equilibrium, *cf.* Chap. 5). The internal temperatures and densities for stars all along the main sequence (MS), in fact, do not differ from those in the sun by enormous factors (see the first footnote in this chapter). This fact suggests that all MS stars must derive their energy from a common energy source. (The fact that nearly all stars are on the MS implies that stars spend most of their lives on or near the MS. If stellar lifetimes are much longer than the corresponding Kelvin times (*cf.* Sect. 17.4), then it follows that all MS stars are in or near thermal equilibrium.)

The observed fact that all MS stars are composed predominantly of hydrogen (H^1) suggests that the energy source probably involves H^1 in some way. We have already shown from energy considerations, in fact, that fusion of the large store of H^1 possessed by MS stars into He^4 can supply the energy radiated by stars over the required times. Provisionally assuming, then, that the basic energy source for MS stars is the fusion of H^1 into He^4 (this assumption is universally accepted), let us consider possible mechanisms by which this fusion might be effected.

There are two general possibilities: (1) the direct (but not simultaneous) combination of four protons to form an α particle; (2) indirect fusion effected by four successive combinations of protons with heavier nuclei, with the eventual emission of an α particle.

Possibility (1) was first studied by von Weizsäcker [vo37] and by Bethe and Critchfield [Be38], and it has come to be known as the "proton chain" or the "p-p chain." Although Bethe and Critchfield concluded that this possibility could account, to order of magnitude, for the rate of energy generation by the sun, it was believed, until 1952, to be less important than an indirect fusion process of the type (2) (see next paragraph) mentioned above. We shall return to the proton chain and discuss it in more detail later in this section.

Possibility (2) was first studied by Bethe [Be39]. Bethe noted that under the conditions of temperature and density prevailing in the solar interior the lifetimes of the light elements H^2, H^3, Li^6, Be^9, B^{10}, and B^{11} against combination with hydrogen, through the reactions $H^2\,(p,\gamma)\,He^3$, $H^3\,(p,\gamma)\,He^4$, $Li^6\,(p,\alpha)\,He^3$, $Li^7\,(p,\alpha)\,He^4$, $Be^9\,(p,\alpha)\,Li^6$, $B^{10}\,(p,\gamma)\,C^{11}$, and $B^{11}\,(p,2\alpha)\,He^4$, turn out to range from some fraction of a second to $\sim 10^4$ years in a predominantly hydrogen environment. These light elements would therefore all be consumed in times of this order, and could then no longer contribute to the hydrogen fusion process. It follows, incidentally, that any such elements initially present in a predominantly hydrogen star will be consumed early in the history of the star if its internal temperature reaches a few times 10^7 °K. Similarly, the mean lives of the C-N-O isotopes against combination with hydrogen under these conditions of temperature, density, and hydrogen abundance turn out to be $\sim 10^6$–10^7 years; these times are still considerably less than the estimated age ($\sim 10^{10}$ years) of the sun, but not by such a large factor as for the very light elements.

On the other hand, Bethe noted, proton-induced reactions with elements heavier than N^{15} have mean lives generally greater than, say, 10^{10}–10^{11} years at the temperatures, densities, and hydrogen abundances of interest. Such small reaction rates would not permit indirect fusion of H^1 into He^4 to occur rapidly enough to supply the necessary energy output of the sun.

Bethe concluded that for indirect fusion to occur under the given conditions, a small abundance of the C-N-O isotopes would have to be continually present in the solar mixture. Because of the short lifetimes of these C-N-O isotopes, they would have to be repeatedly consumed and regenerated by the reactions and therefore would have to serve as "catalysts" for the fusion of H^1 into He^4. The specific cyclic chain of reactions among the C-N-O isotopes which Bethe proposed has come to be known as the "carbon cycle" or the "CN" or "CNO" cycle, and will be discussed in more detail presently.

It was believed prior to 1952 that the carbon cycle was mostly responsible for the energy generation in the sun. However, a calculation by Salpeter [Sa52] showed that the reaction rate of the proton chain was about an order of magnitude larger than had been believed heretofore. This result brought about a reversal in the roles thought to be played by the two mechanisms in the energy generation in the sun. Since 1952 calculations have indicated that the proton chain contributes most of the energy production in the sun. We shall discuss these two mechanisms separately.

Hydrogen burning by the carbon cycle of course cannot occur unless some elements of the C-N-O group are already present and unless the

temperature is high enough so that all the reactions in the chain can be completed in times smaller than the age of the system. This cycle proceeds as follows:

	Q(Mev)		
(1) $C^{12}(p,\gamma)N^{13}$	1.94		
(2) $N^{13}\rightarrow C^{13}+\beta^{+}+\nu$	1.51		
(3) $C^{13}(p,\gamma)N^{14}$	7.54	$\Sigma Q = 25.02$ Mev*	(C1)
(4) $N^{14}(p,\gamma)O^{15}$	7.29		
(5) $O^{15}\rightarrow N^{15}+\beta^{+}+\nu$	1.76		
(6) $N^{15}(p,\alpha)C^{12}$	4.96		

where β^{+} stands for a positron and ν for a neutrino.

Except in the case of the two beta processes, the Q values here and in the following are simply the mass differences (in energy units) of the constituents before and after the reactions. In the case of positron emission we have for the "effective" Q (*i.e.*, the energy actually available to the star)

$$\begin{aligned} Q &= [M_n(Z+1) - M_n(Z) - m_e]c^2 + 2m_e c^2 - E_\nu \\ &= [M_n(Z+1) - M_n(Z) + m_e]c^2 - E_\nu , \\ &= [M_{at}(Z+1) - M_{at}(Z)]c^2 - E_\nu , \end{aligned} \tag{17.265}$$

where E_ν is the average energy carried off by the neutrino and the subscripts n and at denote, respectively, *nuclear* and *atomic* masses. The term $(-m_e c^2)$ represents half of the energy required to create an electron-positron pair; the negative electron is "retained" by the nucleus, thus changing the nuclear charge from $(Z+1)e$ to Ze, while the positron is emitted by the nucleus. The term $(+2m_e c^2)$ is the energy given up when the emitted positron combines with a negative electron from the surrounding gas and yields (in nearly all cases†) two gamma rays.

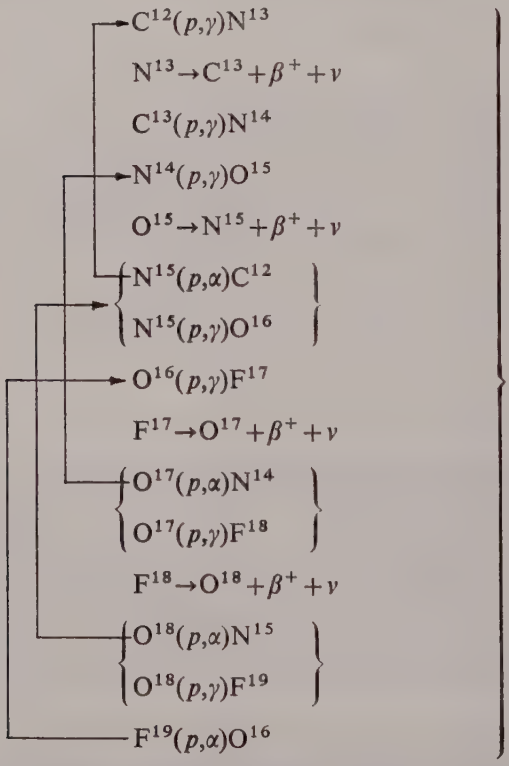
In the case of *electron* emission the Q value is just the mass difference between the constituents (in *atomic*, rather than *nuclear*, masses) before and after the reaction *minus* $E_{\bar{\nu}}$, the average energy carried off by the anti-neutrino.

* This number (as will be true of most of the total Q values we give here) is not exactly equal to the sum of the Q values as listed, but is the number given by Reeves in his review article [Re65]; the difference is presumably due to accumulation of rounding errors.

† We are indebted to Dr. Jane B. Blizard (*private communication*, 1965) for calling our attention to this qualification.

The "cycling time" for the carbon cycle is less than the age of the sun (say 5×10^9 years) if $T_6 \gtrsim 16$. The sum of the Q values (25.02 Mev) in (C1) is slightly smaller than the value (26.73 Mev) that would be obtained from the mass difference between four H^1 atoms and one He^4 atom. The difference between the two Q values is just the energy carried away by the neutrinos in the two β processes.

There is actually a small branching of reaction (6) into O^{16} (one $N^{15}+H^1$ reaction in about 2000 terminates in the formation of O^{16}). Thus, over many cycling times for the carbon cycle, further reactions would take place, as follows:



(C2)

Unless we explicitly state otherwise, however, we shall always mean the cycle (C1) when we refer to the carbon cycle.

The other possible hydrogen burning chain reaction is the proton chain, which begins with the direct fusion of two protons into a deuterium nucleus:

$$H^1+H^1\rightarrow H^2+\beta^+ +\nu. \tag{17.266}$$

Thus this process could operate even in a star composed initially of pure hydrogen. It is interesting that this reaction has never been, and probably never will be, directly observed in the laboratory, because of its extremely

small cross section. The cross section is small because a β-decay must occur during the brief time of encounter between the two protons, an event which is extremely unlikely (on the average, one β-decay process occurs in about 10^{27} proton-proton collisions). Because of the Pauli exclusion principle, the two protons must collide with anti-parallel spins in order to come close enough together to form a compound nucleus.* The compound nucleus He^2 then has a momentary existence (Bohr theory, *cf.* Sect. 17.9) and then (when reaction (17.266) occurs) emits a positron and a neutrino, becoming deuterium, H^2. However, the ground state of deuterium is one in which the proton and neutron have parallel spins (^3S); hence one of the incoming protons must have "flipped" over during the process, causing the total spin angular momentum of the H^2 nucleus to be approximately $\hbar$. Thus the positron and the neutrino must each carry off an amount of angular momentum equal to about $\hbar/2$, with spins both anti-parallel to that of the H^2 nucleus. This transition implies the so-called "Gamow-Teller" selection rules for β-decay. The cross section factor S here is computed from β-decay theory and the calculations are thought to be fairly reliable (error probably considerably less than 50 per cent). The value given by Reeves [Re65] is $S = (3.36 \pm 0.4) \times 10^{-22}$ Kev barns.

The deuterium nucleus then rapidly captures a proton, becoming He^3. The succeeding steps then depend on the temperature and the abundance of He^4 present (the succeeding steps, however, will not proceed to any appreciable degree if $T_6 \lesssim 8$). If the abundance of He^4 is very small, the proton chain proceeds as follows:

$$\begin{array}{lc} & Q(\text{Mev}) \\ \left[\begin{array}{l} H^1 + H^1 \rightarrow H^2 + \beta^+ + \nu \\ H^2(p,\gamma)He^3 \end{array}\right. & \begin{array}{c} 1.18 \\ 5.49 \end{array} \Big] \times 2 \quad \Sigma Q = 26.20\ \text{Mev} \quad \text{(P1)} \\ He^3 + He^3 \rightarrow He^4 + 2H^1 & 12.86, \end{array}$$

* This can be seen as follows: Let the wave function for the two protons be denoted by $\psi = \psi(1,2)$, where "1" ($\equiv \mathbf{r}_1, \mathbf{s}_1$) denotes the space coordinates and spin of particle 1, "2" those for particle 2. Since protons are fermions, the wave function must be anti-symmetric to an interchange of the space and spin coordinates of the two particles (see any book on quantum mechanics); *i.e.*,

$$\psi(2,1) = -\psi(1,2). \qquad (17.267)$$

Now suppose that both space and spin coordinates of the two particles are identical, *i.e.*, that "1" $\equiv$ "2", so that $\psi(1,2) \equiv \psi(1,1)$. It then follows from (17.267) that $\psi(1,1) = -\psi(1,1)$, which can only be true if $\psi(1,1) = 0$. Thus two fermions with identical spins cannot both simultaneously occupy the same spatial position and so cannot come close together. If the spins are anti-parallel, however, the spatial coordinates of the two particles can be identical and the wave function will not vanish (in general), so that the particles can come close together in this case.

where ΣQ is reduced by about 2 per cent (due to neutrino losses) from the usual ΣQ computed only from mass differences. We note that since *two* He^3 nuclei are consumed in the third reaction, the first two reactions occur twice for each of the third.

Prior to 1958 it was believed that the chain (P1) would proceed under most conditions, even if an appreciable abundance of He^4 were present. However, measurements of Holmgren and Johnston [Ho58] of the cross section for the reaction $He^3(\alpha,\gamma)Be^7$ showed that this cross section was some 2500 times the previously accepted value, so that this reaction will compete with the reaction $He^3(He^3, 2p)He^4$ in the completion of the proton chain under some conditions. Hence, the proton chain must now be written not only as (P1), but in two additional alternative forms (*cf.* W. A. Fowler [Fo58] and Reeves [Re65]):

	Q(Mev)		
$H^1(p,\beta^+\nu)H^2$	1.18*		
$H^2(p,\gamma)He^3$	5.49		
$He^3(\alpha,\gamma)Be^7$	1.59	$\Sigma Q = 25.67$† Mev (4% neutrino loss)	
$Be^7(\beta^-,\nu)Li^7$	0.06		(*P*2)
$Li^7(p,\alpha)He^4$	17.35		

or

	Q(Mev)		
$H^1(p,\beta^+\nu)H^2$	1.18*		
$H^2(p,\gamma)He^3$	5.49		
$He^3(\alpha,\gamma)Be^7$	1.59	$\Sigma Q = 19.2$† Mev (29% neutrino loss)	
$Be^7(p,\gamma)B^8$	0.13		(*P*3)
$B^8 \rightarrow Be^8 + \beta^+ + \nu$	10.78*		
$Be^8 \rightarrow 2He^4$	0.09		

If a star has a small He^4 abundance, then (*P*2) and (*P*3) may contribute negligibly to the energy production. However, if a star contains appreciable He^4, then these chains may make significant contributions to the energy production, depending on the temperature. We see from (17.208), (17.200), and (17.176) that the temperature dependence of the rate of energy production for a given chain enters mainly through the term $e^{-\tau}$, where $\tau \propto (\mathscr{A}/T)^{1/3}$, $\mathscr{A}$ being the reduced atomic mass of the reacting particles. Thus, since the reduced mass of (He^3,He^4) is greater than that of (He^3,He^3), the reaction rate for $He^3 + He^4$ is a more rapidly increasing function of

* and †: See comments and footnotes pertaining to (C1).

temperature than is that for $He^3 + He^3$. We would then expect (as quantitative considerations show (see next section)) that for stars with appreciable He^4 abundance ($P1$) would dominate over ($P2$) and ($P3$) in cool stars ($T_c \lesssim 10^7$ °K), whereas the latter cycles should predominate over ($P1$) in hotter stars ($T_c > 10^7$ °K).

If reactions ($P2$) or ($P3$) occur, then the first two steps in the chain do not have to occur twice before the next one can occur. Hence the rate of He^4 production per $H^1 + H^1$ reaction can be up to a factor of 2 times that for reaction ($P1$), depending on the temperature and the He^4 abundance.

The choice between reactions ($P2$) and ($P3$) also depends on the temperature and is determined by the competition between the reactions $Be^7(\beta^-,\nu)Li^7$ and $Be^7(p,\gamma)B^8$. According to Reeves [Re65], reaction ($P3$) is likely to predominate over ($P2$) only for $T_6 \gtrsim 20$. If only reaction ($P3$) occurred, then the available energy released per He nucleus formed would be reduced in the ratio 19.2/26.2 from what it would be if only reaction ($P1$) occurred, because of the heavy neutrino losses in reaction ($P3$).

All these factors taken together imply that the rate of energy production per unit mass for the proton chain will be increased over that for reaction ($P1$) alone by some factor less than 2 (see Sect. 17.17).

Other possible modes of completion of the proton chain are discussed by Parker, Bahcall, and Fowler [Pa64a] and Bahcall and Wolf [Ba64a]. In the second of these papers, on termination of the proton chains at high densities, the possibility of completion of the proton chain by the reactions

$$He^3(\beta^-,\nu)H^3 \tag{17.267a}$$

$$H^3(p,\gamma)He^4 \tag{17.267b}$$

or

$$H^3(He^3, np)He^4 \tag{17.267c}$$

or

$$H^3(H^3, 2n)He^4 \tag{17.267d}$$

is discussed. These reactions can occur with appreciable probability at high densities at temperatures considerably less than 6×10^6 °K since there is no Coulomb barrier for reaction (17.267a) and the Coulomb barriers in the other reactions are not large. However, because reaction (17.267a) is endothermic by 18 Kev, densities of the order of 2×10^4 gm/cm^3 (corresponding to a Fermi energy (see Chap. 24) of about 18 Kev) or greater are required for this reaction to proceed at an appreciable rate. These reactions are of interest for very low-mass stars (masses considerably less than one solar mass).

The net result of both the proton chain and the carbon cycle is the following:

$$4H^1 \rightarrow He^4 + 2\nu + \Sigma Q . \tag{17.268}$$

That (17.268) must be the net result of fusion of H^1 into He^4 regardless of the detailed mechanism follows from the fact that two protons must have been transformed into two neutrons by undergoing β^+ emission; according to the rule of lepton number conservation (see one of the footnotes in Sect. 17.9), two neutrinos must then have been emitted. The neutrinos here carry off a certain fraction of the total energy released. Because of the weak interaction of neutrinos with matter, however, this energy carried by the neutrinos is essentially completely lost to the star. Although the total number of neutrinos produced in a given cycle is always the same (two), the amount of energy carried off by them depends on the particular cycle in which they were formed. Thus we see why the total effective Q values are different for each branch of the proton chain and the carbon cycle.

It is interesting to consider the flux of neutrinos emitted by the sun and their fate. Knowing the total energy leaving the sun in the form of neutrinos and the average energy carried by each neutrino, one may compute the flux of neutrinos leaving the sun. The result is that at the earth $\sim 10^{11}$ neutrinos cross each cm^2/sec from the sun.

The cross section for absorption of neutrinos by the reaction

$$n + \nu \rightarrow p + \beta^- \tag{17.269}$$

is $\sigma \sim 10^{-44}$ cm^2 at neutrino energies of the order of a few Mev, which implies a mean free path in lead of several light-years.* In fact, the mean free path of a neutrino is given by $\ell \sim 1/(n\sigma)$, where n is the number of absorbers/cm^3. But $n = (N_0/\mu)\rho$, so that $\ell = \mu/N_0\rho\sigma$, where μ is the mean molecular weight of the material, or

$$\ell \sim \frac{2 \times 10^{20}\mu}{\rho} \text{cm}$$

$$\sim \frac{200\mu}{\rho} \text{light-years} . \tag{17.270}$$

For matter in the universe at large we may take $\mu \simeq 1$, $\rho \lesssim 10^{-28}$ gm/cm^3, which gives $\ell \gtrsim 2 \times 10^{30}$ light-years. Since $v = c = 1$ light-year/year, then the

* Neutrino "opacities" have been computed by Bahcall and Frautschi [Ba64a']. For very high energy neutrinos the interaction of neutrinos with matter may not be negligible; see Colgate and White [Co66] for an interesting application of such interactions of neutrinos with matter in a theory of supernovae.

mean life of a typical neutrino "free" in the universe should be $t \gtrsim 2 \times 10^{30}$ years.

Some of the attempts to detect neutrinos from the sun (for example, Davis and Harmer [Da59]) are based on the reactions

$$\nu + \mathrm{Cl}^{37} \rightarrow \mathrm{A}^{37} + \beta^-$$

$$\mathrm{A}^{37} \rightarrow \mathrm{Cl}^{37} + \nu + \beta^+ . \tag{17.271}$$

The A^{37} is detected by means of its radioactivity. Further discussion regarding detection of solar neutrinos may be found, for example, in Bahcall [Ba65] and Reeves [Re65].

17.17 Rates of Energy Production by Hydrogen Burning

Much of the numerical data appearing in this section, as well as that in the remaining sections of this chapter, have been taken from Reeves' [Re65] review article.

17.17a Carbon Cycle

The carbon cycle consists not of a single reaction, but of a set of six reactions (considering only the cycle (C1), see Sect. 17.16). To obtain the rate of energy release from the carbon cycle, then, we must sum the rates of energy release over all reactions (*cf.* (17.176b)):

$$\varepsilon = \frac{1}{\rho} \sum_{i=1}^{6} \frac{N_i}{t_i} Q_i , \tag{17.272}$$

where ρ is mass density and the summation is extended over the six reactions of the cycle (C1). Here N_i denotes the number density of the reacting particles of type i (other than hydrogen) involved in the i^{th} reaction, t_i denotes the mean lifetime of that type of particle against reaction with hydrogen, and Q_i denotes the energy given off in the i^{th} reaction. If the reaction under consideration is a β-decay, then N_i and t_i refer, of course, to the decaying isotope. After a sufficiently long time (which decreases with increasing temperature and density but is typically several millions of years for the carbon cycle if $\rho \sim 10^2$ gm/cm^3 and $T \sim (25\text{–}30) \times 10^6$ °K), the elements entering into the cycle will have reached a condition of "secular equilibrium." In this condition the abundance of each isotope (excluding hydrogen and helium) will be statistically constant in time and proportional to its mean life:

$$N_i = \text{const.} \cdot t_i = C t_i , \text{say} . \tag{17.273}$$

Thus, also,

$$\sum_i N_i = C \sum_i t_i \tag{17.274}$$

(excluding hydrogen and helium from the summation), whence

$$\sum_i \frac{N_i Q_i}{t_i} = C \sum_i Q_i = \frac{\sum_i N_i}{t} \sum_i Q_i \tag{17.275}$$

where

$$t \equiv \sum_i t_i \simeq t_{\text{slowest}} \tag{17.276}$$

is the "cycling time" for the carbon cycle. t_{slowest} is the longest lifetime of any of the isotopes (excluding H^1 and He^4) in the cycle. The approximate equality in (17.276) is sufficiently accurate if t_{slowest} is several times larger than the mean reaction time for the next slowest reaction in the cycle. We then have

$$\varepsilon = \frac{1}{\rho}\left(\sum_i N_i\right)\cdot\left(\sum_i Q_i\right)/t \simeq \frac{1}{\rho}\left(\sum_i N_i\right)\left(\sum_i Q_i\right)/t_{\text{slowest}}\,. \tag{17.277}$$

Thus, when we have secular equilibrium, we can group the C-N-O elements together in our expression for the rate of energy release and can use the reaction rate of the slowest reaction in the chain if this rate is appreciably smaller than that of any of the other reactions in the chain. The case where secular equilibrium among the nuclei is not attained is considered at the end of this subsection.

We write $Q \equiv \sum_i Q_i$ and let $N_i = N_0 \rho x_i / \mathscr{A}_i$, where N_0 is Avogadro's number and x_i and $\mathscr{A}_i$ are the relative mass abundance and atomic mass of element i. Also, we set $t_{\text{slowest}} = t_{14}$, the mean life of N^{14}, since t_{14} is roughly an order of magnitude greater than t_{12}, the next largest mean life in the carbon cycle.* We then write (*cf.* (17.174)):

$$p_{14} = 1/t_{14} = N_0 \rho x_1 \langle 14,1 \rangle\,, \tag{17.278}$$

where x_1 is the relative mass abundance of H^1 and $\langle 14,1 \rangle \equiv \langle \sigma v \rangle_{14,1}$ is the value of $\langle \sigma v \rangle$ for the reaction $N^{14}(p,\gamma)O^{15}$. It will be recalled that the temperature dependence of the reaction is contained in $\langle \sigma v \rangle$ (*cf.* (17.209)):

$$\langle \sigma v \rangle \propto \tau^2 e^{-\tau}$$

* The *relative* lifetimes of the isotopes in the carbon cycle (excluding the two β^+ emitters) are slowly varying functions of temperature T and are practically independent of density ρ.

for a non-resonant reaction, where $\tau \propto T^{-1/3}$. We then obtain

$$\varepsilon_{CN} = N_0^2 \rho x_1 (\sum_i x_i/\mathscr{A}_i) \cdot Q \cdot \langle 14,1 \rangle \tag{17.279}$$

for the rate of energy production per unit mass from the carbon cycle (C1).

Reeves [Re65] writes this expression in the form

$$\varepsilon_{CN} = 7.94 \times 10^{27} f_{14,1} \cdot g_{14,1} \cdot \rho \cdot x_{CN} x_1 \cdot e^{-152.313/T_6^{1/3}} / T_6^{2/3} \text{ erg/gm/sec}, \tag{17.280}$$

where $x_{CN} = x_C + x_N$ is the initial combined relative mass abundance of carbon and nitrogen and is the most significant part of the appropriate factor in (17.279) (an average value of $\mathscr{A}_i$ has been absorbed into the numerical factor in (17.280)). $f_{14,1}$ is the screening factor, $g_{14,1}$ corrects for replacing the Gamow peak by a Gaussian, $T_6 \equiv T(°K)/10^6$, and ρ is in gm/cm^3. Reeves also writes this in the form

$$\varepsilon_{CN} = \rho x_1 x_{CN} \cdot f_{14,1} \cdot g_{14,1} \cdot \varepsilon_0 (T/T_0)^\nu \text{ergs/gm/sec}, \tag{17.281}$$

where T_0 is an arbitrary reference temperature, ν is the temperature exponent given by $\nu = (1/3)(\tau_0 - 2)$ (*cf.* (17.212)) for the reaction $N^{14}(p,\gamma)O^{15}$, and ε_0 is defined by requiring that (17.281) give the correct value for ε_{CN} under the conditions of interest. Because $f_{14,1}$ and $g_{14,1}$ are functions of temperature, the exponent ν is not exactly equal to the temperature exponent for ε_{CN} itself; however, their effect on the temperature exponent is very small and may be neglected. Reeves gives the following formulae for $f_{14,1}$ and $g_{14,1}$:

$$f_{14,1} = 1 + 1.75\, \rho^{1/2}/T_6^{3/2} \text{ (weak screening)}, \tag{17.282}$$

$$g_{14,1} = 1 + 0.0027\, T_6^{1/3} - 0.0037\, T_6^{2/3} - 0.0007\, T_6. \tag{17.283}$$

Table 17.3 [adapted from Re65] gives some values of the quantities of interest.

Table 17.4 [adapted from Re65] gives the cross section factors $S(0)$ (extrapolated to zero energy) for reactions pertinent to the carbon cycle.

Since the "cycling time" of the carbon cycle becomes less than a few billion years for $T_6 \gtrsim 16$, the branching (by about one part in 2000) of the reaction $N^{15} + H^1$ into O^{16} takes place at such temperatures, so that the more involved chain (C2) (*cf.* Sect. 17.16) partakes in the energy production. The rate of energy production, however, is only a few per cent different from that of the chain (C1) and we do not consider it further here.

If the nuclei participating in the chain have not reached a condition of secular equilibrium, then it is necessary to consider each reaction of the chain separately, and the abundances of these nuclei must be computed at each

Table 17.3

PARAMETERS FOR ENERGY GENERATION BY THE CARBON CYCLE †

T_{06}	$g_{14,1}$	ε_0* (c.g.s.)	ν
10	0.99	3.35(−4)	22.9
15	.98	1.90(0)	19.9
20	.98	4.51(+2)	18.0
25	.97	2.16(+4)	16.7
30	.97	4.11(+5)	15.6
35	.97	4.30(+6)	14.8
40	.96	2.98(+7)	14.1
45	.96	1.52(+8)	13.6
50	.96	6.20(+8)	13.1
60	.95	6.24(+9)	12.3
70	.94	3.93(+10)	11.6
80	.94	1.78(+11)	11.1
90	.93	6.39(+11)	10.6
100	.93	1.91(+12)	10.2

* Numbers in parentheses are the powers of ten by which the corresponding entries are to be multiplied.

† Adapted from Reeves [Re65].

Table 17.4

CROSS SECTION FACTORS FOR THE CARBON CYCLE*

Reaction	$S(0)$ (Kev-barns)
$C^{12}(p,\gamma)N^{13}$	1.20 ± 0.15
$C^{13}(p,\gamma)N^{14}$	5.52 ± 0.7
$N^{14}(p,\gamma)O^{15}$	3.12 ± 0.25
$N^{15}(p,\alpha)C^{12}$	5.34×10^4
$N^{15}(p,\gamma)O^{16}$	2.74×10^1
$O^{16}(p,\gamma)F^{17}$	$(1.06 \pm 0.18) \times 10^1$

* From Reeves [Re65].

instant of time from the reaction rates. We let p_{ij} be the probability per unit time that a nucleus of type i will be transformed into one of type j. Then, if N_i/ρ is the number of nuclei of type i per unit mass, we have for the time rate of change of N_i/ρ

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{N_i}{\rho}\right) = \sum_j \left(\frac{N_j}{\rho}\right) p_{ji} - \left(\frac{N_i}{\rho}\right) \sum_k p_{ik} \quad (\text{all } i), \tag{17.283a}$$

where the first summation is taken over all processes that may create nuclei of type i, the second over all processes that may destroy nuclei of type i. The p_{ij} may represent either "collision" or "spontaneous" processes. In the former case we have, in general,

$$p_{ij} = \sum_\alpha N_\alpha \langle \sigma v \rangle_{i\alpha j}, \tag{17.283b}$$

where the summation is taken over all types of particles (of number density N_α) which, by reacting with a nucleus of type i, transform it into one of type j (in the case of photo-disintegration the relevant "particles" would be photons, and the corresponding term in (17.283a) would have to be replaced by an integration over frequency). In the case of "spontaneous" processes (for example, β decay) we have

$$p_{ij} = 1/\tau_{ij}, \tag{17.283c}$$

where τ_{ij} is the mean life for decay of a nucleus of type i into one of type j*. If all the (N_i/ρ) are known at time $t = 0$, then all $\mathrm{d}(N_i/\rho)\mathrm{d}t$ can be computed from (17.283a) at $t = 0$. All the (N_i/ρ) can then be computed as functions of time by integrating the coupled equations (17.283a). These (N_i/ρ) can then be used to compute the rates of energy production from the individual reactions in the chain, and hence the total rate of energy production from the entire chain.

Calculations of this kind for the carbon cycle, as applied to stars settling down onto the main sequence, have been carried out, for example, by Iben [Ib65].

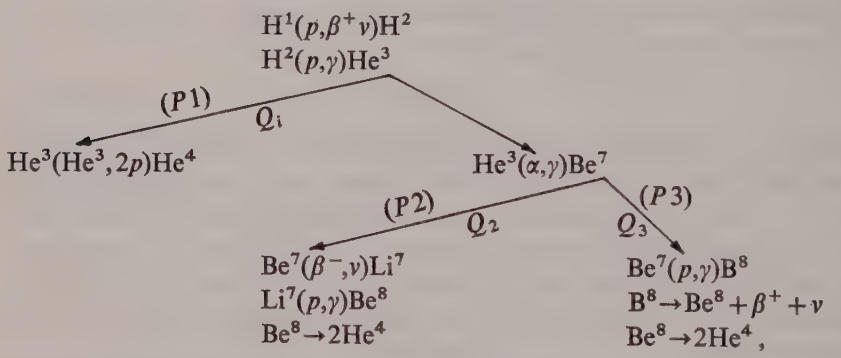
17.17b Proton Chain

The calculation of the rate of energy production from the $p-p$ reaction is considerably more involved than is the case with the carbon cycle, because

* The τ_{ij} are normally considered to be independent of the prevailing conditions of temperature and density. If the electron gas is degenerate or partially degenerate, however (*cf.* Chap. 24), the diminution of the availability of cells in phase space for the emitted electron will diminish the decay probability, *i.e.*, increase τ_{ij}. Such inhibition of beta decay rates in stellar interiors has been calculated by Peterson and Bahcall [Pe63].

of the three possible modes of termination of this reaction under normal conditions (*cf.* Sect. 17.16); in particular, the rate of energy production is not simply proportional to the rate of the $H^1(p,\beta^+\nu)H^2$ reaction but depends, in addition, on the density and temperature and the abundances of the various isotopes that enter into the various reactions. If the temperature is high enough, however, so that all reactions in each mode of termination can occur in times short compared to the age of the system of interest ($T_6 \gtrsim 8$ will assure this situation in the case of the sun and similar stars), then a condition of secular, or near-statistical, equilibrium among the intermediate products of the chains will be established, in which the rates of creation and destruction of each such isotope are approximately equal. In such a case (which we assume in the remainder of this section) the calculation is much simplified, and the rate of energy production can be expressed in terms only of the density, temperature, and relative abundances of H^1 and He^4. If secular equilibrium is not attained, the calculation proceeds in the general way described at the end of Sect. 17.17a.

We summarize here the three relevant proton chains:



where the branching of the various reactions is explicitly shown. It is clear that the rate of energy production per unit mass from the general branch (Pi) is given by

$$\varepsilon_i = Q_i \cdot (\text{number of } He^4 \text{ nuclei created per unit mass and time by branch } i), \tag{17.284}$$

where Q_i is the total energy that is released with the formation of one He^4 nucleus by the branch i; clearly, the creation of one new He^4 nucleus by a particular branch signifies the completion of that branch. The total rate of energy production per unit mass from all branches acting together is then

$$\varepsilon = \sum_i \varepsilon_i . \tag{17.285}$$

We have for (*P*1)

$$\varepsilon_1 = (1/\rho)\cdot Q_1 \cdot (N_3^2/2)\cdot \langle 3,3\rangle , \tag{17.286}$$

where N_3 is the number density of He^3 nuclei and $\langle 3,3\rangle \equiv \langle \sigma v\rangle_{3,3}$. The quantity $(N_3^2/2)\langle 3,3\rangle$ is the number of $He^3(He^3,2p)He^4$ reactions per unit volume and time (equal to the rate of creation of He^4 nuclei per unit volume by this branch); the factor (1/2) accounts for the fact that a reaction between identical particles always involves *two* of these particles (*cf.* Sect. 17.12).

In the chains (*P*2) and (*P*3) we recognize that, because of our secular equilibrium assumption, the net rate of creation of He^4 nuclei in each chain is equal to the rate of *any* of the reactions occurring exclusively in the respective chain. Choosing the rates of $Be^7(\beta^-,\nu)Li^7$ and $Be^7(p,\gamma)B^8$ to represent the net respective rates of He^4 production in the two chains, we see that the rates of energy production per unit mass are given by

$$\varepsilon_2 = (1/\rho)\cdot Q_2 \cdot N_7 p_{7,\beta^-} \tag{17.287}$$

and

$$\varepsilon_3 = (1/\rho)\cdot Q_3 \cdot N_7 N_1 \langle 7,1\rangle , \tag{17.288}$$

where N_7 and N_1 are the number densities of Be^7 and H^1, respectively; p_{7,β^-} is the probability per unit time per Be^7 nucleus of electron capture; and $\langle 7,1\rangle \equiv \langle \sigma v\rangle_{7,1}$ is the reaction rate of the $Be^7(p,\gamma)B^8$ reaction. The total rate of energy production per unit mass from all three branches is then

$$\begin{aligned}\varepsilon_{pp} &= \varepsilon_1+\varepsilon_2+\varepsilon_3 \\ &= (1/\rho)\left[Q_1(N_3^2/2)\langle 3,3\rangle + Q_2 N_7 p_{7,\beta^-} + \right. \\ &\quad \left. + Q_3 N_7 N_1 \langle 7,1\rangle\right].\end{aligned} \tag{17.289}$$

If only (*P*1) occurred, we would have $(N_3^2/2)\langle 3,3\rangle = (1/2)(N_1^2/2)\langle 1,1\rangle$, since each $H^1(p,\beta^+\nu)H^2$ reaction produces only *one* He^3 nucleus, whereas *two* He^3 nuclei are consumed in each $He^3(He^3,2p)He^4$ reaction. In this case the rate of energy production per unit mass would be (*cf.* (17.286)):

$$\varepsilon_1' = (1/\rho)\cdot Q_1 \cdot \frac{1}{2}(N_1^2/2)\langle 1,1\rangle . \tag{17.290}$$

Following Reeves [Re65], we write (17.289) as

$$\varepsilon_{pp} = \varepsilon_1'\left[\frac{2N_3^2\langle 3,3\rangle}{N_1^2\langle 1,1\rangle} + \frac{Q_2}{Q_1}\frac{4N_7 p_{7,\beta^-}}{N_1^2\langle 1,1\rangle} + \frac{Q_3}{Q_1}\frac{4N_7 N_1\langle 7,1\rangle}{N_1^2\langle 1,1\rangle}\right] \tag{17.291}$$

$$\equiv \varepsilon_1' \Psi_{pp}, \tag{17.292}$$

which defines the quantity ψ_{pp}.

Before we can compute the function ψ_{pp}, we must know the abundances of the "intermediate" nuclei He^3 and Be^7. These are obtained from the

assumption of secular equilibrium among the intermediate nuclei. Equating to zero the net rate per unit mass of creation of He^3 nuclei, we have (see (17.283a))

$$(N_1^2/2)\langle 1,1\rangle - 2(N_3^2/2)\langle 3,3\rangle - N_3 N_4 \langle 3,4\rangle = 0, \tag{17.293}$$

where N_4 is the number density of He^4 nuclei and $\langle 3,4\rangle \equiv \langle \sigma v\rangle_{3,4}$ is the reaction rate of the $He^3(\alpha,\gamma)Be^7$ reaction. The factor 2 in (17.293) accounts for the fact that *two* He^3 nuclei are consumed in each $He^3(He^3,2p)He^4$ reaction. Similarly, for Be^7 we have

$$N_3 N_4 \langle 3,4\rangle - N_7 p_{7,\beta-} - N_7 N_1 \langle 7,1\rangle = 0. \tag{17.294}$$

We write (17.293) in the form

$$\left(\frac{N_3}{N_1}\right)^2 \langle 3,3\rangle + \left(\frac{N_3}{N_1}\right)\left(\frac{N_4}{N_1}\right)\langle 3,4\rangle - \frac{1}{2}\langle 1,1\rangle = 0, \tag{17.295}$$

which is a quadratic in (N_3/N_1). Defining [following Re65] the quantity

$$\alpha \equiv \frac{\langle 3,4\rangle^2}{\langle 1,1\rangle\langle 3,3\rangle}\left(\frac{N_4}{N_1}\right)^2, \tag{17.296}$$

which is seen to be a function only of temperature and the He^4/H^1 abundance ratio, we may write the solution of (17.295) in the form

$$\frac{N_3}{N_1} = -\frac{1}{2}\left(\frac{\langle 1,1\rangle}{\langle 3,3\rangle}\right)^{1/2}\alpha^{1/2} + \frac{1}{2}\left(\frac{\langle 1,1\rangle}{\langle 3,3\rangle}\right)^{1/2}\sqrt{\alpha+2}. \tag{17.297}$$

In terms of another function,

$$\gamma \equiv \alpha^{1/2}\left[\sqrt{\alpha+2} - \alpha^{1/2}\right] = \alpha\left[\sqrt{1+2/\alpha} - 1\right], \tag{17.298}$$

we may write

$$\frac{N_3}{N_1} = \frac{1}{2}\left(\frac{\langle 1,1\rangle}{\langle 3,3\rangle}\right)^{1/2}\cdot\frac{\gamma}{\alpha^{1/2}}. \tag{17.299}$$

Note that γ varies from 0 to 1, and $(\gamma/\alpha^{1/2})$ varies from $\sqrt{2}$ to 0, as α varies from 0 to ∞.

Solving (17.294) for the ratio (N_7/N_1), we obtain

$$\frac{N_7}{N_1} = \frac{N_1}{p_{7,\beta-}}\cdot\frac{N_3}{N_1}\frac{N_4}{N_1}\langle 3,4\rangle/(w+1), \tag{17.300}$$

where

$$w \equiv N_1\langle 7,1\rangle/p_{7,\beta-} \tag{17.301}$$

is the relative probability of occurrence of the ($P3$) branch to that of the ($P2$) branch. Using (17.299) for (N_3/N_1), we have

$$\frac{N_7}{N_1} = \frac{N_1}{p_{7,\beta^-}} \cdot \frac{1}{2} \frac{\langle 1,1\rangle \gamma}{w+1}$$

$$= \frac{1}{2} \gamma \frac{\langle 1,1\rangle}{\langle 7,1\rangle} \cdot \frac{w}{w+1}. \tag{17.302}$$

Using (17.299) and (17.302), we have for Ψ_{pp} (*cf.* (17.292)):

$$\Psi_{pp}(\alpha, w) = \frac{1}{2}\frac{\gamma^2}{\alpha} + \frac{2Q_2}{Q_1} \cdot \frac{\gamma}{w+1} + \frac{2Q_3}{Q_1} \cdot \gamma \cdot \frac{w}{w+1}$$

$$= (1-\gamma) + \frac{2Q_2}{Q_1} \cdot \frac{\gamma}{w+1} + \frac{2Q_3}{Q_1} \cdot \gamma \cdot \frac{w}{w+1}, \tag{17.303}$$

which shows explicitly the separate contributions of the three branches. We may also write (17.303) in the form

$$\Psi_{pp}(\alpha, w) = 1 + \gamma \left[\frac{2Q_2/Q_1 - 1 + (2Q_3/Q_1 - 1)w}{w+1} \right]. \tag{17.304}$$

Using the values 26.20, 25.67, and 19.2 Mev for Q_1, Q_2, and Q_3, respectively, we have

$$\varepsilon_{pp} = \varepsilon_1' \Psi_{pp}(\alpha, w), \tag{17.305}$$

where

$$\Psi_{pp}(\alpha, w) = 1 + \gamma \left(\frac{0.959 + 0.47w}{w+1} \right). \tag{17.306}$$

We note that, since $0 \leq \gamma \leq 1$, $1 \leq \Psi_{pp} \leq 1.959$ if ($P2$) predominates over ($P3$) ($w \to 0$); *i.e.*, ($P2$) can increase the rate of energy production over that which would obtain if only ($P1$) occurred by a factor up to almost 2. If ($P3$) predominates over ($P2$) ($w \gg 1$), then $1 \leq \Psi_{pp} \leq 1.47$; *i.e.*, ($P3$) can increase the rate of energy production over that which would obtain if only ($P1$) occurred by a factor up to about 1.5. In general, then, we see that the occurrence of the chains ($P2$) and ($P3$) can increase the rate of energy production over that of ($P1$) alone by a factor between 1 and 2.

We may evaluate α (*cf.* (17.296)) by combining (17.201) and (17.208):

$$\alpha \equiv \frac{\langle 3,4\rangle^2}{\langle 1,1\rangle\langle 3,3\rangle} \left(\frac{N_4}{N_1} \right)^2 = \left(\frac{49}{48} \right)^{1/3} \frac{S_{3,4}^2}{S_{1,1} S_{3,3}} e^{-100/T_6^{1/3}} \left(\frac{N_4}{N_1} \right)^2.$$

If we approximate the numerical factor by unity, we obtain the expression given by Reeves [Re65]:

$$\alpha = \frac{S_{3,4}^2}{S_{1,1}S_{3,3}} e^{-100/T_6^{1/3}} \left(\frac{x_4}{4x_1}\right)^2 = 5.48 \times 10^{17} e^{-100/T_6^{1/3}} \left(\frac{x_4}{4x_1}\right)^2, \quad (17.307)$$

where x_4 and x_1 are the relative mass abundances of He^4 and H^1, respectively, and where the appropriate values of the S's are given in Table 17.6 below. The quantity $\log_{10}[\alpha/(x_4/4x_1)^2]$ is plotted in Fig. 17.12 [Fig. 18 in Re65].

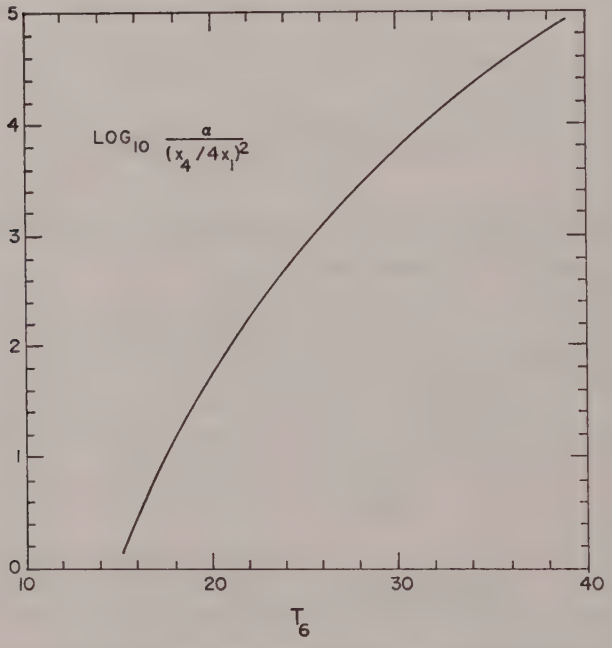


Fig. 17.12 $\text{Log}_{10}[\alpha/(x_4/4x_1)^2]$ as a function of temperature.

The probability/sec/Be^7 nucleus, $p_{7,\beta-}$, for the reaction $Be^7(\beta^-,\nu)Li^7$ has been accurately calculated by J. N. Bahcall [Ba62]; his result is

$$p_{7,\beta-} = \frac{3.06 \times 10^{-9} \rho(1+x_1)}{T_6^{1/2}} \text{sec}^{-1}, \quad (17.308)$$

with ρ in gm/cm^3. (Reeves gives the numerical factor as 2.12×10^{-9}.) Reeves [Re65] then obtains for the quantity w (*cf.* (17.301)) the expression

$$w \equiv \frac{N_1\langle 7,1\rangle}{p_{7,\beta^-}} = 1.22 \times 10^{16} f_{7,1} g_{7,1} e^{-102.6/T_6^{1/3}} (T_6^{-1/6})(1+1/x_1)^{-1}, \quad (17.309)$$

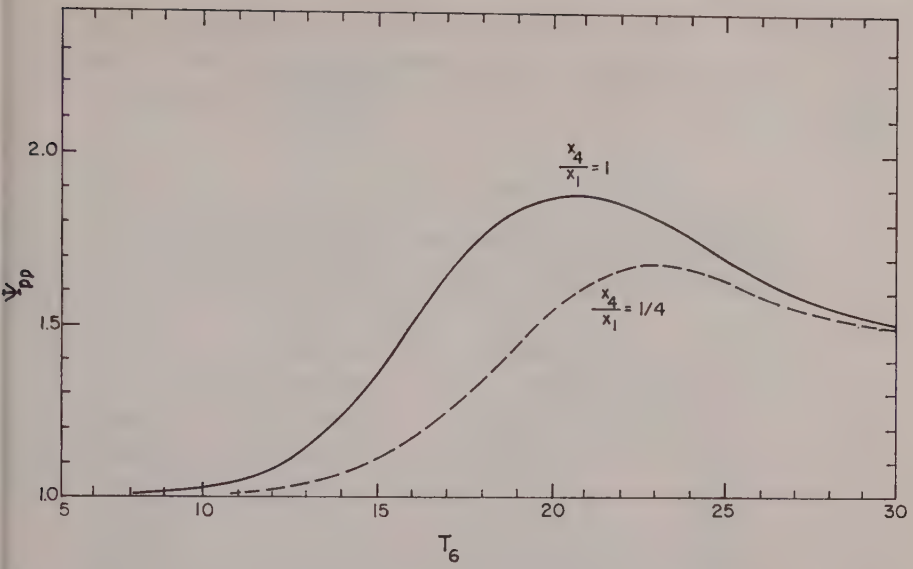


Fig. 17.13 $\Psi_{pp}(\alpha,w)$ as a function of temperature.

where $f_{7,1}$ and $g_{7,1}$ are, respectively, the screening factor and the correction for replacing the Gamow peak by a Gaussian for the reaction $Be^7(p,\gamma)B^8$. Note that w is a function only of the temperature. It is seen from the equation that $w \ll 1$ for $T_6 \lesssim 20$, so that the contribution of $(P3)$ to the energy production is negligible for such temperatures.

The quantity $\Psi_{pp}(\alpha,w)$ is plotted as a function of temperature for $(x_4/x_1) = (1/4)$ and 1 in Fig. 17.13 [Fig. 17 in Re65].

For ε_1' we have [equation (3.7) of Re65]

$$\varepsilon_1' = (2.06 \pm 0.2) \times 10^6 f_{1,1} g_{1,1} \cdot \rho x_1^2 e^{-33.810 T_6^{-1/3}} / T_6^{2/3}, \qquad (17.310)$$

where $f_{1,1}$, the screening factor, and $g_{1,1}$, the correction for replacing the Gamow peak by a Gaussian, are given by the equations

$$f_{1,1} = 1 + 0.25\rho^{1/2}/T_6^{3/2}, \qquad (17.311)$$

$$g_{1,1} = 1 + 0.0012 T_6^{1/3} + 0.0078 T_6^{2/3} + 0.0006 T_6 . \qquad (17.312)$$

Reeves re-writes (17.310) in such a form that (17.305) becomes

$$\varepsilon_{pp} = \rho x_1^2 \cdot f_{1,1} \cdot g_{1,1} \cdot \Psi_{pp}(\alpha,w) \cdot \varepsilon_0 (T/T_0)^\nu \text{erg/gm/sec}, \qquad (17.313)$$

where ε_0 is numerically equal to the unscreened rate of energy production per unit mass when $T = T_0$, $\rho x_1^2 = 1\,\text{gm/cm}^3$, and $x_4 = 0$. The exponent ν is

very nearly equal to the temperature exponent of ε_{pp} itself since the effect of the functions $f_{1,1}$, $g_{1,1}$, and Ψ_{pp} on the temperature exponent is very small. Values of $g_{1,1}$, ε_0, ν, and $\alpha(x_4/4x_1)^2$ are given in Table 17.5 (adapted from [Re65]).

Table 17.5 *

PARAMETERS FOR ENERGY PRODUCTION BY THE PROTON CHAIN†

$T_{0\,6}$	$g_{1,1}$	ε_0 (c.g.s.)	ν	$\alpha/(x_4/4x_1)^2$
5	1.05	1.82(−3)	5.95	2.19(−8)
10	1.07	6.79(−2)	4.60	3.80(−3)
15	1.09	3.77(−1)	3.95	1.34(0)
20	1.10	1.09(0)	3.54	5.48(1)
25	1.12	2.29(0)	3.25	7.68(2)
30	1.13	4.01(0)	3.03	5.77(3)
35	1.15	6.25(0)	2.86	2.89(4)
40	1.16	8.96(0)	2.72	1.09(5)
45	1.17	1.21(1)	2.60	3.37(5)
50	1.19	1.57(1)	2.50	8.90(5)
60	1.21	2.38(1)	2.33	4.41(6)
70	1.23	3.32(1)	2.19	1.59(7)
80	1.25	4.34(1)	2.08	4.56(7)
90	1.27	5.43(1)	1.99	1.11(8)
100	1.29	6.56(1)	1.92	2.41(8)

* Numbers in parentheses are the powers of ten by which the corresponding entries are to be multiplied.

† Adapted from Reeves [Re65].

The values of the cross section factors (extrapolated to zero energy) for the reactions connected with the proton chain are given in Table 17.6 (adapted from [Re65]).

The results presented here (*cf.* especially Fig. 17.13) concerning energy production by the proton chains ($P1$), ($P2$), and ($P3$) can be summarized as follows. If *no* He^4 is present, then only ($P1$) can occur ($\Psi_{pp} = 1$). If He^4 is present in appreciable quantities but if the temperature T is too low, then, again, only ($P1$) occurs. As T increases, ($P2$) and ($P3$) begin to compete with ($P1$), and eventually predominate over ($P1$) ($\Psi_{pp} > 1$). The reason for this is that the rate of the reaction He^3 $(\alpha,\gamma)Be^7$, which is the initiating reaction for

Table 17.6

CROSS SECTION FACTORS FOR THE PROTON CHAIN*

Reaction	$S(0)$ (Kev-Barns)
$H^1(p,\beta^+\nu)H^2$	$(3.36 \pm 0.4) \times 10^{-22}$
$H^2(p,\gamma)He^3$	$(2.5 \pm 0.4) \times 10^{-4}$
$He^3(He^3, 2p)He^4$	1.2×10^3
$He^3(\alpha,\gamma)Be^7$	0.47 ± 0.07
$Li^7(p,\alpha)He^4$	100 ± 25
$Be^7(p,\gamma)B^8$	$(2 \pm 1) \times 10^{-2}$

* From Reeves [Re65].

(*P*2) and (*P*3), increases more strongly with increasing T than does that of the reaction $He^3(He^3, 2p)He^4$, which is the concluding reaction of (*P*1) (*cf.* the discussion of (*P*2) and (*P*3) in Sect. 17.16). In the temperature range where (*P*2) and (*P*3) are more important than (*P*1), (*P*2) predominates over (*P*3) at the lower temperatures, while the converse is true at the higher temperatures. The maximum in each Ψ_{pp} curve in Fig. 17.13 occurs in the temperature region where (*P*2) (with only 4 per cent neutrino energy loss) predominates over (*P*3) (with 29 per cent neutrino energy loss). The reversal of the relative importance of (*P*2) and (*P*3) with increasing T arises because the reaction $Be^7(\beta^-,\nu)Li^7$ (the initiating one for (*P*2)), consisting of an electron capture, is much less temperature sensitive than the competing reaction $Be^7(p,\gamma)B^8$ (the initiating one for (*P*3)), consisting of a proton capture (*cf.* (17.308) and (17.309)). Because in electron capture, in contrast to the case of proton capture, there is no potential barrier to impede the penetration of particle a into the nucleus X (*cf.* Sects. 17.11 and 17.12), the second of the above two reactions increases more strongly with increasing T than does the first. Hence (*P*3) predominates over (*P*2) at the highest temperatures.

We conclude this section by noting that the rates of energy production given here by the formulae and tables for hydrogen burning are clearly of the right order of magnitude to account for the observed rates of energy release by the stars (recall that $1 \lesssim \bar{\varepsilon}_{\text{core}}(\text{erg/gm/sec}) \lesssim 10^4$ for main sequence stars). For example, representative central conditions in a Population I star like Sirius A (mass $M \simeq 2M_\odot$) would be $T_c = 20 \times 10^6\,^\circ\text{K}$, $\rho_c = 45\ \text{gm/cm}^3$, $X(\equiv x_1) = 0.60$, $Z = 0.03$, $x_{\text{CN}} = 0.25Z = 0.0075$. Using the information contained in this section, we find that for these conditions $\varepsilon_{\text{CN}} \simeq 90\ \text{erg/gm/sec}$

and $\varepsilon_{pp} \simeq 30$ erg/gm/sec. We note that the carbon cycle becomes more and more dominant over the proton chain as the temperature increases. Hence hotter (*i.e.*, more massive) main sequence stars derive their energy from the carbon cycle, while cooler (less massive) stars derive their energy from the proton chain (the two sources are of comparable effectiveness for main sequence stellar masses around 1.5 to 2 solar masses). Note also the great temperature sensitivity ($\nu \simeq 15$–20) for the carbon cycle as compared to that ($\nu \simeq 4$) for the proton chain.

17.18 Helium Burning Reactions

17.18a Triple Alpha Reaction

In Sections 17.16 and 17.17 we discussed mechanisms by which a star derives energy from the conversion of H^1 into He^4. We consider now the question of what happens when a star has converted all the H in its central regions into He^4. The details will be discussed in Chap. 26; but, clearly, no effective nuclear energy sources are immediately present (we are also assuming that any shell sources of H burning have also become ineffective), so the star will shrink slowly and release gravitational energy. This shrinkage will normally bring about a heating of the stellar material, as we have seen from our study of the virial theorem (*cf.* Sects. 17.4 and 25.3), and also an increase in the density. When the temperature and density become sufficiently high (say $T \sim 10^8$ °K and $\rho \sim 10^4$–10^5 gm/cm^3), the He^4 itself begins to "burn", becoming transformed into C^{12} with the release of a considerable amount of energy:

$$3\alpha \rightarrow C^{12} + \text{“}2\gamma\text{”} + 7.275 \text{ Mev} \tag{17.314}$$

(using Reeves' [Re65] Q value), where the quotation marks mean that emission of two γ-rays is sometimes replaced by emission of a positron-electron pair. This process will therefore provide an important source of energy at certain phases of a star's evolution. Note that this process is also the first step in the build-up of chemical elements heavier than He^4 in stars.

It was generally believed, prior to about 1952, that reaction (17.314) would proceed too slowly to be of importance, because it appeared to require a triple collision of three α particles, which would be a highly unlikely event. However, Salpeter [Sa52a] suggested that reaction (17.314) might proceed in two separate steps:

$$2He^4 + (99 \pm 6)\text{Kev} \rightleftarrows Be^8 \tag{17.315}$$

$$Be^8 + He^4 \rightarrow C^{12} + \text{“}2\gamma\text{”}, \tag{17.316}$$

where the 99 Kev in reaction (17.315) is the energy excess of Be^8 in its ground state over the rest-mass energy of two α particles. The Be^8 nucleus is therefore highly unstable, with a lifetime of the order of 10^{-17} sec, and reaction (17.315) is therefore endothermic. Because the lifetime of Be^8 is long compared to the duration of a collision between two α particles (say $\sim 10^{-21}$ sec), there should exist a small amount of Be^8 in a state of secular equilibrium with the He^4 nuclei. If this were the case, then the reaction (17.316) might take place and thus complete the reaction (17.314). The real importance of Salpeter's work, however, lay in his recognizing that E_{max} (*cf.* Sects. 17.13 and 17.14) for two α particles would be about 100 Kev at temperatures of about $(1–2)\times 10^8$ °K (in fact, $E_{max} = 84$ Kev for $T = 1.0\times 10^8$ °K). At such temperatures, then, reaction (17.315) would be expected to be a *resonant* reaction, proceeding through the ground level of Be^8 (recall that the resonance energy E_r is measured relative to the total rest-mass energy of the two interacting particles). In such a case the reaction rate for the reaction $2He^4 \rightarrow Be^8$ would be immensely enhanced and the abundance of Be^8 in secular equilibrium might be sufficiently large to be significant. In 1952, however, very little was known regarding the level structure of C^{12} and it was not certain that reaction (17.316) would proceed rapidly enough to be important, even with a relatively large abundance of Be^8.

A year or two later F. Hoyle [Ho54] was investigating processes of element synthesis in stars and he noted that unless reaction (17.314) proceeded fairly rapidly, there would be very little C^{12} present in the universe, in marked contrast to observations (and, in fact, to our own existence!). This conclusion is based on the fact that the reaction $C^{12}(\alpha,\gamma)O^{16}$ (which does not involve any unlikely triple collisions) would rapidly destroy any C^{12} produced by reaction (17.314). Hoyle came to the conclusion that the only way the relatively large abundance of C^{12} in the universe could be accounted for was for the reaction (17.316) *also* to be a resonant reaction. On this basis he predicted the existence of a level lying about 7.7 Mev above the ground state of C^{12} and suggested that experimentalists look for such a level at about this energy. A level having the required properties and lying (7.656 ± 0.008) Mev above the ground level was indeed found a short time later by Dunbar *et al.* [Du53]. Other experimental studies of this level have shown that the inverse of the $3He^4 \rightarrow C^{12}$ process, *i.e.*, break-up of C^{12} into three α particles, is possible, which proves that the direct process of fusion of He^4 into C^{12} can indeed occur.

Since the energy of this level exceeds the combined rest-mass energy of a Be^8 and a He^4 nucleus by (282 ± 6) Kev, then reaction (17.316) is endothermic by this amount. Since, however, E_{max} for the reaction $Be^8 + He^4$ is about 140 Kev for $T = 1\times 10^8$ °K, this is close enough to the resonance

energy of 282 Kev for reaction (17.316) to proceed through the 7.656 Mev level of C^{12}.

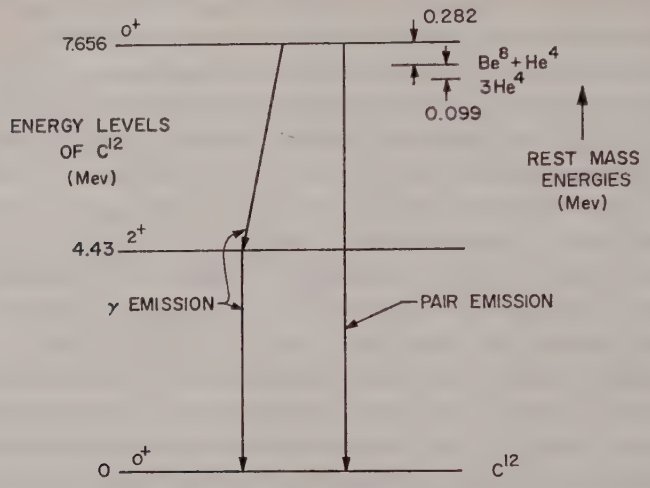


Fig. 17.14 Energy level diagram of C^{12}.

The energy level diagram of C^{12} is shown in Fig. 17.14, where it is seen that the 7.656 Mev level is actually the second excited level of C^{12}. Since reaction (17.316) is endothermic, one possible mode of decay of the compound nucleus C^{12*} from the second excited level is simply break-up back into three α particles. It happens (*cf.* later in this section) that this is, in fact, overwhelmingly the most likely mode of decay. The two other possible modes of decay are cascading to the ground level with the emission of two γ rays and direct transition to the ground state with the emission of a positron-electron pair with zero total angular momentum (a direct transition to the ground level with the emission of photons is forbidden since both levels have zero spin and the same parity, *cf.* Salpeter [Sa57]). It is of course just this small leakage into these last two modes of decay that makes reaction (17.316) a possibility at all. Because of the overwhelming probability of break-up of C^{12*} back into three α particles and of the relatively long lifetime of C^{12*}, a small abundance of C^{12*} nuclei also exists in a state of secular equilibrium with the He^4 nuclei.

We may now re-write reactions (17.315) and (17.316) more explicitly as follows:

$$2He^4 + (99 \pm 6)\text{Kev} \rightleftarrows Be^8 \tag{17.317a}$$

$$Be^8 + He^4 + (282 \pm 6)\text{Kev} \rightleftarrows C^{12*} \tag{17.317b}$$

$$C^{12*} \rightarrow C^{12} + \text{“}2\gamma\text{”} + (7.656 \pm 0.008)\text{Mev}, \tag{17.317c}$$

where the net energy released with the formation of each C^{12} nucleus is $Q = 7.656-(0.099+0.282) = 7.275$ Mev. The energy released per nucleon in the $3He^4 \rightarrow C^{12}$ process is then $7.275/12 \simeq 0.61$ Mev/nucleon, or about (1/12) the corresponding value for the $4H^1 \rightarrow He^4$ process. Hence, in the conversion of He^4 entirely into C^{12}, about 0.06 per cent of the mass disappears and is converted into energy.

The rate of energy generation per unit mass from the $3He^4 \rightarrow C^{12}$ process is clearly given by

$$\varepsilon(3\alpha \rightarrow C^{12}) = Q\frac{d}{dt}\left(\frac{N_{12}}{\rho}\right), \tag{17.318}$$

where $Q = 7.275$ Mev and $d(N_{12}/\rho)/dt$ is the rate of production of C^{12} nuclei per unit mass.* In order to calculate $d(N_{12}/\rho)/dt$, we must know the abundances of the "intermediate" nuclei Be^8 and $C^{12}*$. In the case of reactions (17.317) the assumption of secular equilibrium for Be^8 and $C^{12}*$ (*i.e.*, that $d(N_8/\rho)/dt = 0 = d(N_{12*}/\rho)/dt$, where N_8 and N_{12*} are the number densities of Be^8 and $C^{12}*$, respectively) is essentially always justified, because the lifetimes of both Be^8 and $C^{12}*$ are much less than one second. Equating to zero the net rate per unit mass of production of Be^8, we have (*cf.* (17.283a))

$$\frac{1}{2}N_4^2\langle 4,4\rangle + N_{12*}\Gamma_\alpha^{(12)}/\hbar - N_8\Gamma_\alpha^{(8)}/\hbar - N_4N_8\langle 4,8\rangle^* = 0\,, \tag{17.319}$$

where N_4 is the number density of He^4 nuclei; $\langle 4,4\rangle$ and $\langle 4,8\rangle^*$ are, respectively, the values of $\langle \sigma v\rangle$ for the reactions $2He^4 \rightarrow Be^8$ and $He^4 + Be^8 \rightarrow C^{12}*$; and $\Gamma_\alpha^{(8)}$ and $\Gamma_\alpha^{(12)}$ are, respectively, the "α widths" (widths for α particle emission) for Be^8 and $C^{12}*$ (the gammas divided by $\hbar$ give the rates per nucleus of α particle emission).

Similarly, for $C^{12}*$ we have

$$N_4N_8\langle 4,8\rangle^* - N_{12*}\Gamma_\alpha^{(12)}/\hbar - N_{12*}(\Gamma_{\pm\beta}+\Gamma_\gamma)/\hbar = 0\,, \tag{17.320}$$

where $(\Gamma_{\pm\beta} + \Gamma_\gamma)$ is the combined width for decay of $C^{12}*$ into C^{12} by the two possible decay processes. From (17.320) we have

$$N_{12*} = (\hbar/\Gamma)\cdot N_4N_8\langle 4,8\rangle^*\,, \tag{17.321}$$

where

$$\Gamma \equiv \Gamma_\alpha^{(12)} + (\Gamma_{\pm\beta} + \Gamma_\gamma) \tag{17.322}$$

* The time derivative here is, in general, the *Stokes* derivative, taken following the motion of a fluid element; see any book on fluid mechanics.

is the total width of the second excited level of C^{12}. Substituting (17.321) into (17.319), we obtain

$$\frac{1}{2}N_4^2\langle 4,4\rangle = N_8\Gamma_\alpha^{(8)}/\hbar + (1-\Gamma_\alpha^{(12)}/\Gamma)\cdot N_4N_8\langle 4,8\rangle^* . \quad (17.323)$$

However, the last term of (17.323) is from (17.322)

$$\frac{\Gamma_{\pm\beta}+\Gamma_\gamma}{\Gamma}N_4N_8\langle 4,8\rangle^* = N_4N_8\langle 4,8\rangle , \quad (17.324)$$

where $\langle 4,8\rangle$ (without the asterisk) is the value of $\langle\sigma v\rangle$ for the complete reaction $He^4+Be^8\rightarrow C^{12*}\rightarrow C^{12}$ (note that $(\Gamma_{\pm\beta}+\Gamma_\gamma)/\Gamma$ is the "branching ratio" for decay of C^{12*} into C^{12}, *cf.* Sect. 17.10). Thus (17.323) becomes

$$\frac{1}{2}N_4^2\langle 4,4\rangle = N_8\Gamma_\alpha^{(8)}/\hbar + N_4N_8\langle 4,8\rangle , \quad (17.325)$$

which is equivalent to (17.319) and which could actually have been written down in the first place. Note also that (17.321) becomes, because of (17.324),

$$N_{12*} = \frac{\hbar}{\Gamma_{\pm\beta}+\Gamma_\gamma}N_4N_8\langle 4,8\rangle . \quad (17.326)$$

Solving (17.325) for (N_8/N_4), we have

$$\frac{N_8}{N_4} = N_4\cdot\frac{\frac{1}{2}\langle 4,4\rangle}{(\Gamma_\alpha^{(8)}/\hbar)}\cdot\frac{1}{1+r} , \quad (17.327)$$

where

$$r \equiv \frac{N_4\langle 4,8\rangle}{(\Gamma_\alpha^{(8)}/\hbar)} \quad (17.328)$$

is the ratio of the probability of α particle capture by Be^8 and subsequent formation of C^{12} to the probability of α particle emission by Be^8. The number of C^{12} nuclei created per unit mass and time is then

$$\frac{d}{dt}\left(\frac{N_{12}}{\rho}\right) = N_4N_8\langle 4,8\rangle/\rho , \quad (17.329)$$

which also follows from (17.326), since $(\Gamma_{\pm\beta}+\Gamma_\gamma)/\hbar$ is the probability per unit time for decay of C^{12*} to C^{12}. Using (17.327), (17.329) becomes

$$\frac{d}{dt}\left(\frac{N_{12}}{\rho}\right) = \frac{1}{2}\left(\frac{N_4^3}{\rho}\right)\frac{\langle 4,4\rangle\langle 4,8\rangle}{\Gamma_\alpha^{(8)}/\hbar}\frac{1}{(1+r)} . \quad (17.330)$$

Equation (17.330) is general in the sense that it applies under conditions of secular equilibrium whether the relevant reactions are resonant or non-resonant. Let us now assume that reactions (17.317a) and (17.317b) are both resonant reactions (according to Reeves [Re65], this will be true for $T_8(\equiv T(°K)/10^8)$ between 0.85 and 50). Setting the statistical factor ω equal to unity in (17.223) (because the spins are all zero, *cf.* (17.119)), we have

$$\langle 4,4\rangle = 2\left(\frac{2\pi}{m_{4,4}}\right)^{3/2} \hbar^2 \Gamma_\alpha^{(8)} \frac{1}{(kT)^{3/2}} e^{-\chi_1/kT}, \tag{17.331}$$

$$\langle 4,8\rangle = \left(\frac{2\pi}{m_{4,8}}\right)^{3/2} \hbar^2 \frac{\Gamma_\alpha^{(12)}(\Gamma_{\pm\beta}+\Gamma_\gamma)}{\Gamma} \frac{1}{(kT)^{3/2}} e^{-\chi_2/kT}, \tag{17.332}$$

where $m_{4,4}(=(1/2)m_4, m_4 = \alpha$ particle mass) and $m_{4,8}(=(2/3)m_4)$ are reduced masses, $\chi_1(=99$ Kev) and $\chi_2(=282$ Kev) are the corresponding resonance energies, and the factor 2 in (17.331) is the "symmetry" number (see Sect. 17.10), arising from the identity of the two reacting particles. In (17.332) we have $\Gamma_\alpha^{(12)} \gg (\Gamma_{\pm\beta}+\Gamma_\gamma)$, so the factor involving the gammas can be replaced by $(\Gamma_{\pm\beta}+\Gamma_\gamma)$. In (17.331) only $\Gamma_\alpha^{(8)}$ appears since Be^8 in its ground state can decay (barring the possibility of collisions) only by break-up into two α particles, the inverse of the mode of formation (entrance and exit channels the same, *cf.* Sect. 17.9b).

We shall first estimate the value of the ratio r (see (17.328)). We have

$$r = N_4\left(\frac{2\pi}{m_{4,8}}\right)^{3/2} \hbar^3 \frac{\Gamma_{\pm\beta}+\Gamma_\gamma}{\Gamma_\alpha^{(8)}} \frac{1}{(kT)^{3/2}} e^{-\chi_2/kT}. \tag{17.333}$$

Writing $m_4 \simeq 4/N_0 (N_0 =$ Avogadro's number) and $N_4 \simeq (1/4)N_0\rho x_4$ ($x_4 =$ relative mass abundance of He^4), we obtain

$$r = 1.2 \times 10^{-10} \frac{\Gamma_{\pm\beta}+\Gamma_\gamma}{\Gamma_\alpha^{(8)}} \rho x_4 T_8^{-3/2} 10^{-14.0/T_8}, \tag{17.334}$$

where ρ is in gm/cm^3. According to Reeves [Re65], $(\Gamma_{\pm\beta}+\Gamma_\gamma) \simeq 2.5 \times 10^{-3}$ev; from the lifetime of $\sim 10^{-17}$ sec for Be^8, we get $\Gamma_\alpha^{(8)} \simeq 70$ ev, whence $(\Gamma_{\pm\beta}+\Gamma_\gamma)/\Gamma_\alpha^{(8)} \sim 4 \times 10^{-5}$. We see, then, that $r \ll 1$ under all reasonable stellar conditions, which means that only a minute fraction ($\sim 10^{-23}$ for $\rho = 10^4$ gm/cm^3 and $T_8 = 1$) of all Be^8 nuclei capture an α particle and form C^{12}; practically all Be^8 nuclei simply decay back into two α particles. In other words, there is almost perfect detailed balancing between the process

$2He^4 \rightarrow Be^8$ and its inverse. We shall accordingly neglect r in the following.

Using (17.331), (17.327) becomes

$$\frac{N_8}{N_4} = N_4 \frac{h^3}{(2\pi m_{4,4} kT)^{3/2}} e^{-\chi_1/kT} \tag{17.335a}$$

$$\simeq 1.6 \times 10^{-10} \rho x_4 \frac{e^{-8.53/T_8}}{T_8^{3/2}} \tag{17.335b}$$

if ρ is in gm/cm^3, and we note that the α particle width $\Gamma_\alpha^{(8)}$ has cancelled out. For $x_4 = 1$, $\rho = 10^4$ gm/cm^3, and $T_8 = 1$, we have $N_8/N_4 \simeq 4 \times 10^{-10}$, so that the equilibrium abundance of Be^8 is very small indeed.

Using (17.331) and (17.332) for the resonant reaction rates, we have for the rate of production of C^{12} nuclei per unit mass (*cf.* (17.330))

$$\frac{d}{dt}\left(\frac{N_{12}}{\rho}\right) = 3^{3/2}\left(\frac{N_4^3}{\rho}\right)\left(\frac{2\pi\hbar^2}{m_4}\right)^3 \frac{\Gamma_{\pm\beta} + \Gamma_\gamma}{\hbar} \frac{1}{(kT)^3} e^{-(\chi_1+\chi_2)/kT} f(3\alpha \rightarrow C^{12}), \tag{17.336}$$

where we have inserted the screening factor $f(3\alpha \rightarrow C^{12}) = f_{4,4} f_{4,8}$ (which is of this form because the two reactions $2He^4 \rightarrow Be^8$ and $Be^8(\alpha,\gamma)C^{12}$ are successive).

Before proceeding further, we note an alternative derivation of (17.336), which was used, in fact, by Salpeter [Sa57] in his original calculation of the rate of the triple α reaction. Salpeter recognized the circumstance that, because the reactions $2He^4 \rightarrow Be^8$ and its inverse are in almost perfect detailed balancing, the equilibrium abundance of Be^8 should be given by Saha's equation (see (3.29)). Regarding Be^8 as a compound system composed of two He^4 nuclei, we have (setting the statistical factor equal to unity since all spins are zero)

$$\frac{N_4^2}{N_8} = \frac{(2\pi m_{4,4} kT)^{3/2}}{h^3} e^{+\chi_1/kT}. \tag{17.337}$$

This equation is seen to be identical to (17.335), which was derived directly, assuming that the reaction $2He^4 \rightleftarrows Be^8$ was a resonant one. (The Saha equation gives the correct result if the reaction is resonant (but not otherwise), since in this case the Be^8 nucleus is always formed in a state of fixed energy, namely in the ground state.) Proceeding from this point just as before, we again recover (17.336).

Still another way of proceeding would be to apply the Saha equation also to the C^{12}* nuclei. This is justified since $(\Gamma_{\pm\beta} + \Gamma_\gamma) \simeq 2.5 \times 10^{-3}$ ev and $\Gamma_\alpha^{(12)} \simeq 7$ ev, *i.e.*, $\Gamma_\alpha^{(12)} \gg (\Gamma_{\pm\beta} + \Gamma_\gamma)$, which means that break-up of C^{12}* into

$He^4 + Be^8$ is overwhelmingly more probable than decay into C^{12} (by the ratio $\Gamma_\alpha^{(12)}/(\Gamma_{\pm\beta}+\Gamma_\gamma) \simeq 3\times 10^3$). Hence the reaction $He^4 + Be^8 \rightarrow C^{12}*$ and its inverse are also in very good detailed balancing. Thus we have (again setting the statistical factor equal to unity since all spins are zero here also)

$$\frac{N_4 N_8}{N_{12*}} = \frac{(2\pi m_{4,8} kT)^{3/2}}{h^3} e^{+\chi_2/kT} . \tag{17.338}$$

Multiplying (17.338) by (17.337) and solving for N_{12*}, we obtain

$$N_{12*} = N_4^3 \frac{h^6}{(m_{4,4}\, m_{4,8})^{3/2}} \frac{1}{(2\pi kT)^3} e^{-(\chi_1+\chi_2)/kT} .$$

Multiplying this last equation by $(\Gamma_{\pm\beta}+\Gamma_\gamma)/\hbar$, the probability per unit time of $C^{12}*$ decaying into C^{12}, and dividing by the density ρ, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{N_{12}}{\rho}\right) = \left(\frac{N_4^3}{\rho}\right) \frac{h^6}{(m_{4,4}\, m_{4,8})^{3/2}\,(2\pi kT)^3} \frac{\Gamma_{\pm\beta}+\Gamma_\gamma}{\hbar} e^{-(\chi_1+\chi_2)/kT} , \tag{17.339}$$

which is, aside from the screening factor $f(3\alpha \rightarrow C^{12})$, the same as (17.336).

The rate of energy production per unit mass from the triple α reaction is then given by

$$\varepsilon(3\alpha \rightarrow C^{12}) = Q \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{N_{12}}{\rho}\right)$$

$$= 3^{3/2}\left(\frac{N_0}{4}\right)^3 \left(\frac{2\pi\hbar^2}{m_4}\right)^3 \left(\frac{\Gamma_{\pm\beta}+\Gamma_\gamma}{\hbar}\right) x_4^3 \rho^2 \frac{1}{(kT)^3} e^{-(\chi_1+\chi_2)/kT} Q f(3\alpha \rightarrow C^{12}) , \tag{17.340}$$

where we have used the relation $N_4 \simeq (1/4) N_0 \rho x_4$. Using the value $(\Gamma_{\pm\beta}+\Gamma_\gamma) = 2.5\times 10^{-3}$ ev (accurate to within a factor of 4, *cf.* Re65), we obtain

$$\varepsilon(3\alpha \rightarrow C^{12}) = 3.46\times 10^{11} \rho^2 x_4^3 f(3\alpha \rightarrow C^{12}) \frac{e^{-43.2/T_8}}{T_8^3} \text{ erg/gm/sec} ,* \tag{17.341}$$

where ρ is in gm/cm^3. The screening factor* is given (*cf.* Re65) by

$$f(3\alpha \rightarrow C^{12}) = \exp\left(2.4\times 10^{-3}\, \rho^{1/2}/T_8^{3/2}\right) . \tag{17.342}$$

It is important to note that (17.340) and (17.341) are valid only if no further reactions (such as $C^{12}(\alpha,\gamma)O^{16}(\alpha,\gamma)Ne^{20}$, etc.) follow the production

* It has recently been pointed out by J.W. Truran and C.J. Hansen (*private communication*, 1967) that application of the screening factor in the usual way (as in (17.340) and (17.341)) to the reaction rates for *resonant* reactions may not be correct.

of C^{12}. The presence of such reactions (*cf.* later in this section) would increase the rate of energy production as given by the above equations. Estimates of the factors by which the rate of energy production is increased by these further α particle capture reactions are given by Deinzer and Salpeter [De64].

We write (17.341) in the form (*cf.* Re65)

$$\varepsilon(3\alpha \rightarrow C^{12}) = \rho^2 x_4^3 f(3\alpha \rightarrow C^{12}) \varepsilon_0 (T/T_0)^{\nu}, \tag{17.343}$$

where ε_0 is the unscreened rate of energy production per unit mass when $\rho^2 x_4^3 = 1(\mathrm{gm/cm^3})^2$ and $T = T_0$, and

$$\nu \equiv \frac{\mathrm{d} \ln \varepsilon(3\alpha \rightarrow C^{12})}{\mathrm{d} \ln T}\Big|_{T=T_0} = -3 + \frac{43.2}{T_{8,0}}. \tag{17.344}$$

Some values of these parameters are given in Table 17.7, adapted from [Re65]. The temperature range of most interest in "helium burning" stars with masses in the range, say, 0.5 to 10 solar masses is $1 \lesssim T_8 \lesssim (2 \text{ or } 3)$. Note the high temperature sensitivity in this range ($\nu \simeq 30$–40 for $T_8 \simeq 1.0$–1.2) and the relatively rapid decrease in ν as T increases (*cf.* (17.344)).

17.18b Further Alpha Particle Capture Reactions

As soon as some C^{12} has been produced by the triple α reaction in the core of a star, further α particle captures can occur, such as

	Q(Mev)	
$C^{12} + He^4 \rightarrow O^{16} + \gamma$	7.162	(17.345)
$O^{16} + He^4 \rightarrow Ne^{20} + \gamma$	4.730	(17.346)
$Ne^{20} + He^4 \rightarrow Mg^{24} + \gamma$		(17.347)

etc. These reactions are not very important as far as energy production is concerned. For example, the energy released per unit mass from reaction (17.345) is only about (3/4) of the energy released per unit mass from the triple α reaction. The main importance of these reactions is their role in determining the chemical composition of the stellar core after the He^4 has become exhausted.

Recent experimental results on the $O^{16}(\alpha,\gamma)Ne^{20}$ reaction (Reeves [Re64]) show that the reaction rate of this process is very small; yet the reaction rate of the reaction $Ne^{20}(\alpha,\gamma)Mg^{24}$ may be relatively large. This suggests that only negligible amounts of Ne^{20} may be formed by this α particle capture process in stars. Hence the abundance question involves essentially the C^{12}/O^{16} abundance ratio at the time of He^4 exhaustion in the core.

*Table 17.7**

PARAMETERS FOR HELIUM BURNING REACTIONS †

$T_{0\,8}$	$\frac{p_{12,4}}{(\rho x_4) f_{12,4}}$ $(\text{sec}^{-1}/(\text{gm/cm}^3))$	$\frac{p_{16,4}}{(\rho x_4) f_{16,4}}$ $(\text{sec}^{-1}/(\text{gm/cm}^3))$	ε_0 (c.g.s.)	ν
0.8	–	–	1.8(−12)	49
1.0	3.3(−21)	3.0(−28)	4.4(−8)	41
1.2	1.4(−19)	4.1(−26)	3.6(−5)	33
1.4	2.6(−18)	2.1(−24)	4.0(−3)	29
1.6	3.0(−17)	5.4(−23)	1.3(−1)	24
1.8	2.3(−16)	8.4(−22)	1.9(0)	21
2.0	1.3(−15)	8.8(−21)	1.5(+1)	19
2.2	6.2(−15)	2.7(−19)	8.3(+1)	17
2.4	2.4(−14)	1.3(−17)	3.3(+2)	15
2.6	7.9(−14)	3.4(−16)	1.1(+3)	14
2.8	2.3(−13)	5.6(−15)	2.8(+3)	13
3.0	6.1(−13)	6.3(−14)	6.4(+3)	12
3.2	1.5(−12)	5.1(−13)	1.3(+4)	11
3.4	3.3(−12)	3.3(−12)	2.4(+4)	9.8
3.6	7.0(−12)	1.7(−11)	4.2(+4)	9.1
3.8	1.4(−11)	7.3(−11)	6.7(+4)	8.5
4.0	2.7(−11)	2.7(−10)	1.0(+5)	7.9
4.5	1.1(−10)	4.2(−9)	2.4(+5)	6.7
5.0	3.8(−10)	3.8(−8)	4.6(+5)	5.7

* The numbers in parentheses are the powers of ten by which the corresponding entries are to be multiplied.

† Adapted from Reeves [Re65].

Reeves [Re64] points out that Ne^{20} is an abundant element in the universe (its abundance is approximately equal to the combined abundances of C^{12} and O^{16}). If appreciable quantities of Ne^{20} are not produced by these α particle capture reactions, then there may be some difficulty in accounting for the large amounts of Ne^{20} present in the universe, as other known mechanisms do not produce large quantities of this element (see, for example, Bashkin [Ba65e] or Burbidge, Burbidge, Fowler, and Hoyle [Bu57]).

The rates of reactions (17.345) and (17.346) are given in Table 17.7 (accurate to within a factor of the order of 10 to 20, say, adapted from Re65).

In this table $p_{12,4}$, for example, is the probability per sec per C^{12} nucleus of being transformed into an O^{16} nucleus by α particle capture; x_4 is the relative mass abundance of He^4; and $f_{12,4}$, for example, is the screening factor for the $C^{12}(\alpha,\gamma)O^{16}$ reaction. The rate of energy production per gram from this reaction is then given by

$$\varepsilon_{12,4} = (N_{12}/\rho)p_{12,4}Q_{12,4} \tag{17.348a}$$

$$\simeq N_0(x_{12}/12)p_{12,4}Q_{12,4}$$

$$= 5.74\times 10^{17}x_{12}p_{12,4}\,\text{erg/gm/sec} \tag{17.348b}$$

and from the reaction $O^{16}(\alpha,\gamma)Ne^{20}$

$$\varepsilon_{16,4} = 2.85\times 10^{17}x_{16}p_{16,4}\,\text{erg/gm/sec}. \tag{17.349}$$

The rate of reaction (17.345) can be approximated to an accuracy of about 30 per cent (as compared with the tabulated results in Table 17.7) except at the highest temperatures given in Table 17.7 by the formula [Re65]

$$\frac{p_{12,4}}{\rho x_4 f_{12,4}} = \frac{4.6\times 10^9\times 10^{-(30.05/T_8^{1/3}-\delta)}}{T_8^2[(1-4)\times 10^{-3}T_8 + 0.2T_8^{-2/3}]^2}\,\text{cm}^3/\text{gm/sec}, \tag{17.350}$$

where

$$\delta \equiv 0.06T_8^{2/3}(1+0.07T_8^{1/3}) \tag{17.351}$$

and $f_{12,4}$ is the same screening factor as for the $3\alpha\rightarrow C^{12}$ reaction, given by (17.342). Equation (17.350) is based on the assumption that reaction (17.345) is non-resonant. Since, however, the combined rest mass energy of C^{12} and He^4 lies about 40 Kev above the 7.118 Mev level of O^{16}, then this is a reaction in which the resonance energy $E_r(=-40$ Kev) is negative and therefore in which all energies of interest are *larger* than the resonance energy (recall that energies which enter into the Breit-Wigner dispersion formula (17.129) are measured relative to the combined rest mass energy of the incoming particles). This fact accounts for the somewhat unexpected temperature dependence of $p_{12,4}$ as given in (17.350).

The tabulated results for the $O^{16}(\alpha,\gamma)Ne^{20}$ reaction (*cf.* Table 17.7) may be approximated by the following three equations [Re65]:
$1\leq T_8\leq 2.1$:

$$\frac{p_{16,4}}{\rho x_4 f_{16,4}} = \frac{4.5\times 10^9\times 10^{-37.2/T_8^{1/3}}}{T_8^{2/3}}\,\text{cm}^3/\text{gm/sec}; \tag{17.352}$$

$2.1 \leq T_8 \leq 8$:

$$\frac{p_{16,4}}{\rho x_4 f_{16,4}} = \frac{6.3 \times 10^3 \times 10^{-45.9/T_8}}{T_8^{3/2}} \text{ cm}^3/\text{gm/sec} ; \tag{17.353}$$

$8 \leq T_8$:

$$\frac{p_{16,4}}{\rho x_4 f_{16,4}} = \frac{6.3 \times 10^3 \times 10^{-54.4/T_8}}{T_8^{3/2}} \text{ cm}^3/\text{gm/sec} . \tag{17.354}$$

Also,

$$f_{16,4} = \exp(3.2 \times 10^{-3} \rho^{1/2}/T_8^{3/2}) . \tag{17.355}$$

Equation (17.352) (accurate to within a factor of about 10) is the non-resonant contribution from the 5.80 and 6.72 Mev levels of Ne^{20}; (17.353) (accurate to within a factor of about 2) is the resonant contribution from the 5.64 Mev level; and (17.354) (accurate to within a factor of about 10) is the resonant contribution from the 5.80 Mev level.

It is clear that the C^{12}/O^{16} abundance ratio at the end of helium burning will depend on the relative rates of the $3He^4 \rightarrow C^{12}$ reaction (which creates C^{12}) and the $C^{12}(\alpha,\gamma)O^{16}$ reaction (which destroys C^{12}) during the helium burning stage. Calculations of this "final" C^{12}/O^{16} abundance ratio have yielded conflicting results due to the uncertainty in the rate of the $C^{12}(\alpha,\gamma)O^{16}$ reaction. Calculations by Hayashi, Hōshi, and Sugimoto [Ha62a] and Deinzer and Salpeter [De64] show that at least for stars with masses less than, say, some 4–5 solar masses, the final abundances of C^{12} and O^{16} are comparable with each other (the Reeves' [Re65] rate of the $C^{12}(\alpha,\gamma)O^{16}$ was used by Deinzer and Salpeter). The final abundances have also been discussed by Fowler and Hoyle [Fo64], who conclude that O^{16} is the main product of helium burning in *massive* stars (say $M \sim 30\ M_\odot$). On the other hand, calculations of the evolution of nearly pure helium stars by Divine [Di65] show that O^{16} is the main product of helium burning at least for masses $M \geq 0.5\ M_\odot$. However, the rate of the $C^{12}(\alpha,\gamma)O^{16}$ reaction used by Divine (based on material given in Fo64) was several times larger than that used by Deinzer and Salpeter. We conclude that the question of the final C^{12}/O^{16} abundance ratio must await improvements in our knowledge of the relevant nuclear cross sections.

17.18c Neon-Sodium Cycle

If a source of protons becomes available among the products of successive α particle capture reactions (*i.e.*, C^{12}, O^{16}, Ne^{20}, Mg^{24},...), and if some

Ne^{20} is present, the "neon-sodium" cycle may occur, at temperatures $\gtrsim 5\times 10^7\,°K$:

$$\begin{aligned} &Ne^{20}(p,\gamma)Na^{21}\\ &Na^{21}\rightarrow Ne^{21}+\beta^+ +\nu\\ &Ne^{21}(p,\gamma)Na^{22}\\ &Na^{22}(p,\gamma)Mg^{23}\\ &Mg^{23}\rightarrow Na^{23}+\beta^+ +\nu\\ &Na^{23}(p,\alpha)Ne^{20}. \end{aligned} \tag{17.356}$$

The main importance of this cycle is not energy production, but rather the transformation of some of the Ne^{20} into Ne^{21}, which may be an important source of neutrons if α particles are available:

$$Ne^{21}(\alpha,n)Mg^{24}. \tag{17.357}$$

Neutrons play an important role in theories of the synthesis of elements heavier than iron in stars (*cf.* Bashkin [Ba65e] and Burbidge, Burbidge, Fowler, and Hoyle [Bu57]).

We note that if hydrogen burning previously took place via the carbon cycle, a certain equilibrium abundance of C^{13} will have remained after hydrogen exhaustion, and C^{13} may similarly serve as a source of neutrons:

$$C^{13}(\alpha,n)O^{16}. \tag{17.358}$$

(Note that C^{13} and Ne^{21} both have one neutron in excess of an integral number of α particles; such nuclei readily undergo exothermic (α,n) reactions and may accordingly serve as neutron sources under suitable conditions.)

17.19 Carbon, Oxygen, and Neon Burning

After the transformation of He^4 into C^{12}, O^{16}, and possibly Ne^{20} in the core of a star, gravitational contraction may again result in an increase in the central temperature (provided that the star is sufficiently massive, (*cf.* Sect. 25.3) and density until these nuclei begin reacting with themselves. The first such reaction to occur (at $T_9 \equiv T(°K)/10^9 \simeq 0.6$–$0.7$) is carbon burning:

		Q(Mev)	
$C^{12}+C^{12}\rightarrow$	$Mg^{24}+\gamma$	13.930	(17.359a)
	$Na^{23}+H^1$	2.238	(17.359b)
	$Mg^{23}+n$	−2.605	(17.359c)
	$Ne^{20}+He^4$	4.616	(17.359d)
	$O^{16}+2He^4$	−0.114	(17.359e)

According to Reeves [Re65], reactions (17.359b) and (17.359d) are the most likely to occur, and these occur with about equal likelihood (reactions (17.359c) and (17.359e), being endothermic, are quite improbable). The important feature of the reactions (17.359) is that light particles (H^1, He^4) are produced and introduced into the mixture. These light particles will rapidly combine with other nuclei already present, thus resulting in a complicated series of isotopes, mostly lying in the range of mass number $20 \lesssim A \lesssim 28$.

If an appreciable abundance of C^{12} is present (say near the beginning of the carbon burning stage), the light particles will probably react with C^{12}. We may then have reactions such as $C^{12}(\alpha,\gamma)O^{16}$ or $C^{12}(p,\gamma)N^{13}(\beta^+,\nu)C^{13}(\alpha,n)O^{16}$, which can serve as a source of neutrons. According to A. G. W. Cameron (1964, unpublished lecture notes), if the temperature is high enough, photodisintegration of N^{13} may compete with the $N^{13}(\beta^+,\nu)C^{13}$ process and thereby partially "throttle" the neutron production: $N^{13}(\gamma,p)C^{12}$, $Q = -1.941$ Mev. Proton capture by Na^{23}, $Na^{23}(p,\gamma)Mg^{24}$, and α particle capture by Ne^{20}, $Ne^{20}(\alpha,\gamma)Mg^{24}$, are also fairly likely. Proton capture by other isotopes can also build up elements, but with small abundances, somewhat beyond Mg^{24}.

The reaction rate $p_{12,12}$ per C^{12} nucleus (related to the number of reactions per unit volume and time by $r_{12,12} = (1/2)N_{12}p_{12,12}$) for the reaction $C^{12}+C^{12}\rightarrow Mg^{24}*$ (formation of Mg^{24} in an excited state) is given by Reeves [Re65], to within a factor of 10, as follows:

$$\log_{10}\left(\frac{p_{12,12}}{\rho x_{12} f_{12,12}}\right) = 27.0 - \frac{36.55}{T_9^{1/3}}(1 + 0.1T_9)^{1/3} - \frac{2}{3}\log_{10}T_9\,, \quad (17.360)$$

where ρ is in gm/cm^3, x_{12} is the relative mass abundance of C^{12}, $p_{12,12}$ is in sec^{-1}, and the screening factor $f_{12,12}$ is given by

$$\left.\begin{aligned} f_{12,12} &= \exp(U_0/kT)\,, \\ U_0/kT &= \begin{cases} 1.2 \times 10^{-2}\rho^{1/2}/T_8{}^{3/2} & (U_0/kT \leq 1) \\ 3.5 \times 10^{-2}\rho^{1/3}/T_8 & (U_0/kT \geq 1)\,. \end{cases} \end{aligned}\right\} \quad (17.361)$$

Since the net amount of energy released with each $C^{12}+C^{12}$ reaction is about 13 Mev [Re65], the rate of energy production per unit mass from carbon burning is

$$\varepsilon_C = 10^{17.7}x_{12}p_{12,12} = \rho x^2{}_{12} f_{12,12}\varepsilon_0(T/T_0)^\nu \text{erg/gm/sec}\,, \quad (17.362)$$

where values of ε_0 and ν are given in Table 17.8 (adapted from Re65).

Models of carbon burning stars have been constructed by Deinzer and Salpeter [De65] and possibly by others.

At temperatures near $(1\text{–}1.5)\times 10^9\,°\mathrm{K}$ oxygen burning can take place:

$$\mathrm{O}^{16}+\mathrm{O}^{16}\rightarrow \begin{cases} \mathrm{S}^{32}+\gamma & Q(\mathrm{Mev})\ 16.539 & (17.363a)\\ \mathrm{P}^{31}+\mathrm{H}^{1} & 7.676 & (17.363b)\\ \mathrm{S}^{31}+n & 1.459 & (17.363c)\\ \mathrm{Si}^{28}+\mathrm{He}^{4} & 9.593 & (17.363d)\\ \mathrm{Mg}^{24}+2\mathrm{He}^{4} & -0.393 & (17.363e)\\ \mathrm{Si}^{30}+2\mathrm{H}^{1} & 0.424\,. & (17.363f) \end{cases}$$

Again, the H^1 and He^4 released are rapidly captured by the prevailing isotopes, and elements in the mass number range $25\lesssim A\lesssim 32$ are built up (mainly Si^{28} and S^{32}, according to A.G.W. Cameron (unpublished lecture notes, 1964)).

The reaction rate for formation of S^{32}* is given by Reeves [Re65] as follows (accurate to within a factor of 10 or 20):

$$\log_{10}\left(\frac{p_{16,16}}{\rho x_{16} f_{16,16}}\right)-40.5-\frac{59.02(1+0.14T_9)^{1/3}}{T_9^{1/3}}-\frac{2}{3}\log_{10}T_9\,, \qquad (17.364)$$

where

$$\left.\begin{aligned} f_{16,16} &= \exp\left(U_0/kT\right),\\ U_0/kT &= \begin{cases} 2.5\times 10^{-2}\rho^{1/2}/T_8^{3/2} & (U_0/kT\le 1)\\ 6.2\times 10^{-2}\rho^{1/3}/T_8 & (U_0/kT\ge 1)\,. \end{cases} \end{aligned}\right\} \qquad (17.365)$$

Taking 16 Mev as the average energy released with each $\mathrm{O}^{16}+\mathrm{O}^{16}$ reaction, we have for the energy generation rate

$$\varepsilon_{0x}=10^{17.8}x_{16}p_{16,16}=\rho x_{16}^2 f_{16,16}\varepsilon_0(T/T_0)^{\nu}\,\mathrm{erg/gm/sec}\,, \qquad (17.366)$$

where values of ε_0 and ν are given in Table 17.8 (adapted from Re65).

At temperatures around $T_9\simeq 1-2$ the photodissociation of Ne^{20} will begin to occur and may compete with oxygen burning as an energy source. (At these temperatures most photons begin to acquire energies approaching an Mev.) We have

$$\mathrm{Ne}^{20}+\gamma\rightarrow\mathrm{O}^{16}+\mathrm{He}^{4},\quad Q=-4.730\,\mathrm{Mev}. \qquad (17.367)$$

The α particle which is released will be quickly captured by some of the isotopes present. However, Ne^{20} has a capture cross section which is some

10 times larger than that of the other isotopes present at this stage [Re65]. The most probable reaction following (17.367) is then

$$Ne^{20}+He^{4}\rightarrow Mg^{24}+\gamma. \tag{17.368}$$

The net result of the photodissociation of Ne^{20} is thus seen to be the conversion of Ne^{20} into O^{16} and Mg^{24}:

$$2Ne^{20}\rightarrow O^{16}+Mg^{24}, \quad Q=4.583\ \text{Mev}. \tag{17.369}$$

Reeves [Re65] gives two equations for the photodissociation rate of Ne^{20}. For photodissociation from the 5.64 Mev level of Ne^{20}, we have

$$\log_{10}(p_{20,\gamma})=12.7\begin{pmatrix}+0.2\\-0.5\end{pmatrix}-\frac{28.4}{T_9}\ ; \tag{17.370}$$

Table 17.8 *

PARAMETERS FOR ENERGY PRODUCTION FROM CARBON BURNING, OXYGEN BURNING, AND PHOTODISINTEGRATION OF NEON †

	$C^{12}+C^{12}$		$O^{16}+O^{16}$		$Ne^{20}+\gamma$	
$T_{0\,9}$	ε_0(c.g.s.)	ν	ε_0(c.g.s.)	ν	ε_0(c.g.s.)	ν
0.5	1(−2)	35				
0.6	5(0)	32				
0.7	5(+2)	31				
0.8	2(+4)	29	3(−8)	47	3(−6)	81
1.0	9(+6)	27	4(−4)	43	5(+1)	65
1.2			6(−1)	40	4(+6)	54
1.4	3(+10)	24	2(+2)	38	9(+9)	47
1.5	1(+11)	23	2(+3)	37	3(+11)	44
1.6	5(+11)	22	2(+4)	36	4(+12)	41
1.8	6(+12)	21	7(+5)	34	3(+14)	36
2.0	5(+13)	20	2(+7)	33	2(+16)	33
2.2	3(+14)	20	3(+8)	31		
2.4	1(+15)	19	3(+9)	30		
2.6	5(+15)	18	3(+10)	29		
2.8	2(+16)	18	2(+11)	28		
3.0	5(+16)	17	1(+12)	27	2(+21)	22

* Numbers in parentheses are the powers of ten by which the corresponding entries are to be multiplied.

† Adapted from Reeves [Re65].

and, from the 5.80 Mev level,

$$\log_{10}(p_{20,\gamma}) = 13.40 - \frac{29.3}{T_9} \tag{17.371}$$

(both equations are accurate to within a factor of about 10). The rate of energy production per unit mass from this process is given by

$$\varepsilon_{\mathrm{Ne}} = 10^{17.13} p_{20,\gamma} x_{20} = x_{20} \varepsilon_0 (T/T_0)^{\nu} \text{ erg/gm/sec}, \tag{17.372}$$

where values of ε_0 and ν are given in Table 17.8 (adapted from Re65). The entries in this table are based on a compromise between (17.370) and (17.371) and are accurate to within a factor of 3 to 5.

17.20 Neutrino Energy Losses

Prior to about 1958 the only process known by which neutrinos could conceivably carry away really significant amounts of energy from stars was the "Urca process" of G. Gamow and M. Schönberg [Ga41]. This process involves transitions between levels of two isobars of nearly equal energies by electron capture and subsequent re-emission. One nucleus upon which the Urca process can operate is Cl^{35}. Since the ground state of S^{35} lies only about 0.17 Mev above the ground state of Cl^{35}, Cl^{35} can capture a negative electron and become S^{35} with the emission of a neutrino. However, S^{35} is unstable to β decay, so that S^{35} re-emits the negative electron, along with an antineutrino:

$$\begin{aligned} &Cl^{35} + \beta^- \rightarrow S^{35} + \nu \qquad (Q = -0.17 \text{ Mev}) \\ &S^{35} \rightarrow Cl^{35} + \beta^- + \bar{\nu} \qquad (Q = +0.17 \text{ Mev}). \end{aligned} \tag{17.373}$$

Thus the only effect is that a $(\nu\bar{\nu})$ pair is emitted with each cycle, and these $(\nu\bar{\nu})$ pairs can in principle carry away significant amounts of energy under suitable circumstances. This energy is directly lost from the star because, under all but the most extreme conditions of density and temperature, the neutrino mean free path is orders of magnitude larger than a stellar radius (*cf.* Sect. 17.16). Only certain, very special nuclei can partake of this process, and it is probable that the neutrino loss mechanisms to be discussed below are more important astrophysically than the Urca process. Energy loss rates by the Urca process are discussed, for example, by Reeves [Re65] and Schatzman [Sc65]. A "modified" Urca process has been invoked by Hansen [Ha66] in connection with the question of the cooling of vibrating neutron stars.

Recent developments in experiment and theory in connection with the fundamental β process (for example, Feynman and Gell-Mann [Fe58]) suggest the existence of a "universal Fermi interaction" in which direct interaction between neutrinos and electrons or photons should be possible. Such an interaction has not been directly observed in the laboratory but, if the interaction exists, it may be highly significant astrophysically in certain phases of stellar evolution. Some general discussions of the astrophysical implications of these neutrino energy losses have been given, for example, by Reeves [Re63, Re65] and by Fowler and Hoyle [Fo64].

The different ways by which this interaction may manifest itself are discussed by Reeves [Re65], A.G.W. Cameron (unpublished lecture notes, 1964), and Fowler and Hoyle [Fo64] (references to original sources are given in these works). It appears so far, however, that only three mechanisms are likely to be significant under astrophysical conditions. The region on the $\rho - T$ plane where these neutrino processes may be important may be taken as follows:

$$10^2 \lesssim \rho(\mathrm{gm/cm^3}) \lesssim 10^9$$
$$10^8 \lesssim T(^\circ\mathrm{K}) \lesssim 5 \times 10^9 \,. \tag{17.374}$$

The three processes are the *photo neutrino*, *pair neutrino*, and *plasma neutrino* processes, which we now discuss in turn. We shall omit most of the derivations.

17.20a Photo-Neutrino Process

This process may be symbolized as follows:

$$\gamma + e^- \begin{cases} \nearrow e^- + \gamma \\ \searrow e^- + \nu + \bar{\nu} \,. \end{cases} \tag{17.375}$$

It is therefore a variant of Thomson or Compton scattering of photons by free electrons (represented by the upper branch in (17.375)). In a small fraction of scatterings of photons by a free electron, a $(\nu\bar{\nu})$ pair instead of a photon will leave the electron (lower branch in (17.375)).

The rate of energy production per unit mass for this process, assuming the electrons to be non-degenerate and non-relativistic, and assuming X (hydrogen relative mass abundance) $= 0$, and $A/Z \simeq 2$, is given by Reeves [Re65] as follows:

$$\varepsilon_{\mathrm{photo}} \simeq -10^{7.7} T_9^8 \,\mathrm{erg/gm/sec}\,, \tag{17.376}$$

where $T_9 \equiv T(^\circ\mathrm{K})/10^9$. The rapid increase in $\varepsilon_{\mathrm{photo}}$ with increasing T arises in part from the rapid increase in the photon density as T increases. The fact

that $\varepsilon_{\text{photo}}$ is independent of density ρ is simply a consequence of the fact that the rate of process (17.375) per unit mass is proportional to the number of free electrons per unit mass, which is independent of density if the gas is fully ionized and if all electrons present are "ionization" electrons (*cf.* Sect. 15.1). It is clear that, according to (17.376), $|\varepsilon_{\text{photo}}|$ may at $T_9 \simeq 1$ exceed by several orders of magnitude the rate of energy production by nuclear sources in a star.

If the electron gas is partially degenerate, the rate of energy production from process (17.375) will be diminished below the value given by (17.376). The reason is that, because the electron undergoes a recoil subsequent to the "scattering," it must move into another "cell" in phase space. Partial degeneracy reduces the number of cells in phase space available to the recoiled electron and therefore reduces the cross section for Compton scattering (*cf.* Sect. 16.6a), and hence also for process (17.375).

17.20b Pair Neutrino Process

At temperatures of the order of 10^9 °K or greater, there will be enough photons in a Planck distribution having energies in excess of $2m_e c^2$ that creation of positron-electron (e^+e^-) pairs will proceed at a significant rate. The created (e^+e^-) pairs quickly recombine and produce (in most cases) two photons per pair. Because the lifetime for pair annihilation is very short on most astrophysical time scales, it can be assumed that pair creation and annihilation come to equilibrium, with the result that a certain equilibrium abundance of (e^+e^-) pairs is continuously present in the gas (*cf.* Sect. 24.9). The pair neutrino process owes its origin to the fact that a small fraction of the (e^+e^-) recombinations results in the emission of a ($\nu\bar{\nu}$) pair instead of a pair of photons:

$$\gamma + \gamma \rightleftarrows e^+ + e^- \rightarrow \nu + \bar{\nu}. \tag{17.377}$$

The pair neutrino process (17.377) could therefore occur in the virtual absence of matter, aside from the (e^+e^-) pairs created by the radiation field. (We say the "virtual" absence of matter because (1) *some* matter must be present to enable the radiation field to come to equilibrium and (2) pair production can occur only in the presence of some other particle in order that energy and momentum be conserved (*cf.* Heitler [He54, Sect. 26]).) In such a "vacuum" the rate per unit volume of process (17.377) clearly depends only on the photon density which in a near-Planckian distribution depends only on the temperature and not on the density. It can be shown that the rate per unit volume of process (17.377) is practically independent of density even if "ionization" electrons are present, provided that the electron gas is non-degenerate. (The presence of ionization electrons will affect the equi-

librium concentration of positrons and electrons; if the electron gas is non-degenerate, however, the product of the equilibrium electron and positron concentrations is practically independent of the concentration of ionization electrons, and the rate per unit volume of process (17.377) is proportional to this product.) The rate per unit *mass* of process (17.377) should then be inversely proportional to the density ρ if the electron gas is non-degenerate.

The rate of energy production per unit mass by process (17.377), either for a "vacuum" or in the presence of a non-degenerate electron gas, assuming that $X = 0$ and $A/Z \simeq 2$, is given by Reeves [Re65] as

$$\varepsilon_{\text{pair}} = -10^{18.7} e^{-11.88/T_9} T_9^3/\rho \text{ erg/gm/sec} \qquad (17.378)$$

(valid for $T_9 \lesssim 3$). It is seen from (17.378) that $|\varepsilon_{\text{pair}}| \sim 10^8$–$10^9$ erg/gm/sec for $T_9 \simeq 1$ and $\rho \simeq 10^5$ gm/cm^3, which is, astrophysically, a significant energy loss rate.

Table 17.9

ENERGY LOSS RATES FOR PHOTO AND PAIR NEUTRINOS*

	Photo	*Pair*		
$T_{0\ 9}$	$\log_{10}\varepsilon_0$(erg/gm/sec)	$\log_{10}\varepsilon_0$(erg/cm^3/sec)	ν	*Range* $\log_{10}\rho <$
0.3	3.5	0		4.8
.4	4.5	4.70	32	5.0
.5	5.3	7.7	27	5.2
.6	5.9	9.7	23	5.4
.7	6.5	11.2	20	5.4
.8	6.9	12.3	18	5.6
.9	7.3	13.2	16	5.6
1.0	7.7	14.0	15	5.6
1.2	8.3	15.2	14	5.8
1.4	8.9	16.1	13	6.0
1.6	9.3	16.8	12	6.4
1.8		17.4	11	
2.0		17.9		6.8
2.5		19.0		7.0
3.0		19.7		7.5
3.5		20.4		
4.0		21.0		
5.0		21.9		
6.0		22.6		

* Adapted from Reeves [Re65].

The presence of partial degeneracy in the electron gas would reduce the rate of process (17.377) below the value given by (17.378). Again the degeneracy reduces the number of cells in phase space available to a created negative electron and accordingly reduces the cross section for pair production, and hence also for process (17.377).

In Table 17.9 we present some numerical results relevant to the photo and pair neutrino processes. In this table ε_0, T_0, and ν are defined for the pair process by the equation

$$\varepsilon_{\text{pair}}\rho = -\varepsilon_0(T/T_0)^\nu \text{ erg/cm}^3\text{/sec}. \tag{17.379}$$

Table 17.10

PAIR NEUTRINO ENERGY LOSS RATES ($\log_{10}|\varepsilon_{\text{pair}}|$(erg/gm/sec)) WHEN AFFECTED BY PARTIAL DEGENERACY*

	$\log_{10}\rho$(gm/cm^3)				
$T_{0\,9}$	5	6	7	8	9
0.5	2.6	1.2			
.6	4.6	3.3	0.3		
.7	6.1	4.9	2.3		
.8		6.1	3.8		
.9		7.0	5.0	0	
1.0		7.8	5.9	1.5	
1.2		9.1	7.4	3.7	
1.4		10.0	8.5	5.4	
1.6			9.4	6.6	0.4
1.8			10.1	7.6	2.1
2.0			10.7	8.4	3.5
2.5			11.8	10	6.1
3.0				11.1	7.9

* Adapted from Reeves [Re65].

In Table 17.10 are presented some results for the pair process when partial electron degeneracy affects the results as given by (17.378) or by Table 17.9. Tables 17.9 and 17.10 were adapted from Reeves [Re65].

17.20c Plasma Neutrino Process

This process is most important in the low temperature, high density region of the $\rho - T$ plane, and was suggested by Adams, Ruderman, and

Woo [Ad63], who also worked out the theory of the process. A clear presentation of the calculation and some numerical results are also given by Inman and Ruderman [In64]. This process owes its origin to the possibility that, according to the theory of the weak interaction, the electrons which are accelerated in a plasma (a highly ionized but electrically neutral gas) can radiate neutrinos with a small probability, just as accelerated electrons can radiate electromagnetic waves. One can also say that a "plasmon" decays into a $(\nu\bar{\nu})$ pair. A "plasmon" is the quantum associated with electromagnetic waves in a plasma, just as a photon is the quantum associated with ordinary electromagnetic waves (see later in this subsection).*

An important parameter for the description of a plasma is the *plasma frequency* ω_P, given by the formula (*cf.* (2.157))

$$\omega_P^2 = \frac{4\pi e^2 N_e}{m_e} = \frac{4\pi e^2 c^2 N_e}{(m_e c^2)} \tag{17.380}$$

(valid if $(\hbar\omega_P)^2 \ll 4(m_e c^2)^2$, $2\pi\hbar$ = Planck's constant, m_e = electron rest mass, e = electronic charge, N_e = number density of free electrons). The plasma frequency is the natural frequency of a plasma. If the positive and negative charges making up the plasma are separated momentarily by a small amount, the electrostatic forces of the (essentially stationary) positive ions will exert a restoring force on the electrons which is proportional to the separation of the charges. Thus an oscillation of the electrons about their "equilibrium" positions will be set up having the frequency ω_P.

Equation (17.380) for ω_P was derived in Sect. 2.10d. It can also easily be derived from the following considerations. Consider a homogeneous, uniform plasma and suppose, for example, that all the electrons in the plasma are displaced, relative to the ions, in the same direction by the distance η. The interior of the plasma will still be electrically neutral, but now there will exist a charge density on the surface bounding the plasma, negative on that part of the surface toward which the electrons have been displaced, positive on the opposite. Assuming, for simplicity, that both surfaces can be treated as if they were infinite, parallel planes oriented perpendicularly to the direction of the displacement, the charge density on the two surfaces will clearly be $\pm N_e e\eta$. The resulting electric field strength will then be $E = 4\pi N_e e\eta$; this electric field will be in the direction of the displacement and will extend throughout the plasma. All the displaced electrons will now be subjected to

* In Chap. 2 and elsewhere in this book we have referred to quanta associated with electromagnetic waves in a plasma as "quasi-photons." Strictly speaking, a "quasi-photon" is a "*transverse* plasmon," *i.e.*, the quantum associated with a *transverse* electromagnetic wave in a plasma.

a restoring force which tends to return them to their original positions. This restoring force, per unit area perpendicular to the direction of the displacement, is clearly $-N_e e \ell E = -4\pi N_e^2 e^2 \eta \ell$, where ℓ is the perpendicular distance between the two surfaces. The mass of these displaced electrons, per unit area, is $N_e m_e \ell$, whence the equation of motion of the electrons is

$$\ddot{\eta} = -\frac{4\pi N_e e^2}{m_e}\eta, \tag{17.381}$$

from which (17.380) for the plasma frequency ω_P follows immediately.

The frequency ω_P is also the smallest frequency with which an electromagnetic wave can propagate in a plasma (assuming no steady magnetic field to be present in the plasma and neglecting dissipative effects). In fact, the (real) index of refraction of an electromagnetic wave having angular frequency ω in a plasma containing no permanent magnetic field is given by the approximate formula (*cf.* Sect. 2.10d)

$$\mu_v = \sqrt{1 - \omega_P^2/\omega^2}, \tag{17.382}$$

which shows that $\mu_v < 1$ for $\omega > \omega_P$. For $\omega = \omega_P$, $\mu_v = 0$ in the present approximation of negligible dissipation, which means that the phase velocity of the electromagnetic wave is infinite. The wave is therefore a standing wave and hence does not propagate. For $\omega < \omega_P$, (17.382) for μ_v is no longer valid; according to Sect. 2.10d, $\mu_v = 0$ in the absence of dissipation and we have total reflection of the electromagnetic wave and hence again no propagation.

It was shown in Sect. 2.10d that (17.382) could also be written in the form

$$\omega^2 = c^2k^2 + \omega_P^2, \tag{17.383}$$

which is the "dispersion relation" for an electromagnetic wave whose wave number is $k(= 2\pi/\text{wavelength})$ in a plasma. This equation shows that k is purely imaginary if $\omega < \omega_P$, which again means that the wave is attenuated spatially and does not propagate in this case. It was also shown in Sect. 2.10d that electromagnetic waves in a plasma behave (provided that they are of sufficiently high frequency and that the plasma is completely ionized), when quantized, like material relativistic particles each having a rest mass equal to $\hbar\omega_P/c^2$. Such "particles" are known as "plasmons" or "quasi-photons," and they are energetically unstable against decay into a $(\nu\bar{\nu})$ pair [In64].

For a relativistic, completely degenerate (*i.e.*, zero temperature) plasma the plasma frequency is given by the expression [Ad63]

$$\omega_P^2 = \frac{4c^2e^2p_F^3}{3\pi\hbar^3E_F}, \tag{17.385}$$

where p_F is the "Fermi momentum" of an electron, related to the electron density N_e by (*cf.* Sect. 24.1)

$$N_e = \frac{8\pi}{h^3}\int_0^{p_F} p^2 \mathrm{d}p = \frac{8\pi p_F^3}{3h^3}, \tag{17.386}$$

and E_F is the total energy (including the rest-mass energy) associated with p_F:

$$E_F^2 = p_F^2 c^2 + m_e^2 c^4. \tag{17.387}$$

(Equation (17.385) may be obtained formally from the ordinary expression, (17.380), by using (17.386) for N_e and replacing $m_e c^2$ in (17.380) by the total energy E_F.) Using (17.386) in (17.385), we have, also, the dimensionless quantity

$$\frac{\hbar\omega_P}{m_e c^2} = \frac{2\pi^{1/2} c\hbar e N_e^{1/2}}{(m_e c^2)^{3/2}(E_F/m_e c^2)^{1/2}}. \tag{17.388}$$

Assuming that the gas is completely ionized and that $A/Z \simeq 2$ for elements heavier than hydrogen, we have (*cf.* Sect. 15.2)

$$N_e = \frac{1}{2} N_0 \rho (1 + X), \tag{17.389}$$

where N_0 is Avogadro's number and X is the relative mass abundance of hydrogen. We then obtain

$$\begin{aligned}\frac{\hbar\omega_P}{m_e c^2} &= \left[\frac{(2\pi N_0)^{1/2} c\hbar e}{(m_e c^2)^{3/2}}\right](1+X)^{1/2}\rho^{1/2}\left\{1 + \right. \\ &\quad \left. + \left[\left(\frac{3\pi^2 N_0}{2}\right)^{2/3}\left(\frac{\hbar}{m_e c}\right)^2\right](1+X)^{2/3}\rho^{2/3}\right\}^{-1/4} \\ &= 3.97 \times 10^{-5}\frac{(1+X)^{1/2}\rho^{1/2}}{[1 + 6.40 \times 10^{-5}(1+X)^{2/3}\rho^{2/3}]^{1/4}}\end{aligned} \tag{17.390}$$

if ρ is in gm/cm^3.

Another useful parameter is

$$\frac{\hbar\omega_P}{kT} = \left(\frac{\hbar\omega_P}{m_e c^2}\right)\frac{m_e c^2}{kT} = \left(\frac{\hbar\omega_P}{m_e c^2}\right)\frac{5.930}{T_9}. \tag{17.391}$$

The rates of energy production per unit mass are then given by Adams *et al.* [Ad63] and by Inman and Ruderman [In64] by the following formulae (all employing c.g.s. units)*:

(1) Longitudinal Plasmons:

$$\varepsilon_\ell = -3.15 \times 10^{20} \left(\frac{\hbar\omega_P}{m_e c^2}\right)^4 \frac{1}{(e^{\hbar\omega_P/kT}-1)} \frac{1}{\rho} \text{ erg/gm/sec}. \quad (17.392)$$

(2) Transverse Plasmons:

$$\varepsilon_t = -1.228 \times 10^{22} \left(\frac{\hbar\omega_P}{m_e c^2}\right)^9 \frac{1}{\rho} \mathscr{F}(\hbar\omega_P/kT) \text{ erg/gm/sec}, \quad (17.393)$$

where

$$\mathscr{F}(x) \equiv \sum_{n=1}^{\infty} \int_1^{\infty} \exp(-nx\cosh\xi) \sinh^2\xi \cosh\xi \, d\xi. \quad (17.394)$$

An expansion exists for $\mathscr{F}(x)$ for $x \lesssim 0.5$ (*cf.* In64), of which the leading term is $\mathscr{F}(x) \simeq 2\zeta(3)/x^3 = 2.404/x^3$ ($\zeta(\nu)$ = Riemann zeta function, *cf.* (24.36b)), whence, for $\hbar\omega_P/kT \ll 1$,

$$\varepsilon_t \simeq -2.952 \times 10^{22} \left(\frac{\hbar\omega_P}{m_e c^2}\right)^6 \frac{1}{\rho} \left(\frac{T_9}{5.93}\right)^3 \text{ erg/gm/sec}. \quad (17.395)$$

For $x \gg 1$, we have [In64] $\mathscr{F}(x) \simeq (\pi/2)^{1/2} x^{-3/2} e^{-x}$, whence, in the limit $\hbar\omega_P/kT \gg 1$,

$$\varepsilon_t \simeq -1.539 \times 10^{22} \left(\frac{\hbar\omega_P}{m_e c^2}\right)^{7.5} \left(\frac{T_9}{5.93}\right)^{1.5} e^{-\hbar\omega_P/kT} \frac{1}{\rho} \text{ erg/gm/sec}. \quad (17.396)$$

In order to obtain some idea of the magnitude of these loss rates, we note that for $\rho = 10^8$ gm/cm^3 and $X = 0$, $10^7 \lesssim |\varepsilon_t|$ (erg/gm/sec) $\lesssim 10^9$ for $1 \lesssim T_9 \lesssim 2.5$. Thus the plasma neutrino process may also produce significant energy losses under certain conditions. The influence of the plasma neutrino process on the cooling rates of white dwarf and pre-white dwarf stars (*cf.* Sect. 25.6) has been calculated by Chin, Chiu, and Stothers [Ch65c] and by Vila [Vi65,66].

* Recently, the neutrino loss rates from the plasma process have been recomputed by Zaidi [Za65]. Zaidi's rates are one-fourth as large as those computed by Adams *et al.* [Ad63] and by Inman and Ruderman [In64].

*Table 17.11**

ENERGY LOSS RATES FOR THE TRANSVERSE PLASMA NEUTRINO PROCESS AS A FUNCTION OF TEMPERATURE FOR VARIOUS DENSITIES†

$\log_{10}\rho(\mathrm{gm/cm^3})$	$T_8(^\circ\mathrm{K})$	$\lvert\varepsilon_t\rvert$(erg/gm/sec)	$\lambda=\left(\frac{\partial \ln\lvert\varepsilon_t\rvert}{\partial \ln\rho}\right)_T$	$\nu=\left(\frac{\partial \ln\lvert\varepsilon_t\rvert}{\partial \ln T}\right)_\rho$
4	0.4	3.0491(−3)	1.8	3.16
	1.0	5.2820(−2)	1.9	3.05
	2.0	4.3162(−1)	2.0	3.02
	3.0	1.4638	2.0	3.01
	4.0	3.4762	2.0	3.01
	5.0	6.7958	2.0	3.00
5	0.4	1.4647(−1)	1.4	3.79
	1.0	3.8953	1.7	3.30
	2.0	3.5234(1)	1.8	3.10
	3.0	1.2256(2)	1.9	3.05
	4.0	2.9409(2)	1.9	3.03
	5.0	5.7800(2)	1.9	3.02
6	0.4	8.0711(−1)	−0.15	6.16
	1.0	1.1240(2)	1.1	4.29
	2.0	1.5694(3)	1.4	3.48
	3.0	6.1386(3)	1.5	3.27
	4.0	1.5494(4)	1.5	3.18
	5.0	3.1283(4)	1.6	3.13
7	0.4	2.3503(−2)	−3.2	12.5
	1.0	3.1620(2)	−0.29	7.24
	2.0	1.6879(4)	0.61	4.71
	3.0	9.7028(4)	0.88	3.99
	4.0	2.9100(5)	1.0	3.67
	5.0	6.4591(5)	1.1	3.50
8	0.4	4.7800(−8)	−8.1	26.3
	1.0	1.4708(1)	−2.3	14.0
	2.0	1.9809(4)	−0.43	7.84
	3.0	3.0965(5)	0.17	5.92
	4.0	1.4781(6)	0.45	5.02
	5.0	4.2601(6)	0.61	4.54

* The numbers in parentheses are the powers of ten by which the corresponding entries are to be multiplied.

† According to Zaidi [Za65], the values of $\lvert\varepsilon_t\rvert$ given in this table should all be multiplied by 0.25.

Figure 17.15 and Table 17.11 (adapted from [In64]) give numerical values of $|\varepsilon_t|$ and the density and temperature exponents $\lambda \equiv (\partial \ln |\varepsilon_t|/\partial \ln \rho)_T$, $\nu \equiv (\partial \ln |\varepsilon_t|/\partial \ln T)_\rho$, for $10^4 \leq \rho(\mathrm{gm/cm^3}) \leq 10^8$ and $0.4 \times 10^8 \leq T(^\circ\mathrm{K}) \leq 5 \times 10^8$. (However, see the preceding footnote.)

We note that in all three neutrino emission processes discussed above, a photon is ultimately transformed into a $(\nu\bar{\nu})$ pair, through interactions either

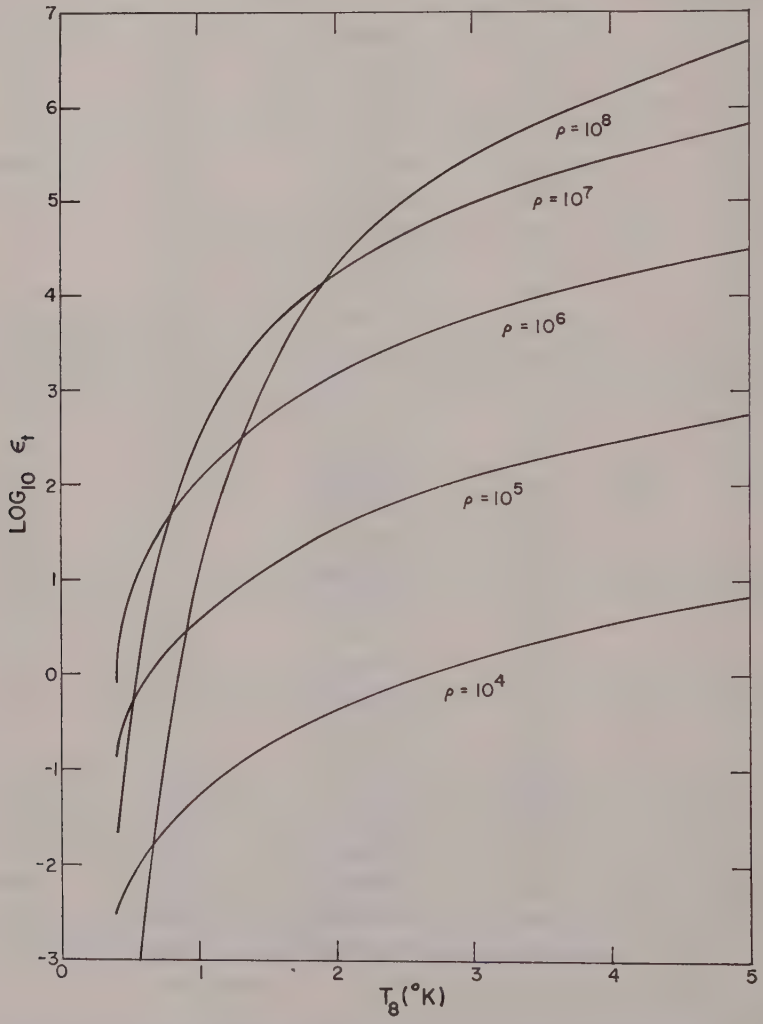


Fig. 17.15 Energy loss rates for the transverse plasma neutrino process as a function of temperature for various densities. According to Zaidi (Za65), all the rates shown in this figure should be multiplied by 0.25.

with positrons or electrons or with particles in a plasma. Direct transformation of a photon into a ($\nu\bar{\nu}$) pair cannot occur (for one reason, at least) because energy and momentum cannot both be simultaneously conserved, just as in the direct formation of an electron-positron pair from a photon (which, similarly, cannot occur; see Heitler [He54, Sect. 23]).

The effect of all these neutrino energy losses on the late stages of stellar evolution can be very drastic indeed since the neutrino luminosity of a star can exceed its photonic luminosity by many orders of magnitude (*cf.* Reeves [Re63], for example). As a rule of thumb, one can say that neutrino losses (from the photo or pair neutrino processes) start to become significant when carbon burning in a star has begun to set in. Practically, the main effect of neutrino losses on stellar evolution is to speed up by a large factor the rate of evolution in the late stages. It is interesting to note that the proof of the existence of a universal Fermi interaction may ultimately come from astrophysically, rather than physically, derived data.

Appendices and Indices

Physical and Astronomical Constants

Values for most of these constants are those given by Allen [Al 63a].

Solar luminosity
$L_\odot = 3.90 \times 10^{33}$ erg/sec

Solar absolute bolometric magnitude
$M_{b,\odot} = +4.77$

Solar mass
$M_\odot = 1.989 \times 10^{33}$ gm

Solar radius
$R_\odot = 6.960 \times 10^{10}$ cm

Solar effective temperature
$T_{e,\odot} = 5800\,^\circ\text{K}$

Solar surface gravity
$g_{s,\odot} = 2.738 \times 10^4$ cm/sec

Velocity of light in vacuo
$c = 2.99793 \times 10^{10}$ cm/sec

Constant of gravitation
$G = 6.668 \times 10^{-8}$ c.g.s. units

Boltzmann constant
$k = 1.3805 \times 10^{-16}$ erg/deg C
$= 8.617 \times 10^{-5}$ ev/deg C

Avogadro's number (phys. scale)
$N_0 = 6.025 \times 10^{23}$

Gas constant (phys. scale)
$\mathscr{R} = 8.317 \times 10^7$ erg/deg C/mole

Planck's constant
$h = 6.625 \times 10^{-27}$ erg sec
$\hbar \equiv h/2\pi = 1.0544 \times 10^{-27}$ erg sec

Electron rest mass
$m_e = 9.108 \times 10^{-28}$ gm
$= 0.5110$ Mev (million electron volts)

Electronic charge
$e = 4.803 \times 10^{-10}$ e.s.u.

Classical electron radius
$e^2/m_e c^2 = 2.818 \times 10^{-13}$ cm

Fine structure constant
$e^2/\hbar c = 1/137.037$

Thomson scattering cross section
$\sigma_0 = (8\pi/3)(e^2/m_e c^2)^2 =$
0.6652×10^{-24} cm^2

Compton wavelength of the electron
$\lambda\!\!\!^{-}_C = \hbar/m_e c = 3.8615 \times 10^{-11}$ cm

Atomic mass unit (AMU)
1 AMU $= H = 1/N_0 =$
$= 1.6598 \times 10^{-24}$ gm
$= 931.1$ Mev

Mass of hydrogen atom
$H^1 = 1.6733 \times 10^{-24}$ gm
$= 1.0081$ AMU

Mass ratio proton/electron
$m_p/m_e = 1836.1$

Electron volt (ev)
1 ev $= 1.602 \times 10^{-12}$ erg
$= 11.605 \times 10^3\,^\circ\text{K}$

Stefan-Boltzmann constant
$\sigma = 5.669 \times 10^{-5}$ erg/cm^2/(deg C)^4/sec

Radiation pressure constant
$a = 4\sigma/c =$
7.564×10^{-15} erg/cm^3/(deg C)^4

Reference List and Author Index

All cited works are listed here alphabetically, by (first) author. Each work is assigned a symbol, such as "Ha62," which is used in the text and in references to other parts of the Reference List and Author Index. The numbers in parentheses following each cited work give the section(s) or chapter(s) where the work is referred to. Mere specification of the chapter number of a chapter which has several sections (*e.g.*, 27) means that the cited work is referred to in the *introduction* to Chap. 27; similarly, the numbers (25.3, 25.3a), for example, mean that the cited work is referred to not only in the subsection 25.3a, but also in the *introduction* to Sect. 25.3. The numbers I or II behind a cited work mean that the work is referred to in the *introduction* to Parts I or II. Authors who are not first authors of works are also listed alphabetically. The cut-off date for the references is mid-1966. The Reference List and Author Index pertains only to the main text, and not to the Supplement.

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